Hillslope-storage Boussinesq model for subsurface flow and variable source areas along complex hillslopes: 1. Formulation and characteristic response

Peter A. Troch

Hydrology and Quantitative Water Management Group, Department of Environmental Sciences, Wageningen University, Wageningen, Netherlands

Claudio Paniconi[1]

Center for Advanced Studies, Research and Development in Sardinia, Cagliari, Italy

E. Emiel van Loon

Hydrology and Quantitative Water Management Group, Department of Environmental Sciences, Wageningen University, Wageningen, Netherlands

Received 17 September 2002; revised 10 July 2003; accepted 6 August 2003; published 15 November 2003.

[1] Hillslope response to rainfall remains one of the central problems of catchment hydrology. Flow processes in a one-dimensional sloping aquifer can be described by Boussinesq's hydraulic groundwater theory. Most hillslopes, however, have complex three-dimensional shapes that are characterized by their plan shape, profile curvature of surface and bedrock, and the soil depth. Field studies and numerical simulation have shown that these attributes are the most significant topographic controls on subsurface flow and saturation along hillslopes. In this paper the Boussinesq equation is reformulated in terms of soil water storage rather than water table height. The continuity and Darcy equations formulated in terms of storage along the hillslope lead to the hillslope-storage Boussinesq (HSB) equation for subsurface flow. Solutions of the HSB equation account explicitly for plan shape of the hillslope by introducing the hillslope width function and for profile curvature through the bedrock slope angle and the hillslope soil depth function. We investigate the behavior of the HSB model for different hillslope types (uniform, convergent, and divergent) and different slope angles under free drainage conditions after partial initial saturation (drainage scenario) and under constant rainfall recharge conditions (recharge scenario). The HSB equation is solved by means of numerical integration of the partial differential equation. We find that convergent hillslopes drain much more slowly compared to divergent hillslopes. The accumulation of moisture storage near the outlet of convergent hillslopes results in bell-shaped hydrographs. In contrast, the fast draining divergent hillslopes produce highly peaked hydrographs. In order to investigate the relative importance of the different terms in the HSB equation, several simplified nonlinear and linearized versions are derived, for instance, by recognizing that the width function of a hillslope generally shows smooth transition along the flow direction or by introducing a fitting parameter to account for average storage along the hillslope. The dynamic response of these reduced versions of the HSB equation under free drainage conditions depend strongly on hillslope shape and bedrock slope angle. For flat slopes (of the order of 5%), only the simplified nonlinear HSB equation is able to capture the dynamics of subsurface flow along complex hillslopes. In contrast, for steep slopes (of the order of 30%), we see that all the reduced versions show very similar results compared to the full version. It can be concluded that the complex derivative terms of width with respect to flow distance play a less dominant role with increasing slope angle. Comparison with the hillslope-storage kinematic wave model of *Troch et al.* [2002] shows that the diffusive drainage terms of the HSB model become less important for the fast draining divergent hillslopes. These results

[1]Now at Institut National de la Recherche Scientifique, Centre Eau, Terre et Environnement, Université du Québec, Sainte-Foy, Quebec, Canada.

0043-1397/03/2002WR001728$09.00

have important implications for the use of simplified versions of the HSB equation in landscapes and for the development of appropriate analytical solutions for subsurface flow along complex hillslopes. *INDEX TERMS:* 1829 Hydrology: Groundwater hydrology; 1894 Hydrology: Instruments and techniques; *KEYWORDS:* subsurface flow, hillslope drainage, Boussinesq equation, groundwater modeling, model simplification, linearization

Citation: Troch, P. A., C. Paniconi, and E. Emiel van Loon, Hillslope-storage Boussinesq model for subsurface flow and variable source areas along complex hillslopes: 1. Formulation and characteristic response, *Water Resour. Res.*, *39*(11), 1316, doi:10.1029/2002WR001728, 2003.

1. Introduction

[2] Hillslopes are the basic landscape elements of many catchments. Understanding the interaction and feedbacks between hillslope forms and the processes responsible for transportation of water, sediments, and pollutants is of crucial importance for catchment scale water and land management. Since the 1950s many hillslope hydrological studies have been conducted. Of particular interest are two landmark books edited by *Kirkby* [1978] and *Anderson and Brooks* [1996]. The mathematical models of hillslope flow processes presented in these works are either complex numerical integrations of the 3-D subsurface flow equations, or simplified hydraulic groundwater equations based on the Dupuit-Forchheimer assumptions applied to unit-width hillslopes. Neither of these references presents models to account for the three-dimensional hillslope form while still using simple flow equations. The geometry of the hillslope exerts a major control on hydrologic response because it defines the domain and the boundary conditions of moisture storage. Models that most fully describe three-dimensional flow processes, based on the 3-D Richards equation, are highly nonlinear and require the solution of large systems of equations even for small-scale problems. Moreover, the parameterization of these models requires detailed information about soil hydraulic properties, information which is generally not at hand at the catchment scale. In order to improve our understanding of the response of hillslopes to atmospheric forcing (precipitation, evapotranspiration), simplified dynamic descriptions of the hydrologic system are needed. The central question of this problem was formulated by *Duffy* [1996]: "Can low dimensional dynamic models of hillslope-scale and catchment-scale flow processes be formulated such that the essential physical behavior of the natural system is preserved?"

