WATER RESOURCES RESEARCH, VOL. 44, W03433, doi:10.1029/2007WR006054, 2008

A new rainfall model based on the Neyman-Scott process using cubic copulas

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Received 20 March 2007; revised 24 September 2007; accepted 5 November 2007; published 29 March 2008.

[1] A classical way to model rainfall is to use a Poisson process. Authors generally employed cluster of rectangular pulses to reproduce the hierarchical structure of rainfall storms. Although independence between cell intensity and duration turned out to be a nonrealistic assumption, only a few models link these variables. In this paper, a Neyman-Scott cluster process considering dependence between cell depth and duration is developed. We introduce this link with a cubic copula. Copulas are multivariate distributions modeling the dependence structure between variables, preserving the marginal distributions. Thanks to this flexibility, we are able to introduce a global concept of dependence between cell depth and duration. We derive the aggregated moments (first-, second-, and third-order moments) from the new model for several families of polynomial copulas and perform an application on Belgium and American data.

Citation: Evin, G., and A.-C. Favre (2008), A new rainfall model based on the Neyman-Scott process using cubic copulas, *Water Resour. Res.*, *44*, W03433, doi:10.1029/2007WR006054.

1. Introduction

[2] A stochastic model of the rainfall process at a point describes the behavior of precipitation in time. In this paper we focus on a specific type of model which uses Poisson-cluster processes to represent rainfall, namely the Neyman-Scott Rectangular Pulses Model (NSRPM). Many applications have proved the effectiveness of such models for the analysis of data collected on a short timescale, e.g., hourly [see *Rodriguez-Iturbe et al.*, 1987b; *Cowpertwait et al.*, 1996; *Onof et al.*, 2000]. Two well-known rectangular pulses models have been extensively used: the Bartlett-Lewis and the Neyman-Scott models. For a detailed review of developments with this type of rainfall model, see *Onof et al.* [2000]. In order to easily derive the theoretical moments of the aggregated intensity of rainfall (mean, variance, autocovariance), most models assume an independent relation between rain cell intensity and duration.

[3] This hypothesis is questionable for many reasons. The model would be more flexible without this condition and more suitable to adequately represent the temporal structure of rainfall. Furthermore, the link between storm duration and intensity is often significantly negative [*De Michele and Salvadori*, 2003]. *Kim and Kavvas* [2006] call the hypothesis of independence into question and review some developments [see *Cordova and Rodriguez-Iturbe*, 1985; *Singh and Singh*, 1991; *Bacchi et al.*, 1994; *Kurothe et al.*, 1997; *Goel et al.*, 2000] where correlated rain cells arrive according to a simple Poisson process (i.e., without clustering). The authors cite some bivariate exponential distributions [*Gumbel*'s [1960] type I, *Downton* [1970]] which are used to consider negative or positive correlation between rainfall intensity and duration. However, these representations do not consider clusters and appear to be limited. Without the cluster approach, the Poisson-based model has difficulties to represent more than one timescale [*Rodriguez-Iturbe et al.*, 1987a]. *Kim and Kavvas* [2006] develop a NSRPM which considers positive and negative correlations with a Gumbel's Type-II bivariate distribution. *Kakou* [1997] includes a dependence in the conditional mean within the Neyman-Scott model. First- and second-order moments are derived and proportions of dry periods seem to be adequately reproduced at all timescales [see *Onof et al.*, 2000]. In our case, a cubic copula links cell intensity and duration. This global structure of dependence in our model includes as a special case the Gumbel's Type-II distribution. In fact, the latter is composed of exponential margins and a dependence structure consisting of the Farlie-Gumbel-Morgenstern copula family (or FGM) discussed by *Morgenstern* [1956], *Gumbel* [1958], and *Farlie* [1960]. The use of cubic copulas is innovative in itself if we consider the scarce applications with this class of copulas. Thus we develop a more general NSRPM and derive first-, second-, and third-order moments which enable us to fit the model with the generalized method of moments. The paper is divided into four more sections as follows. In section 2 the basic theory concerning the copulas and the class of cubic copulas is presented, while section 3 develops the Neyman-Scott model including dependence between rain cell intensity and duration. Aggregated moments are derived up to the third order. Two applications are performed in section 4, and section 5 concludes.

2. Copula Theory

2.1. Copula Definition

[4] Copulas are extremely useful to solve problems that involve modeling the dependence between several random variables. In finance, they can be applied to derivative pricing, portfolio management, the estimation of risk measures (e.g., Value at Risk), risk aggregation, credit modelling

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0043-1397/08/2007WR006054$09.00

[see *Embrechts et al.*, 2002; *Cherubini et al.*, 2004]. We also observe numerous applications in actuarial science [*Frees and Valdez*, 1998]. Recently, copulas appeared in hydrology [e.g., *De Michele and Salvadori*, 2003; *Favre et al.*, 2004a; *Grimaldi and Serinaldi*, 2006; *Genest and Favre*, 2007; *Renard and Lang*, 2007]. The copula representation enables us to separate the dependence structure from the marginal behavior. For a detailed review of the theory of copulas, the authors refer to the monograph by *Joe* [1997] and *Nelsen* [2006].

[5] For the sake of simplicity, we concentrate on the bivariate case. One representation theorem from *Sklar* [1959] highlights the interest of copulas in dependence modeling. Any joint cumulative distribution function (c.d.f.) $H(x, y)$ of any pair (X, Y) of random variables may be written in the form

$$H(x,y) = C\{F(x), G(y)\}, x, y \in \mathbb{R} \quad (1)$$

where the copula C describes the dependence between the random variables X and Y. Hence Sklar's theorem implies that for multivariate distribution functions the univariate margins and the dependence structure (represented by a copula) can be separated. Furthermore, *Sklar* [1959] showed that in the case of continuous distributions F and G, C is uniquely determined.

[6] As is highlighted by *Genest and Favre* [2007], a judicious choice to measure dependence is to use nonparametric measures. Two coefficients, commonly used to this end, are Spearman's rho ρ and Kendall's tau τ. Here ρ and τ are expressible in terms of the copula and are given by

$$\rho = 12 \int_{[0,1]^2} C(u,v) \mathrm{d}v \mathrm{d}u - 3,$$
$$\tau = 4 \int_{[0,1]^2} C(u,v) \mathrm{d}C(u,v) - 1.$$

We say that a copula C is "symmetric" if $C(u, v) = C(v, u)$ for all (u, v) in $[0, 1]^2$.

[7] In our model, the copula C links cell depth and lifetime. The use of a copula increases the complexity of the NSRPM. Consequently, the problem of fitting has to be taken into account to choose the form of the copula. There is no consensus over the view of copulas as being the optimal solution to model dependencies, as illustrated by the recent and heated discussion initiated by *Mikosch* [2006] and followed by numerous authors [see, e.g., *Genest and Rémillard*, 2006; *Joe*, 2006; *Embrechts*, 2006]. In our case, the idea of using cubic copulas is motivated by the need to have algebraic expressions for the theoretical moments. The choice of the bivariate distribution on cell intensity and duration is very limited and nontrivial distributions are necessaries to have both supple dependence structures and algebraically tractable expressions.

[8] Because of the difficulty in obtaining a suitable likelihood function [see, e.g., *Chandler and Onof*, 2005, p. 11], the generalized method of moments is the simplest way to perform the fitting of the NSRPM. With this method, the first- and second-order moments of the aggregated process must be related to the model parameters. The successive steps to reach in order to derive the moments limit the choice of distributions to be used. Exponential families are usually employed to characterize the different random variables (see section 3), e.g., exponential distribution, gamma distribution. Others type of distributions appear to increase the complexity. In particular, we tried to derive the moments with numerous bivariate distributions presented by *Hutchinson and Lai* [1990]. The expressions to be resolved appear to be intractable. Consequently, we study bivariate distributions on cell intensity and duration composed of products of exponentials, i.e., exponential margins with cubic copulas.

2.2. Cubic Copulas

[9] The following results are provided by *Quesada Molina and Rodríguez Lallena* [1995] and *Nelsen et al.* [1997]. The use of quadratic copulas with exponential margins is a priori a judicious choice of bivariate distributions on cell intensity and duration. The aggregated moments can be computed and afterwards the method of moments can be applied. However, the only copula family with quadratic sections in both u and v is the FGM copula which leads to the Gumbel type-II distribution as introduced in the NSRPM by *Kim and Kavvas* [2006]. With this copula family, the degree of dependence is restricted to $[-2/9;2/9]$ for Kendall's tau. That is why *Nelsen et al.* [1997] introduce copulas with cubic sections and prove that these copulas increase the dependence degrees compared to copulas with quadratic sections. As far as we know, this class of copulas has never been used in hydrology.

[10] Following *Nelsen et al.* [1997], let S denote the union of the points in the square $[-1, 2] \times [-2, 1]$ and the points in and on the ellipse $\{(u, v) \in \mathbb{R}^2, u^2 - uv + v^2 - 3u + 3v = 0\}$. Copulas with cubic sections in both u and v are defined by

$$C(u,v) = uv + uv(1-u)(1-v)[A_1 v(1-u) + A_2(1-v)(1-u) + B_1 uv + B_2 u(1-v)] \quad (2)$$

where A_1, A_2, B_1, B_2 are real constants such that the points (A_2, A_1), (B_1, B_2), (B_1, A_1), and (A_2, B_2) all lie in S. The main theorems related to cubic copulas are reported in Appendix A1. Several properties can be obtained from this representation of a cubic copula. *Nelsen et al.* [1997] proved that if a copula C is defined as in (2), we then have:

$$\rho = (A_1 + A_2 + B_1 + B_2)/12$$

and

$$\tau = (A_1 + A_2 + B_1 + B_2)/18 + (A_2 B_1 - A_1 B_2)/450. \quad (3)$$

If C has cubic sections in both u and v, then $|\rho| \leq \sqrt{3}/3$. Simple calculations lead us to find that we also have $|\tau| \leq \sqrt{494\sqrt{19} - 2053}/25$. The proof is given in Appendix A2. Moreover, C is symmetric if and only if $A_1 = B_2$. Numerous subfamilies may be built. When $A_1 = A_2 = B_1 = B_2 = \theta$, we obtain the special case of the Farlie-Gumbel-Morgenstern (FGM) copula defined by $C(u, v) = u\, v \{1 + \theta(1-u)(1-v)\}$. We present several of these families in Table 1 and in Appendix A1. The degrees of dependence they are able to reproduce are reported in Table 2. The graph of the parameter space covered in S for each cubic copula is given in Figure 1.

[11] Families of cubic copulas with one parameter are easy to obtain. The dependence between cell intensity and duration being probably negative, we developed the following copulas in order to attain minimum negative values for the degree of dependence. The cubic copulas already described by *Nelsen et al.* [1997] (i.e., the Sarmanov, Frank

Table 1. Families of One-Parameter Copulas With Cubic Sections in u and v

Name	A_1	A_2	B_1	B_2	$\theta \in$
Sarmanov	$\theta(3-5\theta)$	$\theta(3+5\theta)$	$\theta(3+5\theta)$	$\theta(3-5\theta)$	$[-\sqrt{7}/5, \sqrt{7}/5]$
Frank cubic	$3\theta(1-\theta)$	$3\theta(1+\theta)$	$3\theta(1+\theta)$	$3\theta(1\text{-}\theta)$	$[-\sqrt{3}/3, \sqrt{3}/3]$
Plackett cubic	$\theta(1\text{-}\theta)$	θ	θ	$\theta(1\text{-}\theta)$	$[-1,2]$
AMH cubic	θ	$\theta(1+\theta)$	θ	θ	$[-1,1]$
Sym1	θ^2-3	θ	θ	θ^2-3	$[-1,2]$
Sym2	$\frac{\theta-3-\sqrt{9+6\theta-3\theta^2}}{2}$	θ	θ	$\frac{\theta-3-\sqrt{9+6\theta-3\theta^2}}{2}$	$[-1,3]$
Sym3	$(2+\sqrt{3})\theta$	θ	θ	$(2+\sqrt{3})\theta$	$[1-\sqrt{3},2-\sqrt{3}]$
Asym1	$\theta-2$	$\theta-1$	$\theta-1$	$\theta^2/3-2$	$[0,3]$
Asym2	$\theta(1-\theta)-1$	θ	θ	θ	$[-0.8525,1]$
Asym3	$\theta(1-\theta)+1$	θ	θ	θ	$[-1,0]$

cubic, and Plackett cubic families) are introduced in Appendix A1.

2.2.1. AMH Cubic

[12] We derive the second-order approximation to the Ali-Mikhail-Haq (AMH) family of copulas [see *Ali et al.*, 1978] given by

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}$$

for $\theta \in [-1, 1]$. This family is symmetric and includes the independence case for $\theta = 0$.

2.2.2. Sym1

[13] A copula with cubic sections must respect the conditions defined in Theorem A.2 (Appendix A1). We can define a one-parameter cubic family of copulas by taking $A_2 = B_1 = \theta \in [-1, 2]$ and $A_1 = B_2 = \theta^2 - 3$ in such a way that $(\theta, \theta^2 - 3)$ lies in $[-1, 2] \times [-2, 1]$. This family is symmetric and covers a large interval of dependence but does not include the independence as a special case.

2.2.3. Sym2

[14] Let $A_2 = B_1 = \theta \in [-1, 3]$ and let $A_1 = B_2 = \frac{\theta-3-\sqrt{9+6\theta-3\theta^2}}{2}$. The points (A_2, A_1), (B_1, B_2), (B_1, A_1) and (A_2, B_2) all lie in S and describe the elliptical contour of S from the point $[-1, -2]$ to the point $[3, 0]$. This family is symmetric and attains minimum values for both ρ and τ.

2.2.4. Sym3

[15] The point in S attaining the minimum value for ρ is $[1-\sqrt{3}, -1-\sqrt{3}]$. If we consider a linear relation between $A_2 = B_1$ and $A_1 = B_2$ passing through this point and the point $[0, 0]$, we obtain a family that is symmetric, having linear relations in every parameter, containing independence and attaining the minimum value for ρ.

2.2.5. Asym1-Asym3

[16] Asymmetric families are easily obtained as soon as $A_1 \neq B_2$. The family Asym1 maintains a quadratic relation in θ and attains a large interval of dependence.

3. Neyman-Scott Model

3.1. Model Structure

[17] The same notation as *Rodriguez-Iturbe et al.* [1987a] is used, except for the number of cells which we denote by D in order to avoid confusion with the copula C. Suppose the arrival times of storm origins occur as a Poisson process with rate λ. A random number D of cells is associated with each storm origin and follows a geometric distribution with mean μ_D. The cell origins are independently separated from the storm origin by distances which are exponentially distributed with parameter β. Let a rectangular pulse be associated with each cell origin, representing rainfall cell. Cell intensity is denoted by X and cell duration is denoted by L. The total intensity at time t, $Y(t)$, is the summation of the intensities of cells alive at time t. Let $X_{t-u}(u)$ representing the rainfall intensity at time t due to a cell with starting time t-u. Let us consider process $Y(t)$ defined by

$$Y(t) = \int_{u=0}^{\infty} X_{t-u}(u)\mathrm{d}N(t-u)$$

where $\mathrm{d}N(t-u) = 1$ if a cell has origin at time $t-u$ and 0 if not. We have [see *Cox and Isham*, 1980] $\mathrm{E}\{\mathrm{d}N(t-u)\} = \mathrm{Var}\{\mathrm{d}N(t-u)\} = \lambda\mu_D\mathrm{d}u$. Cell intensity and duration are traditionally linked by the following definition of $X_{t-u}(u)$

$$X_{t-u}(u) = \begin{cases} X & \text{with probability} \quad \mathcal{F}_L(u) = \mathrm{e}^{-\eta u} \\ 0 & \text{with probability} \quad F_L(u) = 1-\mathrm{e}^{-\eta u} \end{cases} \tag{4}$$

where the cell duration is exponentially distributed with parameter η. We want the cell intensity and duration to be dependent. When a link exists between X and L, relation (4) does not hold anymore. We wish to obtain the distribution of cell intensity X with $L > u$ that we denote $F_{X,L}$. We use a copula function C to express joint distribution of X and L. We have

$$\begin{aligned} F_{X,L}(x,u) &= \Pr(X \leq x, L > u) \\ &= \Pr(X \leq x) - \Pr(X \leq x, L \leq u) \\ &= F_X(x) - C\{F_X(x), \mathrm{F}_L(u)\}. \end{aligned} \tag{5}$$

At this point, it is important to underline that relation (5) does not limit the choice of the bivariate distribution on cell intensity and duration. Any bivariate distribution can be expressed with a bivariate copula and marginal distributions according to Sklar's theorem. However, it appears that classical bivariate distributions make very complex and even impossible the derivation of the aggregated moments. The use of cubic copulas and exponential margins is an elegant way to obtain a flexible class of bivariate distributions and for which aggregated moments are easily derivable.

3.2. First- and Second-Order Moments

[18] Let Y_i^h be the aggregated rainfall depth in the ith time interval of length h, so that

$$Y_i^h = \int_{(i-1)h}^{ih} Y(t)\mathrm{d}t.$$

Table 2. Degrees of Dependence for Families of One-Parameter Copulas With Cubic Sections in u and v

Name	$\rho \in$	$\tau \in$
Sarmanov	$[-0.5292,0.5292]$	$[-0.3725,0.3725]$
Frank cubic	$[-0.5774,0.5774]$	$[-0.4003,0.4003]$
Plackett cubic	$[-0.5,0.1667]$	$[-0.34,0.1134]$
AMH cubic	$[-0.25,0.4167]$	$[-0.1688,0.28]$
Sym1	$[-0.5416,0.5]$	$[-0.3778,0.34]$
Sym2	$[-0.5774,0.5]$	$[-0.4006,0.3533]$
Sym3	$[-0.5774,0.2113]$	$[-0.4003,0.1388]$
Asym1	$[-0.5,0.5]$	$[-0.34,0.34]$
Asym2	$[-0.428,0.1667]$	$[-0.2886,0.1155]$
Asym3	$[-0.3333,0.0834]$	$[-0.2222,0.0556]$

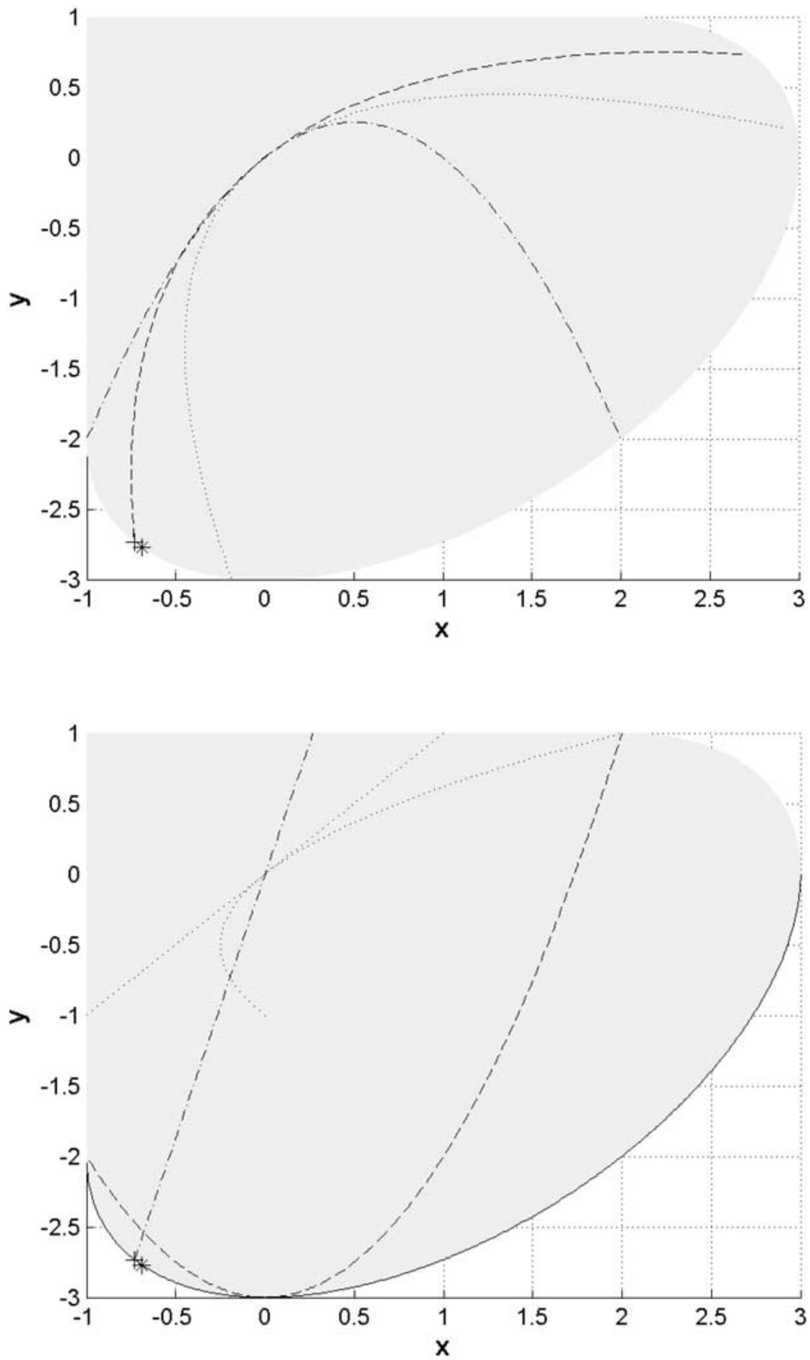

Figure 1. Graph of (A_2, A_1),(B_1, B_2), (B_1, A_1), (A_2, B_2) in S (grey surface) for each cubic copula. The points referring to the minimum values for the Kendall's tau and the Spearman's rho are represented by an asterisk and a plus symbol, respectively. Shown are (a) Sarmanov (dotted line), Frank cubic (dashed line), Plackett cubic (dash-dot line); (b) AMH cubic (dotted line), Sym1 (dashed line), Sym2 (dash-dot line), Sym3 (plain line); and (c) Asym1 (dotted line), Asym2 (dashed line), Sym3 (dash-dot line).

The mth moment is expressed by

$$\begin{aligned}\mathrm{E}\{(Y_i^h)^m\} &= \int_0^h \int_0^h \ldots \int_0^h \mathrm{E}\{Y(t_1)Y(t_2)\ldots Y(t_m)\}\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_m \\ &= m! \int_0^h \int_{t_1}^h \ldots \int_{t_{m-1}}^h \mathrm{E}\{Y(t_1)Y(t_2)\ldots Y(t_m)\} \\ &\quad \cdot \mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_m. \end{aligned} \quad (6)$$

Properties of point processes include the independence between point process generation $\mathrm{d}N(t_m - u_m)$ and cell intensity distribution $X_{t_m-u_m}(u_m)$. We then write:

$$\begin{aligned}&\mathrm{E}\{Y(t_1)Y(t_2)\ldots Y(t_m)\} \\ &= \int_{u_1=0}^{\infty} \int_{u_2=0}^{\infty} \ldots \int_{u_m=0}^{\infty} \mathrm{E}\{X_{t_1-u_1}(u_1)X_{t_2-u_2}(u_2)\ldots X_{t_m-u_m}(u_m)\} \\ &\quad \times \mathrm{E}\{\mathrm{d}N(t_1-u_1)\mathrm{d}N(t_2-u_2)\ldots\mathrm{d}N(t_m-u_m)\}. \end{aligned} \quad (7)$$

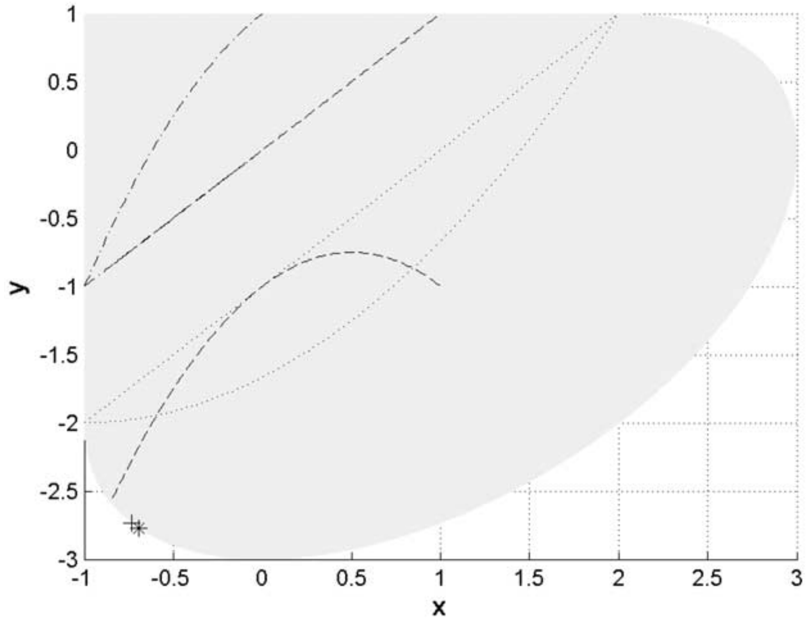

Figure 1. (continued)

The mean of the depth process can be represented as the product of the rate at which cell origins occur and the mean rainfall quantity produced by a cell [*Rodriguez-Iturbe et al.*, 1987a, par. 3.2], that is

$$\begin{aligned} \mathrm{E}\{Y(t)\} &= \int_{u=0}^{\infty} \mathrm{E}\{X_{t-u}(u)\}\mathrm{E}\{\mathrm{d}N(t-u)\} \\ &= \lambda\mu_D \int_0^{\infty} \mathrm{E}\{X_{t-u}(u)\}\mathrm{d}u \\ &= \lambda\mu_D \int_{u=0}^{\infty} \mu_{X,L=u}\mathrm{d}u \end{aligned}$$

where $\mu_{X^k,L=u}$ denotes $\int_0^\infty x^k \frac{\partial F_{X,L}(x,u)}{\partial x}\,\mathrm{d}x$. The covariance with lag τ of the rainfall intensity process is given by

$$\begin{aligned} c_Y(\tau) &= \mathrm{Cov}\{Y(t), Y(t+\tau)\} \\ &= \int_0^{\infty}\int_0^{\infty} \mathrm{E}\{X_{t-u}(u)X_{t+\tau-v}(v)\}\mathrm{Cov}\{\mathrm{d}N(t-u), \\ &\quad \mathrm{d}N(t+\tau-v)\}, \end{aligned} \tag{8}$$

where the covariance of the counting process can be expressed from *Cox and Isham* [1980, p.78] as

$$\begin{aligned} &\mathrm{Cov}\{\mathrm{d}N(t-u), \mathrm{d}N(t+\tau-v)\} \\ &\quad = \lambda\mu_D\Big[\delta(\tau+u-v) + \frac{1}{2}\mu_D^{-1}\mathrm{E}\{D(D-1)\} \\ &\qquad \times \beta\mathrm{e}^{-\beta(\tau+u-v)}\{1-\delta(\tau+u-v)\}\Big]\mathrm{d}u\mathrm{d}v \end{aligned} \tag{9}$$

when $\tau + u - v > 0$. $\delta(.)$ is the Dirac delta function, i.e., $\delta(0) = 1$ and $\delta(x) = 0$ elsewhere. Combining this expression within equation (8) yields (see Appendix B1)

$$\begin{aligned} c_Y(\tau) &= \lambda\mu_D \int_{\tau}^{\infty} \mu_{X^2,L=u}\mathrm{d}u + \frac{\lambda}{2}\mathrm{E}\{D(D-1)\}\beta \\ &\quad \cdot \Big[\int_0^{\infty}\int_0^{u+\tau} \mu_{X,L=u}\mu_{X,L=v}\mathrm{e}^{-\beta(u+\tau-v)}\mathrm{d}v\mathrm{d}u \\ &\quad + \int_0^{\infty}\int_{u+\tau}^{\infty} \mu_{X,L=u}\mu_{X,L=v}\mathrm{e}^{-\beta(v-u-\tau)}\mathrm{d}v\mathrm{d}u\Big]. \end{aligned} \tag{10}$$

Let α and η denote the respective parameters of the distributions on cell intensity and duration. When we choose a cubic copula (with parameter θ) and exponential margins to model the bivariate distribution on cell intensity and duration, an analytical expression is easily obtained. With the FGM copula, we have

$$\begin{aligned} c_Y(\tau) &= \lambda\mu_D\{-3\theta\mathrm{e}^{-2\eta\tau} + 2(4+3\theta)\mathrm{e}^{-\eta\tau}\}/\left(4\alpha^2\eta\right) \\ &\quad + \beta\lambda\mathrm{E}\{D(D-1)\}\left[2\beta\mathrm{e}^{-\eta\tau}\left(\beta^2 - 4\eta^2\right)(2+\theta)(6+\theta)\right. \\ &\quad - \beta\mathrm{e}^{-2\eta\tau}(\beta-\eta)(\beta+\eta)\theta(8+\theta) \\ &\quad + 6\mathrm{e}^{-\beta\tau}\eta\{-2\beta+\eta(4+\theta)\} \\ &\quad \left.\cdot\{2\beta+\eta(4+\theta)\}\right]/\left\{48\alpha^2\eta\left(\beta^4 - 5\beta^2\eta^2 + 4\eta^4\right)\right\}. \end{aligned}$$