[3] In the last few years several breakthroughs in this endeavor were reported. *Salvucci and Entekhabi* [1995] presented a statistical-dynamical methodology for coupling the flows in the saturated and unsaturated zones of planar, converging, and diverging hillslopes at the climatic mean timescale. The limitation of this approach is that small timescale flow processes are ignored, so the method is unsuitable for runoff process studies. *Duffy* [1996] published a two-state integral-balance model for soil moisture and groundwater dynamics in complex terrain. His model is formed by direct integration of the local conservation equation with respect to the partial volumes occupied by unsaturated and saturated storage. The parametric form of the storage-flux or constitutive relationship for this model was determined from numerical experiments based on Richards equation applied to a simple hillslope geometry. Later, *Reggiani et al.* [1998, 1999] extended Duffy's approach to include other subregions of the hydrological cycle (overland flow, saturated areas, and the channel network). The averaging region in this work is called the representative elementary watershed (REW). Parallel to these developments, *Fan and Bras* [1998] presented analytical solutions to a hillslope-based kinematic wave formulation of subsurface storm flow and saturation overland flow applicable to complex hillslopes. They reduced the three-dimensional soil mantle of a hillslope to a one-dimensional profile along which the soil moisture and discharge are modeled. *Troch et al.* [2002] extended the model of Fan and Bras to account for more general profile curvatures of the hillslope. They also presented characteristic response functions for nine basic hillslope types, formed by the combination of three plan shapes (converging, uniform, diverging) with three profile curvatures (concave, straight, convex).

[4] In this paper we develop a more complete "hillslope-storage" equation that accounts also for diffuse drainage and is more generally applicable to the full range of slopes of natural hillslopes. This hillslope-storage Boussinesq (HSB) equation is formulated by expressing the continuity and Darcy equations in terms of soil water storage as the dependent variable. The HSB equation is then used to generate the characteristic response of soil water storage and outflow for seven hillslope types and two bedrock slopes. The different hillslope types are defined by their plan shape: one uniform, three convergent, and three divergent hillslopes are selected for this study. We compute the characteristic response functions of subsurface flow after partial initial saturation, and of subsurface storm flow along with the corresponding degree of saturation from a constant rainfall recharge event of infinite duration. As expected the convergent hillslopes generate pronounced partial saturated areas near the outlet, illustrating how the proposed model also allows the computation of variable source areas induced by topographic controls in a catchment.

[5] In order to further explore the dynamic behavior of the HSB equation a simplified version is also derived. This simplified version is based on the assumption that the hillslope width function varies smoothly along the flow distance, allowing us to neglect certain terms in the full version. The characteristic responses of this simplified version for free drainage are computed and compared to the full version. A similar exercise is performed to study the effect of linearization of the HSB equation. Two linear versions of the HSB are proposed and compared to the full version. Finally, we also examine the assumption of kinematic wave flow dynamics by comparing the HSB response

to the analytical solutions of the hillslope-storage kinematic wave model of *Troch et al.* [2002].

2. Hillslope-Storage Boussinesq Model

2.1. Background

[6] Subsurface flow along a unit-width hillslope with sloping bedrock can be described by the Boussinesq equation (1):

$$\frac{\partial h}{\partial t} = \frac{k}{f}\left[\cos i \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \sin i \frac{\partial h}{\partial x}\right] + \frac{N}{f} \quad (1)$$

where $h(x, t)$ is the elevation of the groundwater table measured perpendicular to the underlying impermeable layer which has a slope angle i, k is the hydraulic conductivity, f is the drainable porosity, x is the distance from the outlet measured parallel to the impermeable layer, and t is time. N represents the rainfall recharge to the groundwater table. We refer to *Childs* [1971] and *Bear* [1972] for a general discussion of the Boussinesq equation.

[7] The limitations of applying (1) to describe subsurface flow in complex hillslopes stem from the fact that it does not account for the three-dimensional soil mantle in which the flow process takes place. Studies reporting observations of the spatial variability of subsurface flow [e.g., *Anderson and Burt*, 1978; *Huff et al.*, 1982; *McDonnell*, 1990; *Woods et al.*, 1997] identified topography as a significant control. The geometry of the hillslope therefore needs to be taken into account to fully capture the dynamics of subsurface flow along complex hillslopes. The equation that describes flow processes in such situations, the three-dimensional Richards equation, is complex and requires the solution of large systems of equations even for small-scale applications [*Paniconi and Wood*, 1993].

[8] *Fan and Bras* [1998] have presented a method to incorporate the topographic control on hydrologic processes into a one-dimensional model formulation. The method essentially reduces the three-dimensional soil mantle into a one-dimensional drainable pore space profile and is based on the following principles. Consider a hillslope with a three-dimensional soil mantle on top of an impermeable layer with given slope angle i (Figure 1). Flow processes in and over this hillslope will be influenced by its geometry as well as by the hydraulic properties of the porous medium and the surface. *Fan and Bras* [1998] introduced the soil moisture storage capacity function $S_c(x)$, defined as:

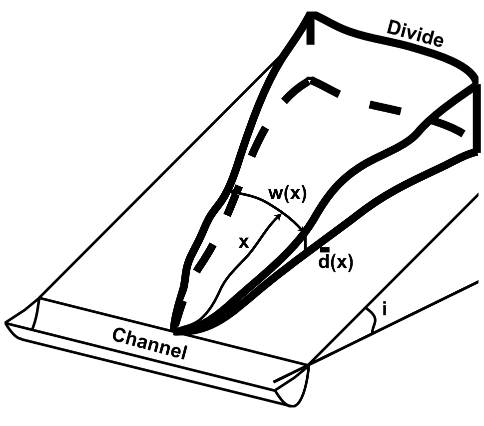

Figure 1. Three-dimensional view of a convergent hillslope overlying a straight bedrock profile.

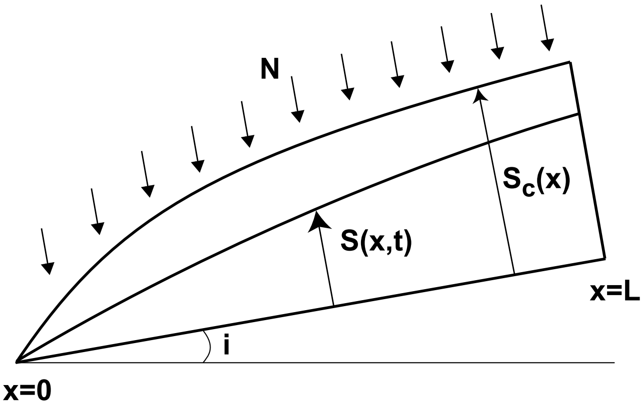

Figure 2. Definition sketch of the cross section of a 1-D hillslope-storage aquifer overlying a bedrock with constant slope angle *i*.

$$S_c(x) = w(x)\bar{d}(x)f \quad (2)$$

where $w(x)$ is the width of the hillslope at flow distance x (the hillslope width function) and $\bar{d}(x)$ is the average soil depth at flow distance x (the hillslope soil depth function). Equation (2) defines the thickness of the pore space along the hillslope and accounts for both plan shape, through the width function, and profile curvature, through the soil depth function. The plan shape and profile curvature are recognized to be the dominant topographic controls on flow processes along hillslopes [e.g., *Betson and Marius*, 1969; *Dunne and Black*, 1970; *Anderson and Burt*, 1978; *Freeze*, 1971]. Essentially, three plan shapes are encountered in nature: convergent, where hillslope width increases with distance x from the channel; divergent, where hillslope width decreases with distance x; and uniform, where the hillslope width remains constant.

[9] By introducing $S(x, t)$, soil moisture storage at a given flow distance x from the outlet at time t, one can transform the three-dimensional flow problem into a one-dimensional flow problem. The soil moisture storage capacity function, $S_c(x)$, now defines the vertical dimension of the hillslope (Figure 2) and the propagation of soil moisture storage in space and time, $S(x, t)$, is constrained by the continuity equation and some form of Darcy's law. *Fan and Bras* [1998] and *Troch et al.* [2002] used a kinematic wave approximation of Darcy's law to derive a quasi-linear wave equation solvable with the method of characteristics. Here we take a more general approach and adopt Darcy's law as the dynamic equation to derive the hillslope-storage Boussinesq equation, accounting in this way for diffuse as well as gravity drainage.

2.2. Derivation

[10] Equation (1) can be modified to describe subsurface flow within a hillslope of arbitrary plan geometry characterized by its width function $w(x)$ with no flow assumed in the lateral direction perpendicular to x (Figure 1). The continuity equation for this configuration is:

$$\frac{\partial}{\partial t}(fwh) = -\frac{\partial}{\partial x}(wq) + Nw \quad (3)$$

where $h = h(x, y, t)$, $q = q(x, y, t)$ is the flux, and y is the perpendicular direction to x. We replace h by $\bar{h}$ and q by $\bar{q}$, where $\bar{h}$ and $\bar{q}$ are defined as

$$\bar{h} = \bar{h}(x,t) = \frac{1}{w(x)} \int_w h(x,y,t)dy$$
$$\bar{q} = \bar{q}(x,t) = \frac{1}{w(x)} \int_w q(x,y,t)dy$$

and introduce the subsurface water storage $S(x, t) = fw\bar{h}$ and the volumetric discharge flux $Q(x, t) = w\bar{q}$, obtaining:

$$\frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial x} + Nw \tag{4}$$

The Darcy equation is now

$$Q = -wk\bar{h}\left(\cos i \frac{\partial \bar{h}}{\partial x} + \sin i\right) = -\frac{kS}{f}\left[\cos i \frac{\partial}{\partial x}\left(\frac{S}{fw}\right) + \sin i\right] \tag{5}$$

Combining equations (4) and (5) yields the hillslope-storage Boussinesq (HSB) equation:

$$f\frac{\partial S}{\partial t} = \frac{k \cos i}{f} \frac{\partial}{\partial x}\left[\frac{S}{w}\left(\frac{\partial S}{\partial x} - \frac{S}{w}\frac{\partial w}{\partial x}\right)\right] + k \sin i \frac{\partial S}{\partial x} + fNw \tag{6}$$

This equation can now be applied to one-dimensional hillslopes as defined in Figure 2.

2.3. Numerical Integration

[11] Equation (6) is solved numerically by discretizing in space by finite differences and applying a multistep ODE solver in time. The code is written in MATLAB. We will present results of equation (6), analyzing the response of the model in terms of subsurface soil moisture storage S, surface saturation (which is triggered when S exceeds its storage capacity given by $S_c(x)$), and outflow at the channel outlet. We will restrict our analysis to the commonly-used boundary conditions $S = 0$ at the channel outlet $x = 0$ and $Q = 0$ at the upslope or water divide boundary $x = L$. It should be noted, however, that when solved numerically, the model can readily incorporate more general boundary conditions as well as spatial (and temporal) variability in recharge, hydraulic model parameters (hydraulic conductivity k and drainable porosity f), and slope angle i [*Hilberts et al.,* 2003].

3. Characteristic Response of the Hillslope-Storage Boussinesq Equation

3.1. Basic Hillslope Types

[12] *Troch et al.* [2002] have used the following general polynomial function to describe the topographic surface of a hillslope:

$$z(x,y) = E + H(x/L)^n + \omega y^2 \tag{7}$$

where y is the distance (perpendicular to x) from the slope center. E defines the reference datum for elevation, H represents the elevation at the top of the hillslope with respect to this reference datum, and L defines the length of the hillslope. By changing the values of the profile curvature parameter n (values less than, equal to, or greater than 1) and the plan curvature parameter ω (either positive, zero or negative values), one can define different geometric relief forms [*Dikau*, 1989]. From these different relief forms and for given values of L, one can derive the drainage area, and thus the plan shape of the hillslope.

Table 1. Parameters for the Seven Hillslopes Used in This Study[a]

Hillslope Identifier	n	$\omega \times 10^{-4}$, m^{-1}	Area, m^2
a	2	5	2496
b	1	5	2160
c	0.31	5	1410
d	2	−5	646
e	1	−5	2161
f	0.31	−5	2386
uniform	1	0	5000

[a]The area column gives the surface area of the hillslope, i.e., the area upslope of the shaded faces shown in Figure 3 (Figure 4 for the straight hillslope). The hillslope identifier corresponds to the hillslope labels used in Figures 3 and 5–10.