[19] Finally, the first- and second-order moments of the aggregated process Y_i^h can be derived [*Rodriguez-Iturbe et al.*, 1984] using the following expressions:

$$\mathrm{E}\left(Y_i^h\right) = h\,\mathrm{E}\{Y(t)\}, \tag{11}$$

$$\mathrm{Var}\left\{Y_i^h\right\} = 2\int_0^h (h-u)c_Y(u)\mathrm{d}u, \tag{12}$$

$$\mathrm{Cov}\left\{Y_i^h, Y_{i+k}^h\right\} = \int_{-h}^{h} c_Y(kh+v)(h-|v|)\mathrm{d}v \tag{13}$$

where k is an integer representing the lag of the autocovariance function. For the FGM copula and exponential margins, simple expressions can be found:

$$
\begin{aligned}
\mathrm{E}\left(Y_i^h\right) &= \frac{h\,\lambda\,\mu_D\,(4+\theta)}{4\,\alpha\,\eta},\\
\mathrm{Var}\left(Y_i^h\right) &= \lambda\mu_D\left[-3\theta+8\mathrm{e}^{\eta h}(4+3\theta)\right.\\
&\quad \left.+\mathrm{e}^{2\eta h}\{-32-21\theta+2\eta h(16+9\theta)\}\right]/\left(8\alpha^2\mathrm{e}^{2\eta h}\eta^3\right).\\
&\quad +\lambda\mathrm{E}\{D(D-1)\}\left[8\beta^3\left(\beta^2-4\eta^2\right)\left(\mathrm{e}^{-\eta h}+\eta h-1\right)\right.\\
&\quad \cdot(2+\theta)(6+\theta)-\beta^3(\beta-\eta)(\beta+\eta)\\
&\quad \cdot\left(\mathrm{e}^{-2\eta h}+2\eta h-1\right)\theta(8+\theta)+24\eta^3\left(\mathrm{e}^{-\beta h}+\beta h-1\right)\\
&\quad \left.\cdot\left\{\eta^2(4+\theta)^2-4\beta^2\right\}\right]/\left\{96\alpha^2\eta^3\left(\beta^5-5\beta^3\eta^2+4\beta\eta^4\right)\right\},\\
\mathrm{Cov}\left(Y_i^h,Y_{i+k}^h\right) &= \lambda\mu_D\left\{-3\left(\mathrm{e}^{2\eta h}-1\right)^2\theta+8\mathrm{e}^{\eta h(1+k)}\left(\mathrm{e}^{\eta h}-1\right)^2\right.\\
&\quad \left.\cdot(4+3\theta)\right\}/\left(16\alpha^2\mathrm{e}^{2\eta h(1+k)}\eta^3\right)\\
&\quad -\lambda\mathrm{E}\{D(D-1)\}\left[-8\beta^3\mathrm{e}^{(\beta+\eta)h(1+k)}\left(\mathrm{e}^{\eta h}-1\right)^2\right.\\
&\quad \cdot\left(\beta^2-4\eta^2\right)(2+\theta)(6+\theta)\\
&\quad +\beta^3\mathrm{e}^{\beta h(1+k)}\left(\mathrm{e}^{2\eta h}-1\right)^2(\beta-\eta)(\beta+\eta)\theta(8+\theta)\\
&\quad +24\mathrm{e}^{2\eta h(1+k)}\left(\mathrm{e}^{\beta h}-1\right)^2\eta^3\\
&\quad \left.\cdot\{2\beta-\eta(4+\theta)\}\{2\beta+\eta(4+\theta)\}\right]\\
&\quad /\left\{192\alpha^2\mathrm{e}^{(\beta+2\eta)h(1+k)}\eta^3\left(\beta^5-5\beta^3\eta^2+4\beta\eta^4\right)\right\}.
\end{aligned}
$$

3.3. Third-Order Moment

[20] Following *Cowpertwait* [1998], we derived expression of the third-order moment in the case of a dependence between cell depth and lifetime. The third-order moment gives information about the asymmetry of a distribution. According to *Cowpertwait* [1998] and *Onof* [2003], the reproduction of extremes is greatly improved when the third-order moment is included in the fitting procedure. To evaluate the third-order moment for the Neyman-Scott process, consider expression (7) with $m = 3$. Let $t_1 < t_2 < t_3$ denote the locations of three cell origins. The expectation of the counting process is derived in [*Cowpertwait* 1998, equation (2.7)]:

$$
\begin{aligned}
&\mathrm{E}\{\mathrm{d}N(t_1)\mathrm{d}N(t_2)\mathrm{d}N(t_3)\}\\
&= \lambda^3\mu_D^3\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3+\frac{1}{2}\lambda^2\beta\mu_D\mathrm{E}\left(D^2-D\right)\\
&\quad \cdot\left\{\mathrm{e}^{-\beta(t_3-t_1)}+\mathrm{e}^{-\beta(t_3-t_2)}+\mathrm{e}^{-\beta(t_2-t_1)}\right\}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&\quad +\frac{1}{3}\lambda\beta^2\mathrm{E}\left\{\left(D^2-D\right)(D-2)\right\}\mathrm{e}^{-\beta(t_3+t_2-2t_1)}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\\
&\quad +\circ(\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3)
\end{aligned}
\qquad (14)
$$

where $t_1 < t_2 < t_3$. Now, we need to express $\mathrm{E}\{X_{t_1-u_1}(u_1)X_{t_2-u_2}(u_2)X_{t_3-u_3}(u_3)\}$. We have to consider different cases (for example, $t_1 - u_1 = t_2 - u_2$ in order to take into account the case where $X_{t_1-u_1}$ and $X_{t_2-u_2}$ refer to the same cell). Hence

$$
E\{X_{t_1-u_1}(u_1)X_{t_2-u_2}(u_2)X_{t_3-u_3}(u_3)\}
$$

$$
=\begin{cases}
\mu_{X^3,L=u_1+t_3-t_1} & \text{when}\quad t_1-u_1=t_2-u_2=t_3-u_3\\
\mu_{X,L=u_1}\mu_{X^2,L=u_2+t_3-t_2} & \text{when}\quad u_3=u_2+t_3-t_2,\,u_2\neq u_1+t_2-t_1\\
\mu_{X,L=u_2}\mu_{X^2,L=u_1+t_3-t_1} & \text{when}\quad u_3=u_1+t_3-t_1,\,u_2\neq u_1+t_2-t_1\\
\mu_{X,L=u_3}\mu_{X^2,L=u_1+t_2-t_1} & \text{when}\quad u_2=u_1+t_2-t_1,\,u_3\neq u_1+t_3-t_1\\
\mu_{X,L=u_1}\mu_{X,L=u_2}\mu_{X,L=u_3} & \text{when}\quad t_1-u_1\neq t_2-u_2\neq t_3-u_3
\end{cases}
$$

[21] Details of these computations are exposed in Appendix B2. With the FGM copula and exponential margins, the expression of the third (central) moment is

$$
\begin{aligned}
\xi_h &= \mathrm{E}\left[\left\{Y_i^h-\mathrm{E}\left(Y_i^h\right)\right\}^3\right] = 3\lambda\mu_D\left\{-21\mathrm{e}^{-2\eta h}(1+\eta h)\theta+24\mathrm{e}^{-\eta h}\right.\\
&\quad \left.\cdot(2+\eta h)(8+7\theta)+3(-128+64\eta h-105\theta+49\eta h\theta)\right\}\\
&\quad /\left(16\alpha^3\eta^4\right)+\{\beta\lambda\mathrm{E}\{D(D-1)\}f(\eta,\beta,\theta,h)\}\\
&\quad /\left\{384\alpha^3\mathrm{e}^{(\beta+4\eta)h}\eta^4\left(\beta^5-5\beta^3\eta^2+4\beta\eta^4\right)^2\right\}\\
&\quad +\{\lambda\mathrm{E}\{D(D-1)(D-2)\}g(\eta,\beta,\theta,h)\}\\
&\quad /\left\{1280\alpha^3\beta\mathrm{e}^{2(\beta+2\eta)h}(\beta-2\eta)^2(\beta-\eta)^2\eta^4(\beta+\eta)(2\beta+\eta)\right.\\
&\quad \left.\cdot(\beta+2\eta)(\beta+3\eta)(\beta+4\eta)\right\}
\end{aligned}
\qquad (15)
$$

where $f(\eta,\ \beta,\ \theta,\ h)$ and $g(\eta,\ \beta,\ \theta,\ h)$ are defined in Appendix B3.

[22] When $\theta = 0$ for the FGM copula, we have $C(u, v) = u\,v$ which corresponds to the case of independence between cell intensity and duration. For $\theta = 0$, equation (15) is consistent with the one obtained by *Cowpertwait* [1998]. General formulas for the aggregated moments (except the third moment whose expression is too cumbersome) using cubic copulas can be found in Appendix B4.

3.4. Probability of Dry Period

[23] The expression for the probability of an interval of given length h to be dry is useful in fitting models. *Cowpertwait* [1991] derived this expression in the case where the rain cells are distributed according to a Poisson distribution. *Favre* [2001] gives this expression when the number of cells D is distributed according to a geometric distribution:

$$
\begin{aligned}
PD(h) &= \exp(-\lambda h)\{\exp(-\beta h)(1-\mu_D)+\mu_D\}^{\frac{\lambda}{\beta(1-\mu_D)}}\\
&\quad \times\exp\left\{-\lambda\int_0^\infty p_t(h)dt\right\},
\end{aligned}
\qquad (16)
$$

where

$$
p_t(h)=\frac{\mu_D[\exp\{-\beta(t+h)\}(\beta-\eta)+\eta\exp(-\beta t)-\beta\exp(-\eta t)]}{(\mu_D-1)[\exp\{-\beta(t+h)\}(\beta-\eta)+\eta\exp(-\beta t)-\beta\exp(-\eta t)]+\eta-\beta}.
$$

[24] Since the sum involved in expression (16) cannot be computed algebraically, it requires a numerical integration of $\int_0^\infty\{1-p_t(h)\}dt$.

4. Model Application

4.1. Data Description

[25] We state that the NSRPM is significantly improved when dependence is included between cell intensity and duration. In this section a large simulation analysis is performed using two data sets. Another successful application can be found in the work of *Evin and Favre* [2006]. The first one is provided by the rainfall station at Uccle, Belgium. Rainfall data used was collected by the Royal Meteorological Institute of Belgium. Hourly recorded rainfall depth are made available for a period of 28 years, from 1968 to 1995. The second series of hourly rainfall is an extract of the CPC Hourly U. S. Precipitation data provided

Table 3. Estimates of the NSRPM Parameters $\hat{\Phi}$ at Uccle, the Dependence Degree (Kendall's tau τ and Pearson's rho ρ), and the Value of the Objective Function $\mathcal{O}(\hat{\Phi})$

	λ	η	β	μ_D	α	γ	τ	θ	f	c	ρ	$\mathcal{O}(\hat{\Phi})$
					Dist. on Cell Int.							
Ind. Exponential	0.0191	2.41	0.100	4.63	0.34		0					1.56e-002
Ind. Gamma	0.0183	2.48	0.116	8.72	0.25	0.40	0					4.08e-003
Ind. Pareto	0.0191	2.34	0.117	6.86	8.59	5.45	0					1.33e-004
					Copula Family							
FGM	0.0192	2.26	0.099	4.47	0.32		−0.08	−0.35				1.47e-002
Sym1	0.0192	2.54	0.115	6.30	0.31		−0.32	0.25				4.80e-004
Sym2	0.0192	2.59	0.115	6.43	0.31		−0.31	0.33				4.08e-004
Asym1	0.0193	2.39	0.105	5.01	0.30		−0.19	0.82				6.58e-003
					DD Model							
DD1	0.0191	2.41	0.100	4.63					2.94	0.00	−0.00	1.56e-002
DD2	0.0173	6.00	0.151	22.75					4.93	0.03	0.58	1.51e-002

by the National Oceanic and Atmospheric Administration/Office of Oceanic and Atmospheric Research/Earth System Research Laboratory-Physical Sciences Division (NOAA/OAR/ESRL PSD), Boulder, Colorado, USA (available from their Web site at http://www.cdc.noaa.gov/cdc/data.cpc_hour.html, accessible on 25 September 2007). NOAA's Climate Prediction Center (CPC) provides hourly U. S. precipitation. Stations were gridded into 2.0 $\times$ 2.5 boxes that extended over the region, 140°W–60°W, 20°N–60°N using a modified Cressman Scheme. In this paper, we present the application of the NSRPM on the point with coordinates 95°W–48°N, situated in the Lower Red Lake (Minnesota). For this data set, 50 years of observations are available from 1949 to 1998. In order to satisfy the temporal homogeneity with respect to rainfall characteristics, the Neyman-Scott model is usually applied on a monthly basis. Therefore without loss of generality, only results on June and February data are shown in this paper for the Belgian and American data, respectively. June and February were selected to be representative months for summer and winter, respectively. The NSRPM is thus applied on two different climates, Uccle having an oceanic climate whereas the Lower Red Lake is typical of a humid continental climate.

4.2. Models Considered

[26] We apply the modified NSRPM on these data sets using the copula families described in Table 1. To avoid overloaded figures and tables, we report the results for only four families, namely FGM, Sym1, Sym2, and Asym1. It must be noticed that several families were giving very similar results. Marginal distributions for cell duration and intensity are exponential. Thus we obtain a three-parameter bivariate distribution to characterize the behavior of cell intensity and duration. We bring to mind that the FGM copula with exponential margins corresponds to the Gumbel's Type-II distribution. Thus this specific model will allow a comparison of performance between the different models considered and the one developed by *Kim and Kavvas* [2006] who apply a Gumbel's Type-II distribution.

[27] To allow a fair comparison of performance according to the number of parameters between the classical model and those including a dependence, we apply a gamma distribution with two parameters on cell intensity in the case of independence, so that the cell intensity X has probability density function $\alpha^\gamma x^{\gamma-1} \mathrm{e}^{-\alpha x}/\Gamma(\gamma)$, $x \geq 0$. We also investigate the case of a Pareto distribution with probability density function $\gamma\alpha^\gamma/(\alpha + x)^{(\gamma+1)}$, $x \geq 0$ on cell intensity, where α is the scale parameter and γ is the shape parameter. The classical case of a simple exponential distribution on cell intensity is also presented.