[13] In this study we have used (7) to define seven basic plan shapes, corresponding to one uniform, three convergent and three divergent hillslope types. The parameters that we used to generate these hillslope types are given in Table 1. The length measured along the bedrock of the hillslopes is kept constant (L = 100 m), and we also assume that the soil depth (measured perpendicular to the bedrock) is constant (D = 2 m), so in this work the effect of the profile curvature on the flow processes is examined only in terms of the effect of changing the bedrock slope angle, using 5% and 30%. The values of H were set such that these two bedrock slope angles were obtained. The soil hydraulic parameters k and f are set to the following values: k = 1 m/h and f = 0.3. In practice these values can best be obtained by means of hydrograph recession analysis in the case when discharge data are available [*Troch et al.*, 1993], or they can be assigned characteristic values for the soil types present in the catchment. Figure 3 shows the three convergent and three divergent hillslope types used in this study. These six hillslopes, together with the uniform hillslope, encompass a broad range of hillslope types generally considered in geomorphology and hydrology.

3.2. Numerical Experiments

[14] In order to study the dynamic behavior of (6) we have conducted the following drainage and recharge numerical experiments. In the drainage scenarios we computed $S(x, t)$ and $Q(0, t)$ for the different hillslope types during free drainage after initial partial saturation (20% of maximum storage capacity). In the recharge scenarios the characteristic responses of the different hillslopes under a constant rainfall recharge of N = 10 mm/d are computed, starting from initially dry conditions $S(x, 0) = 0$. We first present results for the simple case of a uniform hillslope, and follow this with the results of the HSB equation for the convergent and divergent hillslope types of Figure 3. In order to facilitate intercomparison we present results from the drainage and recharge numerical experiments in terms of relative storage, defined as the ratio between actual storage $S(x, t)$ and maximum storage capacity $S_c(x)$, and outflow rates, normalized with respect to drainage area A (i.e., Q/A).

3.2.1. Case 1: Uniform Hillslopes

[15] It is instructive to generate the characteristic responses for straight hillslopes and different slope angles, as

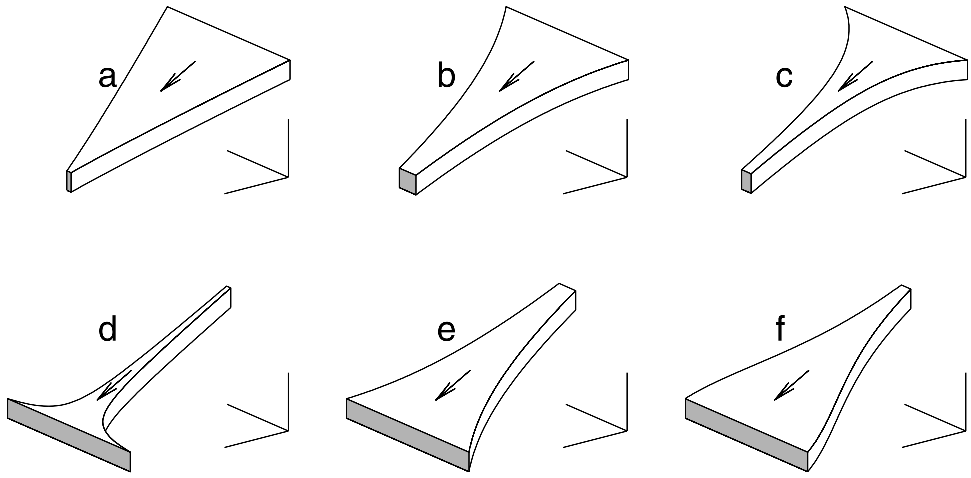

Figure 3. Three-dimensional view of the three convergent and three divergent hillslopes used in this study.

these simulations can be considered a base case to interpret the effect of topographic control on outflow and storage responses for the more complex hillslopes. Moreover, our numerical solution of the HSB equation can be tested by comparing our results with previously published results obtained with the original Boussinesq equation. For uniform hillslopes, $w(x) = w$, the HSB equation reduces to the original Boussinesq equation for sloping aquifers. Figure 4 presents results of the drainage and recharge experiments for the two different slope angles. The steady state storage profile at the end of the recharge experiment (Figure 4c) compares well with previously reported steady state water table profiles computed with an exact solution of the linearized Boussinesq equation [see *Verhoest and Troch*, 2000, Figure 2e]. As the slope angle increases, we observe that the Boussinesq equation behaves more and more as a kinematic wave model of subsurface drainage. For instance, the moisture storage profile in Figure 4b glides down the hill approximately as a pulse function with constant velocity. This results in a hydrograph with pronounced sill (constant outflow). At the time the kinematic wave leaves the hillslope a sudden drop in outflow can be seen, typical of kinematic wave dynamics [e.g., *Troch et al.*, 2002]. This is further illustrated by the outflow rate during constant rainfall recharge which shows the typical ramp function of kinematic wave dynamics (Figure 4d).

3.2.2. Case 2: Convergent and Divergent Hillslopes

[16] Figure 5 shows the spatial and temporal dynamics of the relative storage and the outflow rate for the six basic hillslopes at a 5% slope angle for the drainage scenario. It is clear from these results that the convergent and divergent hillslopes show quite different dynamic behavior. The convergent hillslopes drain much slower due to the reduced flow domain near the outlet of these hillslopes. This is also reflected in the accumulation of storage near the outlet for the convergent hillslopes. This gradual accumulation of moisture storage near the outlet results in a bell-shaped hydrograph. In contrast, the fast draining divergent hillslopes produce highly peaked hydrographs. The difference in dynamic response between hillslopes of a given plan shape is much smaller.