[28] The last type of models considered is the "Dependent Depth-Duration Model" examined by *Kakou* [1997]. This model conserves the storm and cell arrival rates along with cell duration. The cell intensity X has an exponential form. The dependence structure between cell intensity and duration is introduced through a mean cell intensity depending on L, that is $\mathrm{E}(X|L) = g(L)$. The exponential distribution

Table 4. Estimates of the NSRPM Parameters $\hat{\Phi}$ at the Lower Red Lake, the Dependence Degree (Kendall's tau τ and Pearson's rho ρ), and the Value of the Objective Function $\mathcal{O}(\hat{\Phi})$

	λ	η	β	μ_D	α	γ	τ	θ	f	c	ρ	$\mathcal{O}(\hat{\Phi})$
					Dist. on Cell Int.							
Ind. Exponential	0.0052	7.77	0.086	8.59	0.46		0					1.81e-003
Ind. Gamma	0.0053	8.01	0.088	10.25	0.40	0.74	0					8.96e-004
Ind. Pareto	0.0053	8.16	0.089	10.05	18.62	10.74	0					2.17e-004
					Copula Family							
FGM	0.0052	7.27	0.087	8.20	0.41		−0.12	−0.53				1.02e-003
Sym1	0.0053	9.19	0.090	11.27	0.48		−0.12	0.98				4.80e-004
Sym2	0.0053	9.41	0.090	11.97	0.49		−0.14	1.35				6.89e-004
Asym1	0.0053	8.48	0.089	9.75	0.42		−0.14	1.05				1.53e-004
					DD Model							
DD1	0.0052	7.78	0.086	8.60					2.16	0.00	−0.00	1.81e-003
DD2	0.0053	9.75	0.089	14.18					13.62	2.00	0.47	1.92e-003

Table 5. Observed and Estimated Properties at Uccle[a]

	Mean	Probability of No Rain	Variance	Correlation		
				Lag-1	Lag-2	Lag-3
		Level of Aggregation: 1 h				
Observed	**0.107**	**0.881**	**0.339**	**0.395**	0.269	0.176
Ind. Exp.	**0.108(0.105;0.111)**	**0.903(0.901;0.905)**	**0.37(0.36;0.39)**	**0.357**	0.126	0.096
Ind. Gamma	**0.104(0.101;0.107)**	**0.864(0.861;0.866)**	**0.35(0.34;0.37)**	**0.376**	0.154	0.122
Ind. Pareto	**0.108(0.105;0.111)**	**0.876(0.873;0.879)**	**0.34(0.32;0.35)**	**0.394**	0.165	0.130
FGM	**0.109(0.105;0.111)**	**0.904(0.902;0.906)**	**0.37(0.36;0.39)**	**0.359**	0.129	0.098
Sym1	**0.108(0.106;0.112)**	**0.884(0.881;0.886)**	**0.34(0.32;0.36)**	**0.391**	0.160	0.125
Sym2	**0.108(0.105;0.111)**	**0.883(0.880;0.885)**	**0.34(0.32;0.36)**	**0.391**	0.159	0.125
Asym1	**0.109(0.106;0.112)**	**0.897(0.895;0.899)**	**0.36(0.34;0.38)**	**0.369**	0.139	0.107
DD1	**0.108(0.105;0.111)**	**0.903(0.901;0.905)**	**0.37(0.36;0.39)**	**0.358**	0.127	0.097
DD2	**0.106(0.103;0.109)**	**0.825(0.822;0.829)**	**0.33(0.31;0.35)**	**0.377**	0.211	0.180
		Level of Aggregation: 6 h				
Observed	0.643	0.719	4.888	0.262	0.064	0.029
Ind. Exp.	0.646(0.628;0.664)	0.731(0.726;0.736)	4.33(4.10;4.54)	0.249	0.122	0.067
Ind. Gamma	0.623(0.604;0.640)	0.690(0.685;0.695)	4.38(4.11;4.61)	0.280	0.128	0.064
Ind. Pareto	0.647(0.628;0.665)	0.702(0.697;0.708)	4.30(4.04;4.51)	0.287	0.131	0.065
FGM	0.651(0.632;0.669)	0.733(0.728;0.737)	4.36(4.12;4.57)	0.250	0.122	0.067
Sym1	0.651(0.633;0.669)	0.709(0.704;0.714)	4.29(4.02;4.56)	0.283	0.131	0.066
Sym2	0.650(0.631;0.669)	0.707(0.702;0.713)	4.31(4.04;4.57)	0.283	0.130	0.065
Asym1	0.654(0.635;0.672)	0.724(0.719;0.729)	4.35(4.09;4.59)	0.261	0.127	0.068
DD1	0.647(0.629;0.665)	0.731(0.726;0.735)	4.36(4.14;4.59)	0.249	0.123	0.067
DD2	0.636(0.617;0.655)	0.664(0.658;0.670)	4.46(4.17;4.73)	0.339	0.133	0.054
		Level of Aggregation: 12 h				
Observed	1.287	0.601	12.377	0.167	0.049	0.039
Ind. Exp.	1.293(1.255;1.328)	0.618(0.612;0.624)	10.81(10.22;11.38)	0.225	0.064	0.019
Ind. Gamma	1.245(1.208;1.280)	0.589(0.582;0.595)	11.19(10.48;11.77)	0.235	0.057	0.014
Ind. Pareto	1.294(1.256;1.330)	0.597(0.591;0.604)	11.06(10.41;11.65)	0.239	0.057	0.015
FGM	1.302(1.265;1.338)	0.620(0.614;0.626)	10.89(10.32;11.48)	0.224	0.065	0.020
Sym1	1.301(1.267;1.338)	0.602(0.596;0.608)	11.03(10.33;11.70)	0.237	0.058	0.015
Sym2	1.300(1.263;1.338)	0.600(0.594;0.607)	11.06(10.35;11.72)	0.238	0.057	0.015
Asym1	1.308(1.270;1.345)	0.613(0.606;0.619)	10.96(10.31;11.55)	0.231	0.064	0.018
DD1	1.295(1.258;1.330)	0.618(0.612;0.624)	10.91(10.31;11.50)	0.224	0.064	0.019
DD2	1.272(1.235;1.310)	0.578(0.571;0.585)	11.93(11.13;12.63)	0.246	0.040	0.006
		Level of Aggregation: 24 h				
Observed	2.574	**0.460**	**27.611**	**0.146**	0.072	0.089
Ind. Exp.	2.585(2.510;2.655)	**0.474(0.467;0.482)**	**26.47(24.98;27.87)**	**0.152**	0.014	0.002
Ind. Gamma	2.490(2.416;2.559)	**0.459(0.451;0.467)**	**27.62(25.85;29.27)**	**0.148**	0.008	−0.000
Ind. Pareto	2.588(2.512;2.659)	**0.462(0.454;0.469)**	**27.40(25.82;28.82)**	**0.148**	0.009	0.001
FGM	2.605(2.530;2.676)	**0.475(0.468;0.483)**	**26.66(25.23;28.06)**	**0.153**	0.014	0.002
Sym1	2.602(2.533;2.676)	**0.464(0.457;0.472)**	**27.27(25.54;28.95)**	**0.149**	0.010	0.001
Sym2	2.600(2.526;2.675)	**0.463(0.455;0.471)**	**27.38(25.68;29.03)**	**0.149**	0.009	0.000
Asym1	2.615(2.540;2.689)	**0.470(0.463;0.479)**	**27.00(25.40;28.50)**	**0.152**	0.012	0.001
DD1	2.590(2.516;2.661)	**0.474(0.467;0.481)**	**26.70(25.14;28.13)**	**0.152**	0.013	0.002
DD2	2.544(2.469;2.620)	**0.462(0.454;0.470)**	**29.70(27.80;31.35)**	**0.134**	0.003	−0.001

[a]The properties used in the estimation procedure are indicated in bold font.

for the cell duration, L, with parameter η, say, is still retained. Two versions of $g(L)$ are considered: $f\,e^{-cL}$ and $f\,Le^{-cL}$, where f and c are nonnegative scalars. We shall refer to these models as the DD1 and the DD2 model, respectively. In the DD1 model the variables X and L are always negatively correlated, with correlation function

$$\frac{-\zeta}{\zeta+1}\sqrt{\frac{2\zeta+1}{2\zeta^2+2\zeta+1}}$$

where $\zeta = c/\eta$ [*Kakou*, 1997]. With the DD2 model, the correlation function between rain cell intensity and duration is

$$\frac{1-\zeta}{\zeta+1}\sqrt{\frac{(2\zeta+1)^3}{4(\zeta+1)^4-(2\zeta+1)^3}}$$

4.3. Parameter Estimation

[29] We apply the method of moments described by *Rodriguez-Iturbe et al.* [1987a]. The functions are highly nonlinear in the parameters so that an exact solution is not available. Let Φ denotes the set of parameters. A closed solution can be obtained by minimizing $\mathcal{O}$:

Table 6. Observed and Estimated Properties at the Lower Red Lake[a]

				Correlation		
	Mean	Probability of No Rain	Variance	Lag-1	Lag-2	Lag-3
			Level of Aggregation: 1 h			
Observed	**0.013**	**0.970**	**0.014**	**0.216**	0.190	0.168
Ind. Exp.	**0.012(0.012;0.013)**	**0.963(0.961;0.965)**	**0.01(0.01;0.02)**	**0.208**	0.134	0.122
Ind. Gamma	**0.012(0.012;0.013)**	**0.958(0.956;0.959)**	**0.01(0.01;0.02)**	**0.213**	0.139	0.127
Ind. Pareto	**0.013(0.012;0.013)**	**0.958(0.956;0.960)**	**0.01(0.01;0.01)**	**0.217**	0.145	0.133
FGM	**0.012(0.012;0.013)**	**0.964(0.962;0.965)**	**0.01(0.01;0.02)**	**0.210**	0.137	0.125
Sym1	**0.012(0.012;0.013)**	**0.955(0.953;0.957)**	**0.01(0.01;0.01)**	**0.216**	0.146	0.133
Sym2	**0.012(0.012;0.013)**	**0.953(0.951;0.955)**	**0.01(0.01;0.01)**	**0.216**	0.144	0.133
Asym1	**0.013(0.012;0.013)**	**0.959(0.957;0.960)**	**0.01(0.01;0.01)**	**0.216**	0.144	0.132
DD1	**0.012(0.012;0.013)**	**0.963(0.961;0.965)**	**0.01(0.01;0.02)**	**0.208**	0.134	0.123
DD2	**0.012(0.011;0.013)**	**0.948(0.946;0.951)**	**0.01(0.01;0.01)**	**0.214**	0.143	0.131
			Level of Aggregation: 6 h			
Observed	0.075	0.909	0.157	0.331	0.115	0.107
Ind. Exp.	0.074(0.070;0.078)	0.890(0.886;0.895)	0.15(0.13;0.16)	0.336	0.196	0.117
Ind. Gamma	0.074(0.070;0.078)	0.881(0.877;0.885)	0.15(0.13;0.16)	0.342	0.198	0.117
Ind. Pareto	0.075(0.071;0.079)	0.881(0.877;0.886)	0.15(0.13;0.16)	0.349	0.202	0.117
FGM	0.075(0.071;0.079)	0.892(0.887;0.896)	0.15(0.14;0.16)	0.339	0.198	0.117
Sym1	0.075(0.070;0.079)	0.877(0.872;0.881)	0.15(0.13;0.16)	0.349	0.200	0.117
Sym2	0.075(0.070;0.079)	0.874(0.869;0.878)	0.15(0.13;0.16)	0.349	0.201	0.117
Asym1	0.076(0.071;0.079)	0.882(0.878;0.887)	0.15(0.14;0.16)	0.349	0.202	0.117
DD1	0.074(0.070;0.078)	0.890(0.886;0.894)	0.15(0.14;0.16)	0.335	0.196	0.117
DD2	0.073(0.069;0.077)	0.867(0.862;0.873)	0.15(0.13;0.16)	0.347	0.199	0.117
			Level of Aggregation: 12 h			
Observed	0.151	0.857	0.404	0.304	0.112	0.029
Ind. Exp.	0.148(0.140;0.157)	0.844(0.838;0.850)	0.40(0.36;0.44)	0.316	0.111	0.040
Ind. Gamma	0.148(0.140;0.157)	0.834(0.829;0.840)	0.40(0.36;0.44)	0.318	0.110	0.038
Ind. Pareto	0.151(0.143;0.158)	0.835(0.829;0.840)	0.40(0.36;0.44)	0.323	0.109	0.037
FGM	0.150(0.142;0.158)	0.846(0.840;0.852)	0.40(0.36;0.44)	0.318	0.110	0.038
Sym1	0.149(0.141;0.157)	0.830(0.824;0.836)	0.40(0.36;0.43)	0.321	0.108	0.037
Sym2	0.149(0.141;0.157)	0.827(0.821;0.833)	0.40(0.36;0.44)	0.322	0.109	0.037
Asym1	0.151(0.143;0.159)	0.836(0.830;0.842)	0.40(0.36;0.44)	0.322	0.109	0.037
DD1	0.148(0.140;0.156)	0.844(0.838;0.850)	0.40(0.36;0.43)	0.316	0.111	0.039
DD2	0.146(0.138;0.154)	0.821(0.814;0.828)	0.40(0.36;0.43)	0.320	0.109	0.038
			Level of Aggregation: 24 h			
Observed	0.302	**0.776**	**1.058**	**0.220**	0.069	−0.009
Ind. Exp.	0.297(0.279;0.314)	**0.781(0.773;0.789)**	**1.05(0.94;1.15)**	**0.220**	0.028	0.004
Ind. Gamma	0.297(0.280;0.313)	**0.771(0.764;0.779)**	**1.06(0.95;1.15)**	**0.219**	0.026	0.003
Ind. Pareto	0.301(0.285;0.316)	**0.771(0.764;0.779)**	**1.06(0.96;1.16)**	**0.219**	0.025	0.004
FGM	0.300(0.283;0.315)	**0.783(0.775;0.790)**	**1.06(0.95;1.15)**	**0.219**	0.025	0.002
Sym1	0.298(0.282;0.314)	**0.767(0.759;0.776)**	**1.06(0.95;1.16)**	**0.218**	0.025	0.003
Sym2	0.298(0.281;0.314)	**0.764(0.756;0.772)**	**1.06(0.95;1.16)**	**0.219**	0.025	0.004
Asym1	0.302(0.285;0.318)	**0.772(0.765;0.781)**	**1.06(0.96;1.16)**	**0.219**	0.025	0.003
DD1	0.297(0.280;0.313)	**0.781(0.774;0.789)**	**1.05(0.95;1.15)**	**0.219**	0.027	0.002
DD2	0.292(0.276;0.308)	**0.759(0.750;0.768)**	**1.05(0.95;1.15)**	**0.217**	0.026	0.002

[a]The properties used in the estimation procedure are indicated in bold font.

$$\mathcal{O}(\Phi) = \sum_{i=1}^{p} \left(1 - \frac{M_i}{\hat{M}_i}\right)^2, \quad (17)$$

where M_i is a moment or autocovariance function of the Neyman-Scott model given by equations (6) and (11)–(13). It could also be the dry probability at a given scale with theoretical expression given by (16). Here p is the number of moments included in the procedure. The ratios $\frac{M_i}{\hat{M}_i}$ between theoretical expression and observed values have to be close to 1. The use of these ratios ensures that large numerical values do not dominate the fitting procedure.