[17] Figure 6 shows the results from the same drainage scenario for a 30% slope angle. Now the differences in outflow patterns between the convergent hillslopes are more pronounced. The first convergent hillslope (linearly increasing hillslope width with flow distance from the outlet) saturates near the outlet after a few days of drainage. From then on, the outflow rate maintains a constant level, after which a sudden drawdown again suggests kinematic wave dynamics. The other convergent hillslopes do not saturate near the outlet and therefore produce again a bell-shaped hydrograph. The effect of slope angle on the hydrographs produced by the divergent hillslopes is, besides an acceleration of the drainage, negligible.

[18] The results for the recharge experiment are given in Figures 7 and 8. For the 5% slope angle (Figure 7), we see that part of the first convergent hillslope becomes saturated in steady state regime, also illustrated by the amount of subsurface drainage at steady state, which is less than the steady state recharge rate of 10 mm/d (the difference is removed from the solution as surface runoff). Further, it is observed that the steady state storage profile depends on the

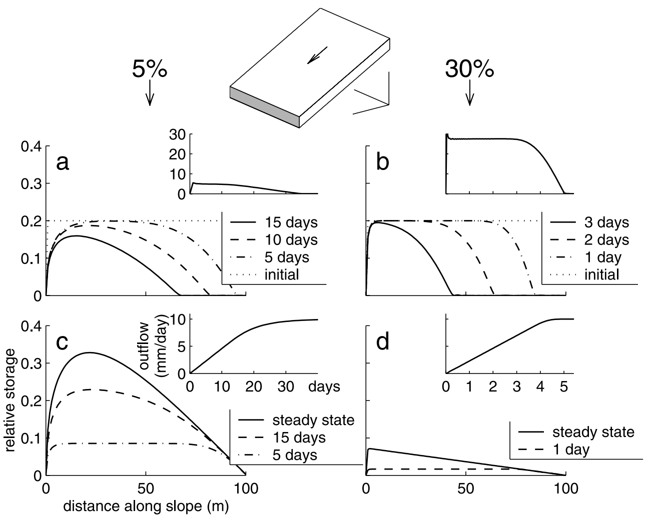

Figure 4. Relative storage profiles along the hillslope and normalized subsurface flow rates (mm/day) at the outlet for the uniform hillslope at (left) 5% and (right) 30% slope angle. (a and b) Drainage scenario results; (c and d) recharge scenario results.

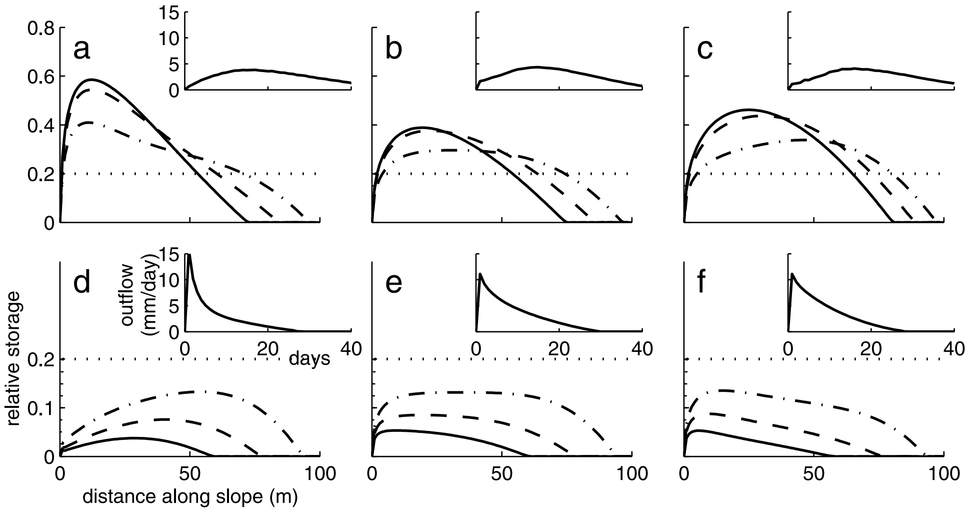

Figure 5. Relative storage profiles along the hillslope and normalized subsurface flow rates (mm/day) at the outlet during the drainage run for the six hillslopes of Figure 3 at a 5% slope angle (the labels "a" to "f" refer to those in Figure 3). For the relative storage plots, dotted line is initial time, t = 0; dash-dotted line is t = 5 days; dashed line is t = 10 days; and solid line is t = 15 days.

plan shape of the hillslope, but that the time to reach steady state outflow is more or less similar for similar hillslope types. The convergent hillslopes reach steady state outflow at about 50 d after the start of the simulation, whereas the divergent hillslopes reach steady state after only 20 d. The 30% slope angle case (Figure 8) shows similar patterns, and the effect of increasing slope angle results in faster response times and shallower storage profiles, preventing any of the hillslopes from saturating for this recharge rate.

4. Simplification and Linearization of the HSB Equation

4.1. Formulation

4.1.1. Simplified Version of the HSB Equation

[19] Expanding the second order derivative term in (6) gives

$$f\frac{\partial S}{\partial t}=\frac{k\cos i}{fw}\left[\left(\frac{\partial S}{\partial x}\right)^2+S\frac{\partial^2 S}{\partial x^2}-\frac{3S}{w}\frac{\partial S}{\partial x}\frac{\partial w}{\partial x}+\frac{2S^2}{w^2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{S^2}{w}\frac{\partial^2 w}{\partial x^2}\right]+k\sin i\frac{\partial S}{\partial x}+fNw \qquad (8)$$

It can be argued that for natural hillslopes the derivative of w with respect to x is small, reflecting smooth increase or decrease of width. Dropping the 3 terms containing $\partial w/\partial x$ yields the simplified form of the hillslope-storage Boussinesq model:

$$f\frac{\partial S}{\partial t}=\frac{k\cos i}{fw}\left[\left(\frac{\partial S}{\partial x}\right)^2+S\frac{\partial^2 S}{\partial x^2}\right]+k\sin i\frac{\partial S}{\partial x}+fNw \qquad (9)$$