[30] It is common to fit models using some combination of means, variances, autocorrelations and proportions of dry intervals, at timescales ranging from 1 to 24 h. For a six-parameter model, six theoretical expressions are sufficient to constrain the optimization problem. However, as suggested by *Cowpertwait et al.* [1996], a larger set of moments guarantees that the model at least fit these properties. We choose hourly mean, hourly variance, hourly autocovariance at lag-1, daily variance, daily autocovariance at lag-1, the third-order moment at an hourly scale, and the probability of dry period at an hourly and a daily scale (thus we obtain $p = 8$). *Cowpertwait* [1998] indicates a good fit of extremes with the insertion of the third-order moment in the estimation procedure. The minimization of $\mathcal{O}$ is not an easy task and must be achieved numerically. Numerous investigations show that the obtained estimates are highly sensitive to the starting point of the search [*Calenda and Napolitano*, 1999]. The optimization proce-

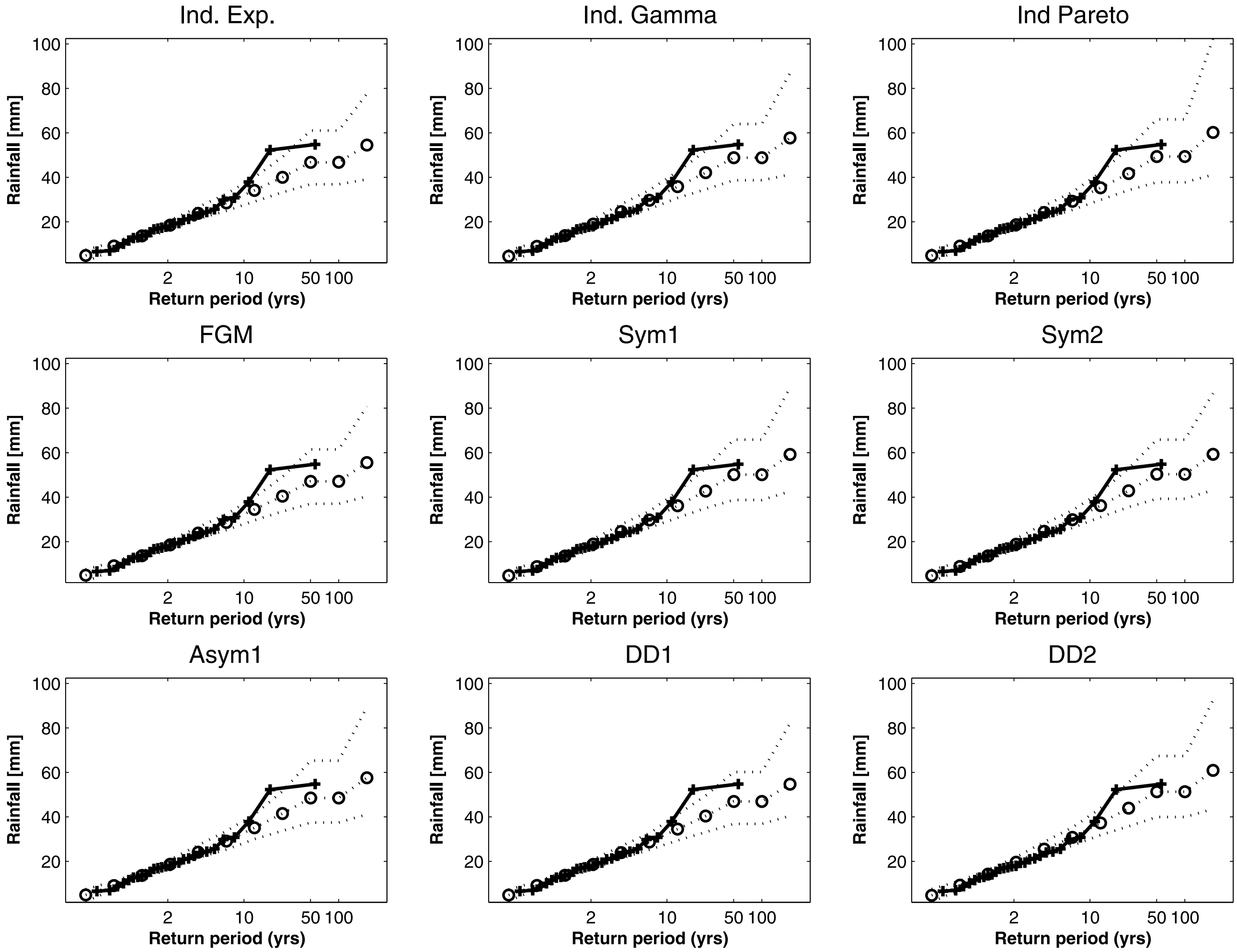

Figure 2. Monthly 1-day maxima at Uccle. X-axis gives the return period in years represented on a Gumbel scale. Solid line with plus symbol represents observed maxima. Dotted lines represent the mean (with o) and a 0.95 confidence interval of simulated maxima obtained from 1000 simulations of 100 years.

dure is derived from the algorithm developed by *Chandler and Onof* [2005] and can be summarized as follows:

[31] 1. A solution in the five dimensional space is provided by the generalized method of moments developed by *Favre et al.* [2004b] in the case of the classical NSRPM with five parameters. This method uses an algebraic computation reducing the number of parameters estimated by minimization. Two parameters are obtained by minimization while the three others can be directly computed.

[32] 2. A starting value for the parameters λ, β, η, μ_D, α are obtained in the first step. The sixth parameter was fixed such that it corresponds to the classical NSRPM. The optimization procedure constrains the parameter space in some realistic regions: $0.00001 < \lambda < 0.1(h^{-1})$; $0.0001 < \beta < 0.99\ (h^{-1})$; $0.01 < \eta < 200\ (h^{-1})$; $1 < \mu_D < 500$; $0.01 < \alpha < 1000$. By restraining the parameters between these bounds, we ensure that the estimates do not get into physically unrealistic parameter spaces. A subjective choice is necessary to set these bounds. Nevertheless, the bounds are wide and experience shows that parameters always lie in a small subspace of this feasible region [*Cowpertwait*, 1998]. A first set of parameters are estimated by minimizing expression (17) using a sequential quadratic programming (SQP) method [*Fletcher*, 2001]. This specific set of parameters is duplicated 100 times to provide a primary set of identical solutions.

[33] 3. Repeat steps 4 to 7 several times (e.g., five times).

[34] 4. Perform 50 constrained optimizations. The starting points are generated via random perturbations about the parameter set identified above. The random perturbation is taken from a beta distribution on the current bounds (the bounds are defined in step 2 and evolute in step 6).

[35] 5. Discard any parameter sets for which the objective function exceeds $10O_{50}$, where O_{50} is the 50th smallest of the objective function values found in step 4.

[36] 6. Bounds evolute according to the best parameter sets found so far (i.e., those minimizing the objective function). If the best sets are close, bounds are narrowed. If they are close to an existing boundary, bounds are loosened.

[37] 7. Keep only the 50 best parameter sets. The overall best set is taken as a starting point for future optimizations.

[38] This algorithm has the first advantage to stride an important subset of the parameter space with the adaptive bounds and the random starting points. His second strength is to avoid failure of minimization algorithm. This could happen when poor starting values are given, the objective

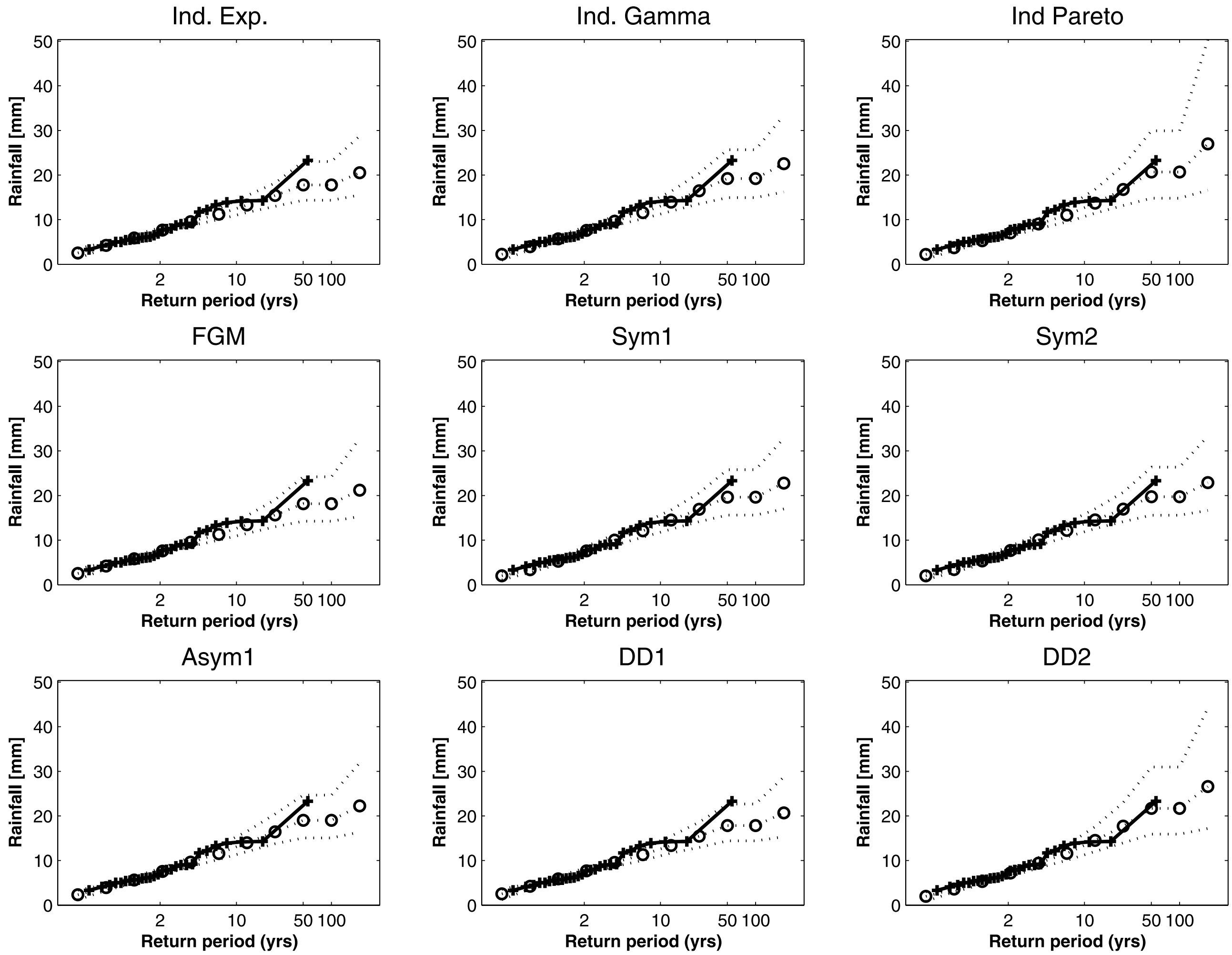

Figure 3. Monthly 1-hour maxima at Uccle. X-axis gives the return period in years represented on a Gumbel scale. Solid line with plus symbol represents observed maxima. Dotted lines represent the mean (with o) and a 0.95 confidence interval of simulated maxima obtained from 1000 simulations of 100 years.

function becoming too flat. The obtained estimates are presented in Tables 3 and 4.

[39] For both applications, the estimated parameters related to the point process, i.e., $\hat{\lambda}$ and $\hat{\beta}$, are quite stable. The parameter related to the number of cells μ_D has an important variation according to the model. It points the identifiability problem of the NSRPM [*Chandler and Onof*, 2005]. The value of the shape parameter of the gamma distribution ($\hat{\gamma}$ = 0.4 at Uccle and $\hat{\gamma}$ = 0.74 at the Lower Red Lake) indicates that the introduction of one extra parameter for the bivariate distribution on cell intensity and duration allows a more heavy-tailed distribution for cell intensity. With the FGM copula, the dependence degree obtained is weak ($\tau = -0.08$ at Uccle and $\tau = -0.12$ at the Lower Red Lake). It is important to bear in mind that when $\tau = 0$, this model is equivalent to the classical model with independent exponential margins. The DD1 model is equivalent to the classical NSRPM for both applications ($c = 0$), while the DD2 model exhibits a positive correlation between cell intensity and duration. It is quite interesting to note that Kendall's tau between the rain cell duration and intensity is negative for the copula-based models and does not attain the lower bounds indicated in Table 2.

[40] We now consider more specifically the estimates at Uccle (see Table 3). The independent Pareto, Sym1, and Sym2 models achieve smaller objective values than the other models. The objective function values give an overall measure of model fit, it thus indicates that these models outperform the other models on this application. On the contrary, the DD1 and DD2 models seem to have difficulties to fit the observations. The same holds for the estimates at the Lower Red Lake, except that the Asym1 model clearly minimizes the objective function. The dependence degree (see Table 4) is almost the same for all the copula-based models. Thus the value of the objective function is minimized thanks to the dependence structure (and not by the increase of the dependence degree).

4.4. Model Performances

[41] For each type of bivariate distribution on cell intensity and duration, 1000 series of hourly rainfall are generated, each sequence containing 100 months of data. Analysis of the variability of the series gives us an indication of uncertainty.

[42] Tables 5 and 6 give the fitting properties of the different models at Uccle and at the Lower Red Lake,

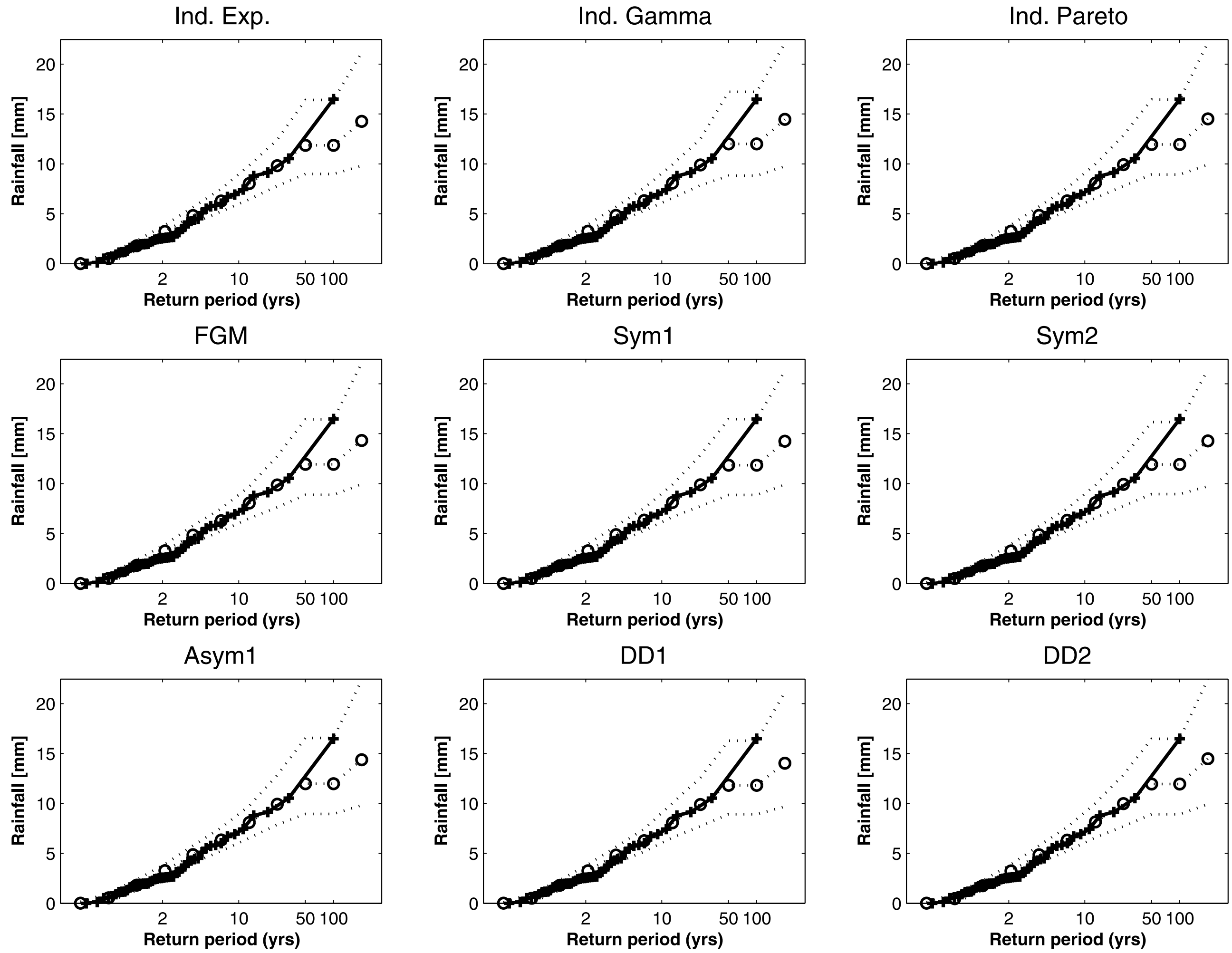

Figure 4. Monthly 1-day maxima at the Lower Red Lake. X-axis gives the return period in years represented on a Gumbel scale. Solid line with plus symbol represents observed maxima. Dotted lines represent the mean (with o) and a 0.95 confidence interval of simulated maxima obtained from 1000 simulations of 100 years.

respectively. For the mean, the probability of dry period and the variance, we indicate a 0.95 empirical confidence interval in brackets, obtained from the 1000 simulations. The properties used in the estimation procedure are indicated in bold font. The method of moments ensures that these properties give good fitted values. As noticed in the previous section, objective values (see Tables 3 and 4) indicate the ability of each model to reproduce the observed statistics. The objective values are directly linked to the differences between the observed and simulated properties of rainfall. At Uccle, the FGM model does not improve the classical NSRPM in this sense, mainly because of the weak dependence degree ($\tau = -0.08$) while the Ind. Pareto, Sym1, and Sym2 models clearly outperform the other models. The Asym1 model is the best model at the Lower Red Lake for this criterion. At Uccle, the variance at a 6 h and 12 h level of aggregation seems to be slightly underestimated for all models, which is not true at the Lower Red Lake. The probability of dry periods are adequately reproduced at the same scales. At a daily scale the lag two and lag three correlations are clearly too weak in an absolute sense for both applications.

[43] It is important to notice that it is difficult to make any comparison of observed and modeled dependence degrees. The NSRPM is a conceptual model where cells are inobservable and an observed measure of dependence between cell intensity and duration is impossible to obtain. Let τ_C be the dependence degree between cell intensity and duration and τ_S the dependence degree between the storm duration and mean intensity. The relation between τ_C and τ_S is very complex for several reasons. First, the parameters μ_D, η and β jointly influence τ_S via a clustering effect, that is to say when cells are superposed and aggregated. Furthermore, the fact that we only observe rainfall intensity on discrete time intervals attenuates τ_C. This point is illustrated in the work of *Rodriguez-Iturbe et al.* [1987b] and is called the smoothing effect. Finally, the need of a method to extract the storms from the observed data is an additive noise. The choice of a dry period to separate different storms [see, e.g., *De Michele and Salvadori*, 2003] is a source of subjectivity.

[44] As pointed out by *Onof et al.* [2000], one main weakness of the classical NSRPM is the reproduction of extremes. Nonetheless, the ability of the model to reproduce extremes is a primary aspect in a risk approach. Synthetic

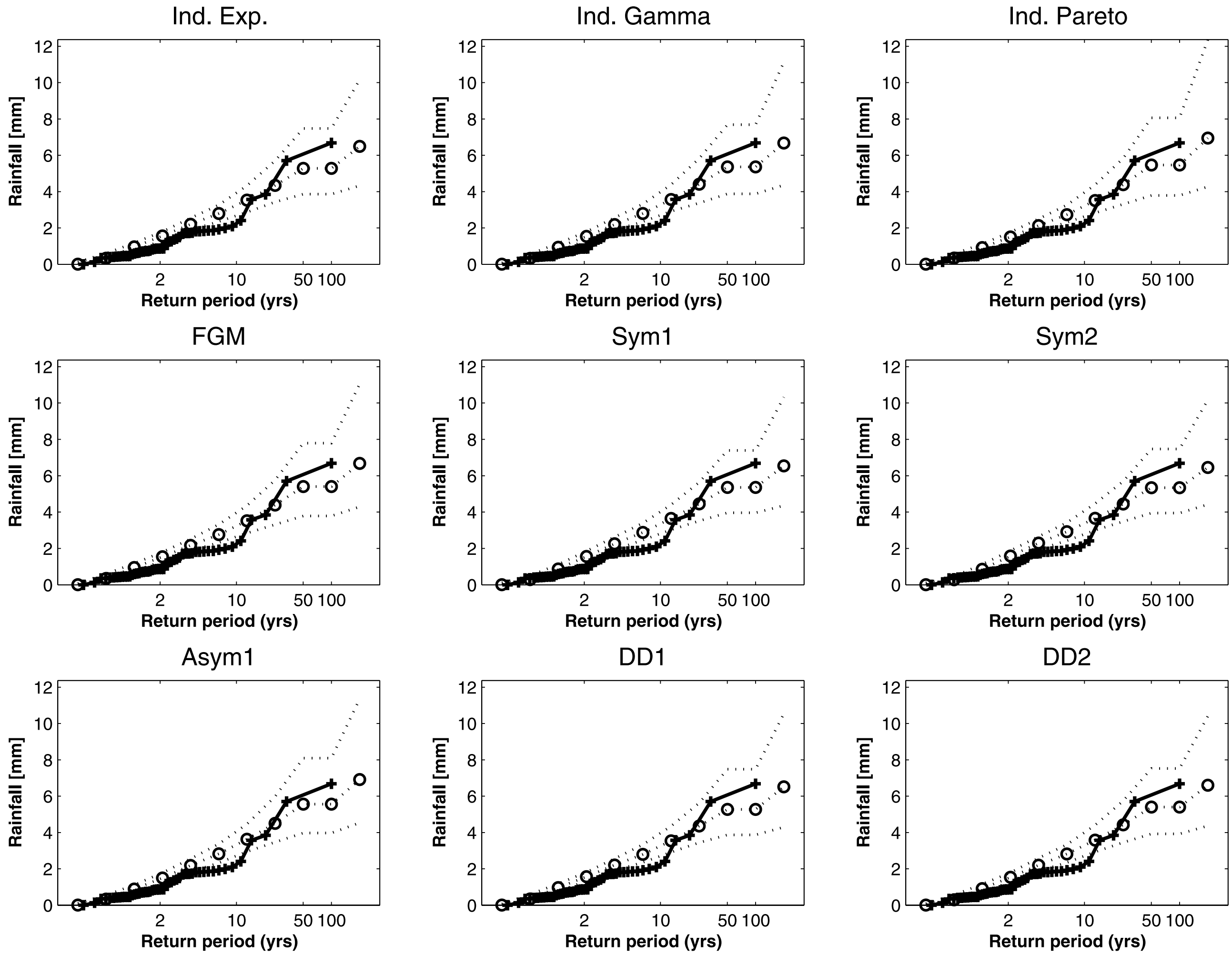

Figure 5. Monthly 1-hour maxima at the Lower Red Lake. X-axis gives the return period in years represented on a Gumbel scale. Solid line with plus symbol represents observed maxima. Dotted lines represent the mean (with o) and a 0.95 confidence interval of simulated maxima obtained from 1000 simulations of 100 years.

series provide monthly maxima at daily and hourly scales, which are represented on Figures 2–5. The X-axis gives the return period in years represented on a Gumbel scale. The empirical quantiles of the maxima are obtained with Hazen's formula $F[Y_{[k]}] = (k - 0.5)/n$ where $Y_{[k]}$ is the monthly 1-d or 1-h maxima corresponding to rank k. A 0.95 empirical confidence interval of simulated maxima are obtained from 1000 simulations of 100 months. We can observe the importance of the type and the degree of dependence by examining the empirical confidence interval. A general tendency in the stochastic point rainfall modeling literature is the underestimation of the extreme values at an hourly scale [*Entekhabi et al.*, 1989].

[45] Let examine the reproduction of the extreme values at Uccle. At a daily scale, extremes are slightly underestimated for the independent exponential, FGM, Asym1, and the DD1 models, as we can see on Figure 2. For these models, a point of the observed maxima is clearly not covered by the confidence interval. The same holds at an hourly scale (see Figure 3). For the independent Pareto model, it is interesting to notice the large uncertainty indicated by the confidence intervals at both the hourly and the daily scale. Figures 4 and 5 report the observed and simulated extremes at the Lower Red Lake. For this application the differences between the various models are very thin and it is difficult to discriminate them on this criterion.

5. Conclusion

[46] A new stochastic point rainfall model is developed which considers correlated rain cell intensity and duration. Various types of dependence structures are possible thanks to cubic copulas. We explore the properties of this class of copulas and suggest several families of this kind attaining a large range of dependence. This more general model includes independent exponential distributions and the Gumbel type-II distribution on cell intensity and duration as special cases. We derive first-, second-, and third-order moments of this modified Neyman-Scott rectangular pulses model. With the expression of the theoretical moments, we can apply the method of moments to fit the model on data.

[47] Two applications on Belgium and American data of hourly rainfall are performed. The model has thus been

applied successfully on two rainfall series with different climates. Long series of synthetic rainfall are generated and compared to the observed rainfall data. With specific cubic families and exponential margins, the fitting of the model can be improved. The independent Pareto distribution on cell intensity also appears to give interesting results. Furthermore, both hourly and daily annual maxima are adequately reproduced by most of the models. The underestimation of the hourly maxima is thus not visible on these applications.

[48] Eventually, it would be interesting to thoroughly analyze the reproduction of extremes. Further analysis on extreme values might include the influence of the type of dependence on their uncertainty and the comparison between simulated and observed data using "peak over thresholds" methods [*Cowpertwait*, 1998].

Appendix A: Cubic Copulas

A1. Necessary Conditions

[49] Theorem A.1 provides necessary conditions for the function $C(u, v)$, defined by relation (1), to be a cubic copula.

[50] Let α, β:[0,1] $\rightarrow \mathbb{R}$ be two functions satisfying $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$, and let $C(u, v)$ be the function defined by

$$C(u,v) = uv + u(1-u)[\alpha(v)(1-u) + \beta(v)u], (u,v) \in [0,1]^2.$$

Then C is a copula if and only if: (1) $\alpha(v)$ and $\beta(v)$ are absolutely continuous, and (2) for almost every $v \in [0, 1]$, the point $\{\alpha'(v), \beta'(v)\}$ lies in S. Moreover, C is absolutely continuous.

[51] Now, we restrict our attention to copulas with cubic sections in u and v, i.e., copulas expressed as a bivariate polynomial with order 3. Theorem A.2 characterizes such types of copulas.

[52] Suppose that C has cubic sections in u and in v; i.e., for all $u, v \in [0,1]$, let C be given by

$$C(u,v) = uv + u(1-u)[\alpha(v)(1-u) + \beta(v)u]$$

and

$$C(u,v) = uv + v(1-v)[\gamma(u)(1-v) + \delta(u)v]$$

where α, β, γ and δ are functions satisfying the hypotheses of Theorem A.1. Then $C(u, v)$ can be expressed as in equation (2). The three following subfamilies belong to the class of cubic copulas and are reported by *Nelsen et al.* [1997].

A1.1. Sarmanov

[53] Case (i) in [*Nelsen et al.*, 1997]. The Sarmanov family [*Sarmanov*, 1974] is symmetric and includes independence as special case.

A1.2. Frank Cubic

[54] Case (iv) in [*Nelsen et al.*, 1997]. This family includes the independence case for $\theta = 0$ and attains extreme values for ρ. The name is justified by the fact that this family is a second-order approximation to the Frank family of copulas [see *Frank*, 1979] given by

$$C(u,v) = -\frac{1}{\theta}\log\left[1 + \frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right]$$

for $\theta \in \mathbb{R}$.

A1.3. Plackett Cubic

[55] Case (v) in [*Nelsen et al.*, 1997]. This family is symmetric and includes the independence case for $\theta = 0$. The name is justified by the fact that this family is a second-order approximation to the Plackett family of copulas [see *Plackett*, 1965] given by

$$C(u,v) = \frac{1 + \theta(u+v) - \sqrt{[1+\theta(u+v)]^2 - 4(\theta+1)\theta uv}}{2\theta}$$

for $\theta \in [-1, 2]$. These cubic copulas have been developed (except for the Sarmanov family) by *Nelsen et al.* [1997].