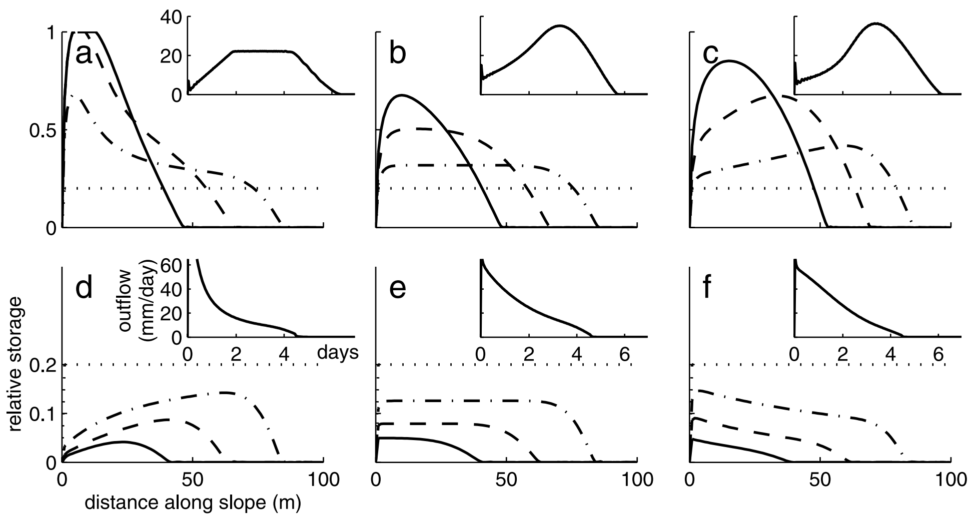

Figure 6. Same as Figure 5 but for 30% slope angle. Dotted line is initial time, t = 0; dash-dotted line is t = 1 day; dashed line is t = 2 days; and solid line is t = 3 days.

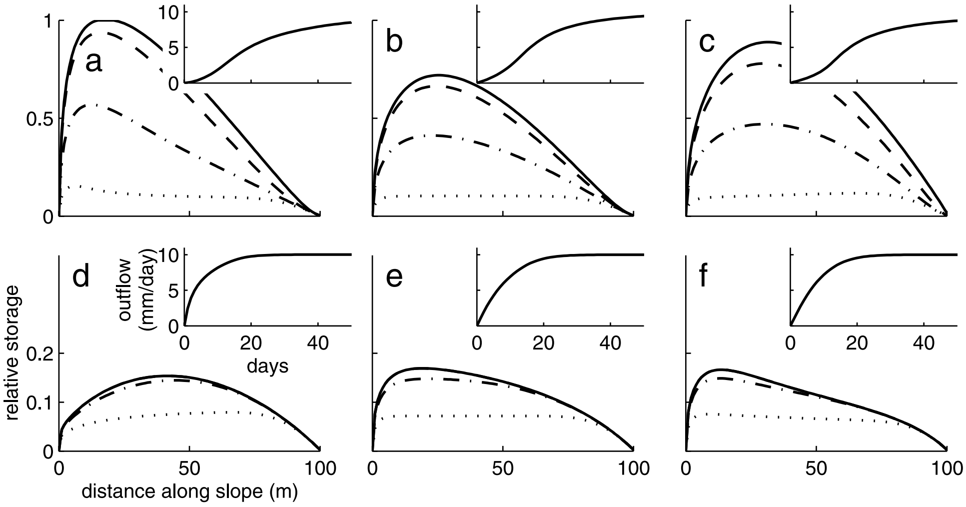

Figure 7. Relative storage profiles along the hillslope and normalized subsurface flow rates (mm/day) at the outlet during the recharge run for the six hillslopes of Figure 3 at a 5% slope angle (the labels "a" to "f" refer to those in Figure 3). For the relative storage plots, dotted line is $t = 5$ days, dash-dotted line is $t = 15$ days, dashed line is $t = 40$ days, and solid line is steady state.

4.1.2. Linearization of the HSB Equation

[20] In general, (8) and (9) do not have analytical solutions. Analytical solutions are useful to increase our insight into the dynamic behavior of these new subsurface flow equations for complex hillslopes. It is therefore interesting to investigate possible routes toward analytical solutions to equations (8) and (9). One approach would be to linearize equation (8) by assuming that the storage per unit width, S/w, can be replaced by

$$\frac{S}{w} \simeq p\frac{\overline{S_c}}{\bar{w}} = pfD \tag{10}$$

where $0 \leq p \leq 1$ is a fitting parameter, $\bar{S}_c$ and $\bar{w}$ are average storage capacity and width of the hillslope, and D represents the average soil depth along the hillslope. pfD defines the average storage per unit width along the hillslope during drainage. Substituting the first S/w term in (8) by (10), the linearized hillslope-storage Boussinesq equation now reads:

$$f\frac{\partial S}{\partial t} = kpD\cos i\left[\frac{\partial^2 S}{\partial x^2} - \left(\frac{1}{w}\frac{\partial w}{\partial x}\right)\frac{\partial S}{\partial x} - \frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right)S\right] + k\sin i\frac{\partial S}{\partial x} + Nwf \tag{11}$$

which is a linear partial differential equation (PDE) with variable coefficients. For certain classes of width functions, this equation can be further simplified. For instance, if we consider the following width function:

$$w(x) = c\exp(ax)$$

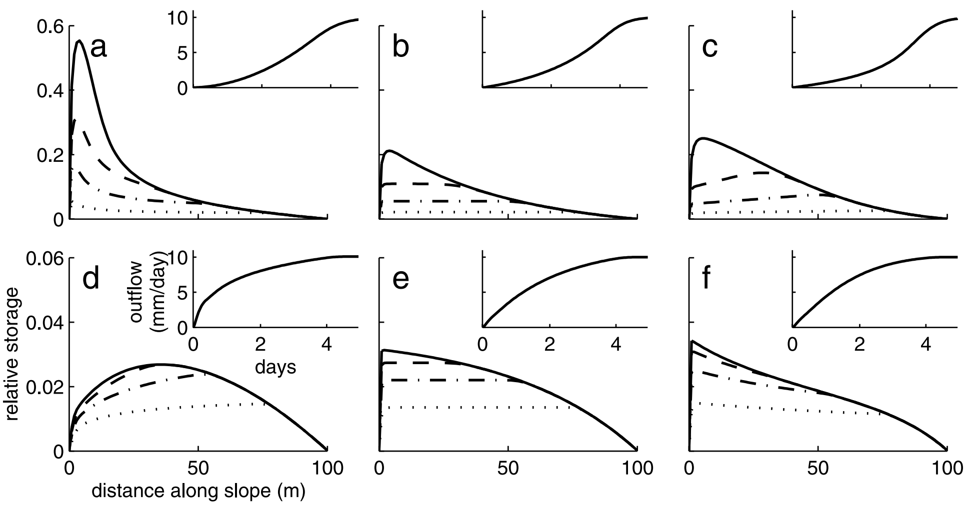

Figure 8. Same as Figure 7 but for 30% slope angle. Dotted line is $t = 1$ day; dash-dotted line is $t = 2$ days; dashed line is $t = 3$ days; solid line is steady state.

where a and c are constants, equation (11) can be written as:

$$f\frac{\partial S}{\partial t} = kpD\cos i\left[\frac{\partial^2 S}{\partial x^2} - a\frac{\partial S}{\partial x}\right] + k\sin i\frac{\partial S}{\partial x} + Nwf \qquad (12)$$

which is a linear PDE with constant coefficients and nonhomogeneous forcing term.

[21] One can also linearize the simplified HSB equation (9). Equation (9) can be rewritten as

$$f\frac{\partial S}{\partial t} = \frac{k\cos i}{fw}\frac{\partial}{\partial x}\left[S\frac{\partial S}{\partial x}\right] + k\sin i\frac{\partial S}{\partial x} + Nwf \qquad (13)$$

If we assume that S within the square brackets can be replaced by

$$S \simeq p\overline{S_c} = pf\bar{w}D$$

we get

$$f\frac{\partial S}{\partial t} = \frac{kpD\bar{w}\cos i}{w}\frac{\partial^2 S}{\partial x^2} + k\sin i\frac{\partial S}{\partial x} + Nwf \qquad (14)$$

which is again a linear PDE but with variable coefficients and nonhomogeneous forcing term. We can further simplify (14) by assuming that $w(x)$ $\bar{w}$, resulting in the following PDE with constant coefficients and nonhomogeneous forcing term:

$$f\frac{\partial S}{\partial t} = kpD\cos i\frac{\partial^2 S}{\partial x^2} + k\sin i\frac{\partial S}{\partial x} + Nwf \qquad (15)$$

Both (12) and (15) assume the general form

$$\frac{\partial S}{\partial t} = K\frac{\partial^2 S}{\partial x^2} + U\frac{\partial S}{\partial x} + Nw \qquad (16)$$

where

$$K = \frac{kpD\cos i}{f}$$

and

$$U = \frac{k\sin i - akpD\cos i}{f} \quad or \quad U = \frac{k\sin i}{f}$$

Solutions to (16) can be sought for the following initial and boundary conditions:

$$S(x,t) = S_c(x) \qquad 0 \le x \le L \quad t = 0$$

$$S(x,t) = 0 \qquad x = 0 \qquad t > 0$$

$$K(\partial S/\partial x) + US = 0 \qquad x = L \qquad t > 0$$

One possible approach to solving (16) analytically is to apply the Laplace transformation, yielding a homogeneous ordinary differential equation with nonhomogeneous forcing term. The derivation of analytical solutions to (16) using the above mentioned initial and boundary conditions and for different functional forms of $w(x)$ is the subject of ongoing research.

4.2. Comparison

[22] Let us now investigate the effect of simplifying and linearizing the HSB equation. Figures 9 and 10 summarize the results from the drainage runs. For the linear versions of the HSB model, a constant value of $p = 0.5$ was used. In Figure 9 we compare the behavior of the simplified HSB equation (9) and the two proposed linearized versions (11) and (15), all solved numerically, with the full HSB equation for a bedrock slope angle of 5%. In general, the relative storage profiles computed from (9) are close to the results presented in Figure 5. The accumulation of storage near the outlet is more pronounced for hillslope a. The storage profiles for hillslope b are quite similar, whereas the results for hillslope c show overestimation of storage near the outlet and underestimation near the drainage divide. For the divergent hillslopes the simplified HSB model is able to reproduce the shape of the storage profiles quite well, but overestimates the storage profiles at given time instances, which results in a delayed hydrograph (not shown). The differences between the HSB results and the linear versions are more pronounced. For the convergent hillslopes, it is observed that the linear models now underestimate the storage profiles. An interesting feature of the linear solutions is that the storage profiles are prevented from sliding down along the bottom of the aquifer in the direction of the outlet (this particular behavior of the linearized Boussinesq equation was already reported by *Brutsaert* [1994]). The linearized versions of the HSB equation reproduce the shape of the storage profiles reasonably well for the convergent hillslopes, but not for the divergent hillslopes. This can be due to the chosen value of p. Further research is planned to investigate the effect of p on the linearized solutions for different hillslope types.