A2. Minimum of Kendall's Tau for a Copula With Both Cubic Sections in *u* and *v*

[56] The minimum value of τ for copulas with cubic sections in both u and v is the solution of the following minimization problem: minimize equation (3) where A_1, A_2, B_1, B_2 fulfill the constraints described in Theorem (A.2). We notice directly that A_1 and B_2 are exchangeable. It also holds for A_2 and B_1. Then, we have to find

$$\min_{(a,b)} (a+b)\{1/9 + (a-b)/450\}$$

with $(a, b) \in S$. We have $(a + b)\{1/9 + (a - b)/450\}$ increasing in both a and b when $(a, b) \in S$ so that the solution lies on the ellipse $a^2 - ab + b^2 - 3a + 3b = 0$. Setting $b = \{a - 3 - \sqrt{-3a^2 + 6a + 9}\}/2$, a is a root of $-1250 - 1400Z + 600Z^2 + 22Z^3 + Z^4$. Finally, the point in S that minimizes τ is $\{(-11 + 3\sqrt{19} - 3\sqrt{-112 + 26\sqrt{19}})/2, (11 - 3\sqrt{19} - 3\sqrt{-112 + 26\sqrt{19}})/2\}$ and we can write $|\tau| \leq \sqrt{494\sqrt{19} - 2053}/25$.

Appendix B: Moments With a Cubic Copula

B1. Autocovariance Function With Dependent Cell Duration and Intensity

[57] The following demonstration derives the covariance with lag τ of the rainfall intensity process

$$\begin{aligned} c_Y(\tau) &= \mathrm{Cov}\{Y(t), Y(t+\tau)\} \\ &= \mathrm{E}\{Y(t)Y(t+\tau)\} - \mathrm{E}\{Y(t)\}\mathrm{E}\{Y(t+\tau)\} \\ &= \int_0^\infty \int_0^\infty \mathrm{E}\{X_{t-u}(u)X_{t+\tau-v}(v)\}\mathrm{E}\{\mathrm{d}N(t-u)\mathrm{d}N(t+\tau-v)\} \\ &\quad - \int_0^\infty \int_0^\infty \mu_{X,L=u}\mu_{X,L=v}\mathrm{E}\{\mathrm{d}N(t-u)\}\mathrm{E}\{\mathrm{d}N(t+\tau-v)\} \\ &= I_1 - I_2 \end{aligned}$$

But,

$$I_1 = \int_\tau^\infty \mathrm{E}\{X^2_{t+\tau-v}(v)\}\mathrm{E}\left\{\mathrm{d}N(t+\tau-v)^2\right\} + \int_0^\infty\int_0^\infty \mathrm{E}\{X_{t-u}(u)X_{t+\tau-v}(v)\}\mathrm{E}\{\mathrm{d}N(t-u)\mathrm{d}N(t+\tau-v)\}$$
$$= \int_\tau^\infty \mathrm{E}\{X^2_{t+\tau-v}(v)\}\mathrm{E}\{\mathrm{d}N(t+\tau-v)\} + \int_0^\infty\int_0^\infty \mu_{X,L=u}\mu_{X,L=v}\mathrm{E}\{\mathrm{d}N(t-u)\mathrm{d}N(t+\tau-v)\}$$

because of the independence between $X_{t-u}(u)$ and $X_{t+\tau-v}(v)$ when $\tau + u - v \neq 0$, and $\mathrm{d}N^2 = \mathrm{d}N$. Moreover,

$$I_2 = \int_0^\infty\int_0^\infty \delta(\tau+u-v)\mu_{X,L=u}\mu_{X,L=v}\lambda^2\mu_D^2\mathrm{d}u\mathrm{d}v + \int_0^\infty\int_0^\infty \mu_{X,L=u}\mu_{X,L=v}\mathrm{E}\{\mathrm{d}N(t-u)\}\mathrm{E}\{\mathrm{d}N(t+\tau-v)\}$$

and $\int_0^\infty\int_0^\infty \delta(\tau + u - v)\mu_{X,\,L=u}\,\mu_{X,\,L=v}\,\lambda^2\,\mu_D^2\,\mathrm{d}u\mathrm{d}v = 0$ because of the Lebesgue measure theory. Then

$$c_Y(\tau) = \int_\tau^\infty \mathrm{E}\{X^2_{t+\tau-v}(v)\}\mathrm{E}\left\{\mathrm{d}N(t+\tau-v)^2\right\} + \int_0^\infty\int_0^\infty \mu_{X,L=u}\mu_{X,L=v}\mathrm{Cov}\{\mathrm{d}N(t-u),\mathrm{d}N(t+\tau-v)\}$$
$$= \lambda\mu_D\int_\tau^\infty \mu_{X^2,L=u}\mathrm{d}u + \frac{\lambda}{2}\mathrm{E}\{D(D-1)\} \cdot \beta\left[\int_0^\infty\int_0^{u+\tau}\mu_{X,L=u}\mu_{X,L=v}e^{-\beta(u+\tau-v)}\mathrm{d}v\mathrm{d}u + \int_0^\infty\int_{u+\tau}^\infty \mu_{X,L=u}\mu_{X,L=v}e^{-\beta(v-u-\tau)}\mathrm{d}v\mathrm{d}u\right]$$

since Cov $\{\mathrm{d}N(t-u), \mathrm{d}N(t+\tau-v)\}$ is expressed by (9) when $v < u + \tau$. Otherwise, we have Cov $\{\mathrm{d}N(t-u), \mathrm{d}N(t+\tau-v)\} = \mathrm{Cov}\{\mathrm{d}N(t+\tau-v), \mathrm{d}N(t-u)\}$ by symmetry of the covariance. This leads to equation (10).

B2. Third-Order Moment Computation

[58] Because equation (14) is valid only when $t_1 < t_2 < t_3$, different cases have to be considered in order to compute the third order moment. For example, when $t_1 = t_2$ and $t_3 \neq t_1$, we have:

$$\mathrm{E}\{\mathrm{d}N(t_1)\mathrm{d}N(t_2)\mathrm{d}N(t_3)\} = \mathrm{E}\left\{\mathrm{d}N(t_1)^2\mathrm{d}N(t_3)\right\} = \mathrm{E}\{\mathrm{d}N(t_1)\mathrm{d}N(t_3)\},$$

which corresponds to the case of two distinct cells. When t_1, t_2 and t_3 belong to the same cell, we have:

$$\mathrm{E}\{\mathrm{d}N(t_1)\mathrm{d}N(t_2)\mathrm{d}N(t_3)\} = \mathrm{E}\left\{\mathrm{d}N(t_1)^3\right\} = \mathrm{E}\{\mathrm{d}N(t_1)\} = \lambda\mu_D.$$

Note that we consider expression (14) replacing t_i by $t_i - u_i$ as in expression (7). We sum $\mathrm{E}\{Y(t_1)Y(t_2)Y(t_3)\}$ according to the order of $t_3 - u_3$, $t_2 - u_2$ and $t_1 - u_1$, since equation (14) is valid only when $t_1 < t_2 < t_3$. Because t_3 and t_2 are exchangeable in (14), we have for example:

$$\begin{cases} t_1 - u_1 < t_2 - u_2 \\ t_1 - u_1 < t_3 - u_3 \end{cases} \text{when} \begin{cases} u_3 \in [0, t_3 - t_1 + u_1] \\ u_2 \in [0, t_2 - t_1 + u_1] \\ u_1 \in \mathbb{R}^+ \end{cases}$$

$$\begin{cases} t_2 - u_2 < t_1 - u_1 \\ t_2 - u_2 < t_3 - u_3 \end{cases} \text{when} \begin{cases} u_3 \in [0, t_3 - t_2 + u_2] \\ u_2 \in [t_2 - t_1 + u_1, \infty] \\ u_1 \in \mathbb{R}^+ \end{cases}$$

$$\begin{cases} t_3 - u_3 < t_1 - u_1 \\ t_3 - u_3 < t_2 - u_2 \end{cases} \text{when} \begin{cases} u_1 \in [0, t_1 - t_3 + u_3] \\ u_2 \in [0, t_2 - t_3 + u_3] \\ u_3 \in [t_3 - t_1, \infty] \end{cases}$$

[59] This point is illustrated in Figure B1.

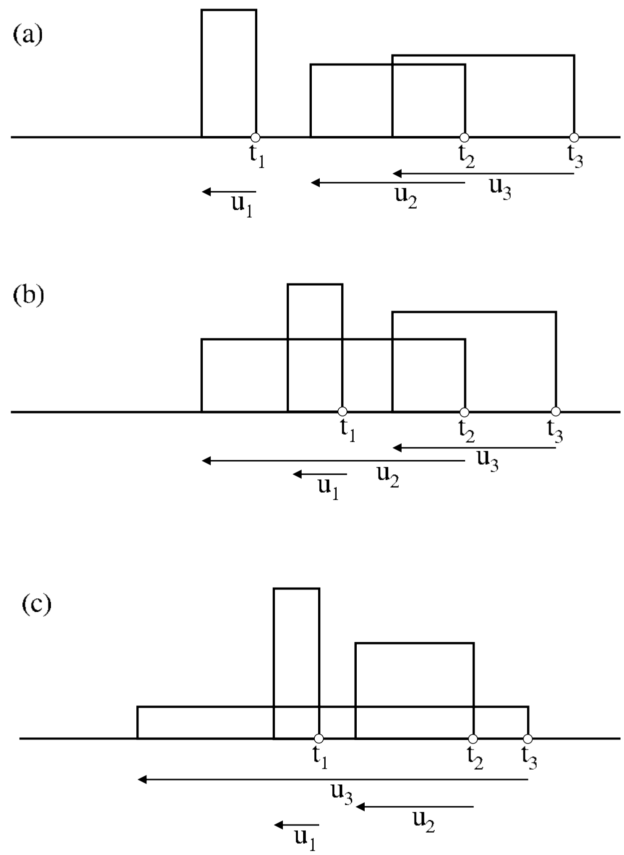

Figure B1. Illustration of the three cases to be distinguished in order to derive the third-order moment of the NSRPM. (a) $t_1-u_1 < t_2-u_2$ and $t_1-u_1 < t_3-u_3$, (b) $t_2-u_2 < t_1-u_1$ and $t_2-u_2 < t_3-u_3$, and (c) $t_3-u_3 < t_1-u_1$ and $t_3-u_3 < t_2-u_2$.

B3. Third-Order Moment With the FGM Copula

[60] In the third (central) order moment with the FGM copula and exponential margins (see equation (15)), $f(\eta, \beta,$

θ, h) and $g(\eta, \beta, \theta, h)$ are given by:

$$
\begin{aligned}
f(\eta,\beta,\theta,h)= &-1152e^{4\eta h}(e^{\beta h}-1)\eta^9(4+\theta)(16+9\theta)+288\beta^6e^{2\eta h}(e^{\beta h}-1)(e^{\eta h}-1)\eta^3\{-3\theta+e^{\eta h}(8+3\theta)\}-144\beta^4e^{2\eta h}(e^{\beta h}-1)\eta^5\{-3(\theta-2)\theta-2e^{\eta h})\times(16+\theta)(4+3\theta)+e^{2\eta h}(160+66\theta+9\theta^2)\}\\
&+288\beta e^{2\eta h}\eta^8(4+\theta)[3\theta-3e^{\beta h}\theta-8e^{\eta h}(4+3\theta)+8e^{(\beta+\eta)h}(4+3\theta)+e^{2\eta h}(32+21\theta)+e^{(\beta+2\eta)h}\{-32-21\theta+4\eta h(16+9\theta)\}]+144\beta^2e^{2\eta h}(e^{\beta h-1})\eta^7[-3\theta^2-8e^{\eta h}(8+\theta)(4+3\theta)+e^{2\eta h}\\
&\cdot\{512+\theta(344+63\theta)\}]+\beta^9(-27e^{\beta h}\theta^2+64e^{(\beta+\eta)h}\theta(5+3\theta)+96e^{(\beta+3\eta)h}\times\{96+\eta h(6+\theta)(4+3\theta)+2\theta(47+9\theta)\}+e^{(\beta+4\eta)h}\{-8064+72\eta h(4+\theta)(16+9\theta)-\theta(7136+1245\theta)\}\\
&-12e^{(\beta+2\eta)h}[96+\theta\{184+54\theta+3\eta h(8+\theta)\}])+6\beta^7\eta^2(27e^{\beta h}\theta^2-16e^{(\beta+\eta)h}\theta(26+15\theta)-48e^{(\beta+3\eta)h}\{304+3\eta h(6+\theta)(4+3\theta)+\theta(298+57\theta)\}+e^{(\beta+4\eta)h}\{12864-120\eta h(4+\theta)(16+9\theta)\\
&+\theta(11504+2013\theta)\}+12e^{(\beta+2\eta)h}[144+\theta\{268+78\theta+3\eta h(8+\theta)\}])+9\beta^5\eta^4(-27e^{\beta h}\theta^2+48e^{2\eta h}\theta(6+\theta)-32e^{3\eta h}\times(2+\theta)(4+3\theta)+32e^{(\beta+\eta)h}\theta(20+11\theta)+16e^{4\eta h}\{16+\theta(2+3\theta)\}\\
&+64e^{(\beta+3\eta)h}\times\{452+4\eta h(6+\theta)(4+3\theta)+\theta(443+84\theta)\}+e^{(\beta+4\eta)h}\{-25856+264\eta h(4+\theta)(16+9\theta)-\theta(23520+4093\theta)\}-12e^{(\beta+2\eta)h}[256+\theta\{456+134\theta+3\eta h(8+\theta)\}])\\
&-4\beta^3\eta^6(-27e^{\beta h}\theta^2-144e^{3\eta h}(4+3\theta)(8+3\theta)+108e^{2\eta h}\theta(14+3\theta)+32e^{(\beta+\eta)h}\theta(28+15\theta)+36e^{4\eta h}\times\{128+3\theta(34+9\theta)\}+48e^{(\beta+3\eta)h}\{1248+8\eta h(6+\theta)(4+3\theta)+5\theta(244+45\theta)\}\\
&+e^{(\beta+4\eta)h}\{-55296+720\eta h(4+\theta)(16+9\theta)-\theta(50552+8661\theta)\}-12e^{(\beta+2\eta)h}[384+\theta\{742+216\theta+3\eta h(8+\theta)\}])
\end{aligned}
$$

and

$$
\begin{aligned}
g(\eta,\beta,\theta,h)= &-960e^{4\eta h}(1-4e^{\beta h}+3e^{2\beta h})\eta^{10}(4+\theta)^3+80\beta e^{4\eta h}\eta^9\{-13+52e^{\beta h}+3e^{2\beta h}(8\eta h-13)\}(4+\theta)^3+40\beta^2e^{4\eta h}\eta^8(4+\theta)\times(-9\{-16+\theta(8+\theta)\}-144e^{\beta h}\{20+\theta(8+\theta)\}+e^{2\beta h}[52\eta h(4+\theta)^2\\
&+9\{304+17\theta(8+\theta)\}])-2\beta^4\eta^6(-64e^{(2\beta+\eta)h}\theta(2+\theta)(10+\theta)+3e^{2\beta h}\theta^2(12+\theta)-80e^{4\eta h}(4+27\theta)+5e^{2(\beta+2\eta)h}(4+\theta)\{-816+340\eta h(4+\theta)^2-295\theta(8+\theta)\}+1152e^{2\beta h+3\eta h}(2+\theta)\\
&\cdot\{20+\theta(10+\theta)\}+960e^{(\beta+3\eta)h}(2+\theta)\{40+\theta(10+\theta)\}+120e^{(\beta+2\eta)h}\theta\{24+\theta(12+\theta)\}+192e^{2(\beta+\eta)h}[80+\theta\{75+2\theta(12+\theta)\}]-40e^{(\beta+4\eta)h}[3040+9\theta\times\{184+3\theta(12+\theta)\}])\\
&+20\beta^7\eta^3(-96e^{(\beta+2\eta)h}\theta+192e^{(\beta+3\eta)h}(2+\theta)-96e^{(\beta+4\eta)h}(4+\theta)-18e^{(2\beta+\eta)h}\theta(2+\theta)(10+\theta)+e^{2\beta h}\theta^2\times(12+\theta)-3e^{2(\beta+2\eta)h}(4+\theta)\{-712+32\eta h(4+\theta)^2-53\theta(8+\theta)\}-34e^{2\beta h+3\eta h}(2+\theta)\\
&\cdot\{160+7\theta(10+\theta)\}+8e^{2(\beta+\eta)h}[292+3\theta\{139+4\theta(12+\theta)\}])-2\beta^{10}e^{2\beta h}(-8e^{\eta h}\theta(2+\theta)(10+\theta)+\theta^2(12+\theta)-5e^{4\eta h}(4+\theta)\{-96+4\eta h(4+\theta)^2-7\theta(8+\theta)\}-8e^{3\eta h}(2+\theta)\{160+7\theta(10+\theta)\}\\
&+4e^{2\eta h}[160+\theta\{240+7\theta(12+\theta)\}])-9\beta^9e^{2\beta h}\eta(-8e^{\eta h}\times\theta(2+\theta)(10+\theta)+\theta^2(12+\theta)-5e^{4\eta h}(4+\theta)\{-96+4\eta h(4+\theta)^2-7\theta(8+\theta)\}-8e^{3\eta h}(2+\theta)\{160+7\theta(10+\theta)\}\\
&+4e^{2\eta h}[160+\theta\{240+7\theta(12+\theta)\}])-4\beta^6\eta^4(160e^{4\eta h}+720\times e^{(\beta+2\eta)h}\theta-2880(2+\theta)\times e^{(\beta+3\eta)h}+24e^{(2\beta+\eta)h}\theta(2+\theta)(10+\theta)-3e^{2\beta h}\theta^2(12+\theta)+80e^{(\beta+4\eta)h}(88+27\theta)-15e^{2(\beta+2\eta)h}(4+\theta)\\
&\cdot\{-456+28\eta h(4+\theta)^2-41\theta\times(8+\theta)\}-24e^{2\beta h+3\eta h}(2+\theta)\times\{680+33\theta(10+\theta)\}+12e^{2(\beta+\eta)h}[320+\theta\{430+13\theta(12+\theta)\}])-4\beta^8e^{2\beta h}\eta^2(12e^{\eta h}\theta(2+\theta)(10+\theta)+\theta^2(12+\theta)+4e^{3\eta h}(2+\theta)\\
&\cdot\{1360+61\theta(10+\theta)\}+5e^{4\eta h}(4+\theta)[20\eta h(4+\theta)^2-3\times\{144+11\theta(8+\theta)\}]-4e^{2\eta h}[560+\theta\{780+23\theta(12+\theta)\}])-40\beta^3\eta^7(-192e^{(\beta+3\eta)h}(2+\theta)\times(4+\theta)(6+\theta)+192e^{2\beta h+3\eta h}(2+\theta)(4+\theta)\\
&\cdot(6+\theta)-6e^{2(\beta+\eta)h}\theta(4+\theta)(8+\theta)+6e^{(\beta+2\eta)h}\theta(4+\theta)(8+\theta)+3e^{2(\beta+2\eta)h}(4+\theta)\{-648+54\eta h(4+\theta)^2-41\theta(8+\theta)\}+e^{4\eta h}[-800+\theta\{-264+\theta(12+\theta)\}]+2e^{(\beta+4\eta)h}\\
&\cdot[4288+\theta\{3072+61\theta(12+\theta)\}])+\beta^5\eta^5(480e^{4\eta h}(\theta-8)+288e^{(2\beta+\eta)h}\theta(2+\theta)(10+\theta)-11e^{2\beta h}\theta^2(12+\theta)+15e^{2(\beta+2\eta)h}(4+\theta)\{-7872+420\eta h(4+\theta)^2-599\theta(8+\theta)\}-960e^{(\beta+3\eta)h}(2+\theta)\\
&\cdot\{48+\theta(10+\theta)\}+32e^{2\beta h+3\eta h}(2+\theta)\{8800+373\theta(10+\theta)\}+480e^{(\beta+2\eta)h}\theta\times\{38+\theta(12+\theta)\}+480e^{(\beta+4\eta)h}[176+\theta\{82+\theta(12+\theta)\}]-4e^{2(\beta+\eta)h}[19840+3\theta\{9440+269\theta(12+\theta)\}])
\end{aligned}
$$

B4. General Formulas of the Aggregated Moments With a Cubic Copula

[61] The general formulas of the aggregated moments with a cubic copula defined as in Theorem (A.2) are

$$\mathrm{E}\left(Y_i^h\right) = (36 + 2A_1 + A_2 + 4B_1 + 2B_2)\,h\,\lambda\,\mu_D/(36\,\alpha\,\eta),$$

$$
\begin{aligned}
c_Y(\tau) = \lambda\mu_D\{&2e^{-3\eta\tau}(5A_1 - 5A_2 + 22B_1 - 22B_2) - 3e^{-2\eta\tau}\\
&\cdot(10A_1 - 5A_2 + 44B_1 - 22B_2) + 6\times e^{-\eta\tau}\\
&\cdot(36 + 5A_1 + 22B_1)\}/(108\alpha^2\eta) + \beta\lambda\mathrm{E}\{D(D-1)\}\\
&\cdot(e^{-3\eta\tau}(A_1 - A_2 + 2B_1 - 2B_2)\times\{A_1 + 2(45 + A_2 + B_1\\
&+ 2B_2)\}\beta(\beta^4 - 5\beta^2\eta^2 + 4\eta^4) - e^{-2\eta\tau}(2A_1 - A_2 + 4B_1\\
&- 2B_2)\times(120 + 2A_1 + 3A_2 + 4B_1 + 6B_2)\beta(\beta^4 - 10\beta^2\eta^2\\
&+ 9\eta^4) + 5e^{-\eta\tau}(6 + A_1 + 2B_1)\{A_1 + A_2 + 2(18 + B_1\\
&+ B_2)\}\beta(\beta^4 - 13\beta^2\eta^2 + 36\eta^4) - 30e^{-\beta\tau}\eta[36\beta^4 - \{468\\
&\cdot -24A_1 + {A_2}^2 - 48B_1 + 96B_2 + 4{B_2}^2 + 4A_2(12 + B_2)\}\\
&\cdot\beta^2\eta^2 + (36 + 2A_1 + A_2 + 4B_1 + 2B_2)^2\eta^4])\\
&/\{2160\alpha^2\eta(\beta^6 - 14\beta^4\eta^2 + 49\beta^2\eta^4 - 36\eta^6)\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(Y_i^h) = \lambda\,\mu_D[&-7776 - 850\,A_1 - 95\,A_2 - 3740\,B_1 - 418\,B_2\\
&+ 8e^{-3\,\eta\,h}\,\{5\,A_1 - 5\,A_2 + 22\,(B_1 - B_2)\} - 27\,e^{-2\,\eta\,h}\\
&\cdot(10\,A_1 - 5\,A_2 + 44\,B_1 - 22\,B_2) + 216\,e^{-\eta\,h}(36 + 5\,A_1\\
&+ 22\,B_1) + 6\,(1296 + 110\,A_1 + 25\,A_2 + 484\,B_1\\
&+ 110\,B_2)\,\eta\,h]/\{1944\,\alpha^2\,\eta^3\} + \lambda\,\mathrm{E}\{D(D-1)\}\\
&\cdot(1080\,e^{-\beta\,h}\eta^3\,[36\,\beta^4 - \{468 - 24\,A_1 + {A_2}^2 - 48\,B_1\\
&+ 96\,B_2 + 4\,{B_2}^2 + 4\,A_2\,(12 + B_2)\}\beta^2\,\eta^2\\
&+ (36 + 2\,A_1 + A_2 + 4\,B_1 + 2\,B_2)^2\,\eta^4]\\
&\cdot\{e^{\beta\,h}(1 - \beta\,h) - 1\} + 180(6 + A_1 + 2\,B_1)\\
&\cdot\{A_1 + A_2 + 2\,(18 + B_1 + B_2)\}\beta^2\,(\beta^5 - 13\,\beta^3\,\eta^2\\
&+ 36\,\beta\,\eta^4)\,(e^{-\eta\,h} + \eta\,h - 1) - 9(2A_1 - A_2 + 4B_1\\
&- 2B_2)(120 + 2\,A_1 + 3\,A_2 + 4\,B_1 + 6\,B_2)\beta^2\\
&\cdot(\beta^5 - 10\,\beta^3\eta^2 + 9\,\beta\,\eta^4)(e^{-2\,\eta\,h} - 1 + 2\,\eta\,h)\\
&+ 4(A_1 - A_2 + 2\,B_1 - 2\,B_2)\,(A_1 + 2\,(45 + A_2 + B_1\\
&+ 2\,B_2))\beta^2\,(\beta^5 - 5\,\beta^3\eta^2 + 4\,\beta\,\eta^4)\,(e^{-3\,\eta\,h} - 1 + 3\,\eta\,h))\\
&/\{38880\alpha^2\,\beta\,\eta^3\,(\beta^6 - 14\,\beta^4\,\eta^2 + 49\,\beta^2\,\eta^4 - 36\,\eta^6)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Cov}(Y_i^h, Y_{i+k}^h) = {} & \lambda\mu_D(e^{\eta h}-1)^2[10A_1(4+8e^{\eta h}+12e^{2\eta h}+8e^{3\eta h} \\
& +4e^{4\eta h}-27e^{\eta h(1+k)}+108e^{2\eta h(1+k)}-54e^{\eta h(2+k)} \\
& -27e^{\eta h(3+k)})-5A_2\{-27e^{\eta h(1+k)}(1+e^{\eta h})^2 \\
& +8(1+e^{\eta h}+e^{2\eta h})^2\}+2\{3888e^{2\eta h(1+k)} \\
& +297B_2e^{\eta h(1+k)}(1+e^{\eta h})^2 \\
& -88B_2(1+e^{\eta h}+e^{2\eta h})^2+22B_1(4+8e^{\eta h} \\
& +12e^{2\eta h}+8e^{3\eta h}+4e^{4\eta h}-27e^{\eta h(1+k)} \\
& +108e^{2\eta h(1+k)}-54e^{\eta h(2+k)}-27e^{\eta h(3+k)})\}] \\
& /\{3888\alpha^2e^{3\eta h(1+k)}\eta^3\}+\lambda\,\mathrm{E}\{D(D-1)\} \\
& \cdot(2e^{-3\eta h(1+k)}(A_1-A_2+2B_1-2B_2) \\
& \cdot\{A_1+2(45+A_2+B_1+2B_2)\}\beta^3(e^{3\eta h}-1)^2 \\
& \cdot(\beta^4-5\beta^2\eta^2+4\eta^4)-(9/2)e^{-2\eta h(1+k)} \\
& \cdot(2A_1-A_2+4B_1-2B_2)(120+2A_1+3A_2 \\
& +4B_1+6B_2)\beta^3(e^{2\eta h}-1)^2(\beta^4-10\beta^2\eta^2+9\eta^4) \\
& +90e^{-\eta h(1+k)}(6+A_1+2B_1)\{A_1+A_2 \\
& +2(18+B_1+B_2)\}\beta^3(e^{\eta h}-1)^2(\beta^4-13\beta^2\eta^2 \\
& +36\eta^4)-540e^{-\beta h(1+k)}(e^{\beta h}-1)^2\eta^3 \\
& \cdot[36\beta^4-\{468-24A_1+{A_2}^2-48B_1+96B_2 \\
& +4{B_2}^2+4A_2(12+B_2)\}\beta^2\eta^2+(36+2A_1+A_2 \\
& +4B_1+2B_2)^2\eta^4])/\{38880\alpha^2\beta\eta^3(\beta^6-14\beta^4\eta^2 \\
& +49\beta^2\eta^4-36\eta^6)\}.
\end{aligned}
$$

Notation

H Joint c.d.f. of (X, Y).
F Univariate c.d.f. of X.
G Univariate c.d.f. of Y.
$\delta(.)$ Dirac delta function.
C copula distribution.
A_1, A_2, B_1, B_2 parameters of the cubic copulas.
D number of cells.
L duration of cells.
X intensity of cells.
λ rate of the Poisson process of storm origins.
μ_D mean number of cells.
α scale parameter related to the intensity of cells.
γ shape parameter related to the intensity of cells.
β parameter of the exponential distribution related to the position of cells.
η parameter of the exponential distribution related to the duration of cells.
θ parameter of the cubic copula.
f, c and ζ parameters related to the intensity/duration of cells in the DD1 and DD2 models.
Φ vector of parameters of NSRPM.
h level of aggregation.
k lag of the autocovariance function.
m order of the moment.
$Y(t)$ precipitation intensity at time t.
$c_Y(\tau)$ covariance with lag τ of the rainfall intensity process.
Y_i^h cumulative rainfall totals in disjoint time intervals of length h.
ξ_h third (central) moment of an interval of length h.
$PD(h)$ probability of no rain in an interval of length h.
$\mathcal{O}$ objective function in the method of moments.
p number of moments included in the objective function.

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