[23] Figure 10 compares the behavior of the simplified HSB and the two proposed linearized versions with the full HSB equation for a bedrock slope angle of 30%. There is an almost perfect match between the simplified HSB equation and the full version for both the convergent and divergent hillslopes. Apparently, the complex derivative terms of w with respect to x play a much less important role with increasing bedrock slope. Even the linearized versions perform well for the 30% slope case. Even though the storage profiles are still prevented from sliding down the bedrock, the profiles computed with the linear models are a close approximation of the HSB profiles. This is in contrast to the findings of *Brutsaert* [1994] concerning unit-width hillslopes, where it was suggested that the inability of the storage profile to slide down along the bottom of the aquifer would cause larger deviations for steeper slopes. Even the linearization (15) is able to capture the main dynamics during the drainage experiment. Again, this suggests that for higher bedrock slope angles the complex derivative terms of w with respect to x do not play an important role in the full version. This finding has signif-

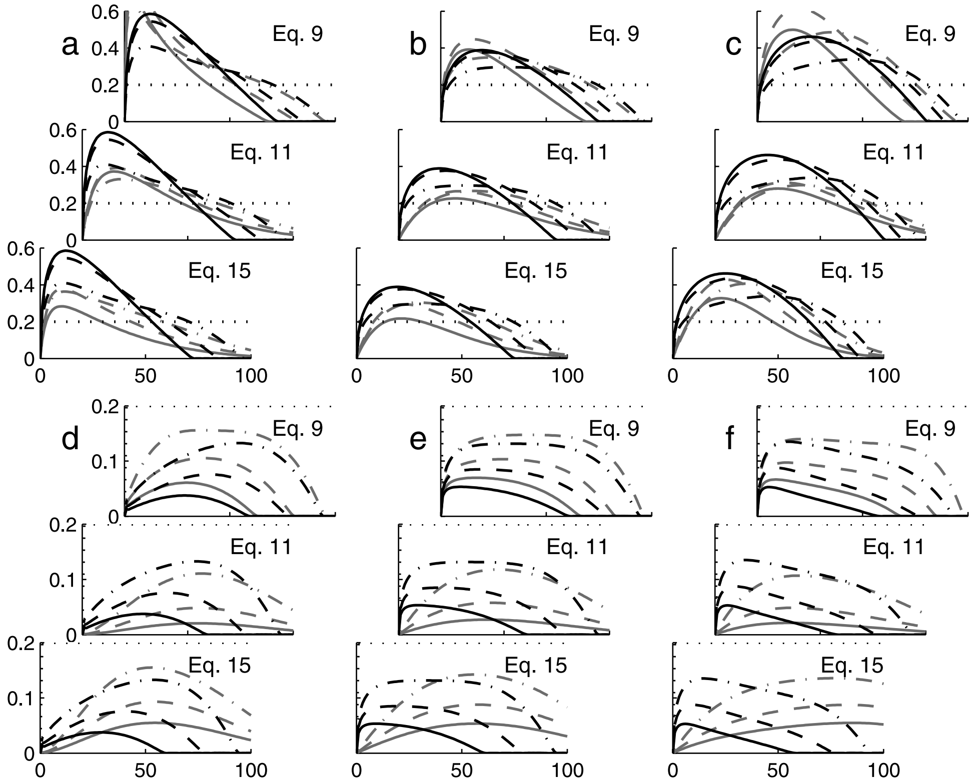

Figure 9. Relative storage profiles along the hillslope for the six hillslopes of Figure 3 computed from equations (9), (11), and (15) during the drainage run at a 5% slope angle (the labels "a" to "f" refer to those in Figure 3). In all plots the red lines are the simplified/linearized model results, and the black lines are the HSB results. Dotted line is initial time, t = 0; dash-dotted line is t = 5 days; dashed line is t = 10 days; and solid line is t = 15 days. See color version of this figure at back of this issue.

icant implications for the use of simplified versions of the HSB equation to simulate subsurface flow dynamics in steep hillslopes.

5. Discussion

5.1. Field Evidence of Topographic Control on Subsurface Flow

[24] *Woods and Rowe* [1996] and *Woods et al.* [1997] investigated the effect of topography and antecedent conditions on patterns of subsurface flow along hillslopes. Thirty subsurface flow troughs were placed end-to-end across the base of a steep (30 degrees), short (60 m) forested hillslope with shallow and highly permeable soils (depth of 0.6 m and saturated hydraulic conductivity on the order of 25 cm/h for the soil matrix). The configuration of the troughs was such that different hillslope types (uniform, convergent, and divergent) were monitored. Rainfall and trough flow measurements were made continuously for 2 years, making this one of the most complete data sets for studying the effect of topography on subsurface flow processes along complex hillslopes. *Woods and Rowe* [1996] observed that spatial variability of runoff between troughs was considerable, and that this variability changed markedly with antecedent conditions. The hillslope-storage Boussinesq model could help in analyzing the spatial variability observed in experimental hillslopes such as these, and in explaining the dominant flow processes that occur along convergent and divergent hillslopes. The model could also provide further theoretical support for the use of topography-based wetness indices [e.g., *Beven and Kirkby*, 1979; *Woods et al.*, 1997; *Western et al.*, 1999] to quantify spatially variable runoff generation at the catchment scale.

5.2. Comments on the Kinematic Wave Approximation

[25] At this point, it is interesting to discuss the difference between the model presented here and the kinematic wave approximation for subsurface flow and variable source areas for complex hillslopes developed by *Troch et al.* [2002]. *Troch et al.* [2002] used a kinematic wave approximation of Darcy's law to derive a quasi-linear wave equation solvable with the method of characteristics. The advantages of the kinematic wave approximation are that, first, analytical solutions exist, and second, the model accounts explicitly for the profile curvature of the bedrock or land surface. It should be noted that the HSB model presented here is also capable of handling the effect of profile curvature on subsurface flow processes, since the soil depth $\bar{d}$ is a function of flow distance and since a variable local bedrock slope angle is readily implemented in the numerical solution of (6). The main disadvantage of the kinematic wave approximation is that it does not account for diffuse drainage, so the model is not applicable for gently to moderately sloping terrains.

[26] Figure 11 compares the relative storage profiles and the resulting hydrographs, computed with the HSB model for the 5% and 30% slope drainage scenario with those

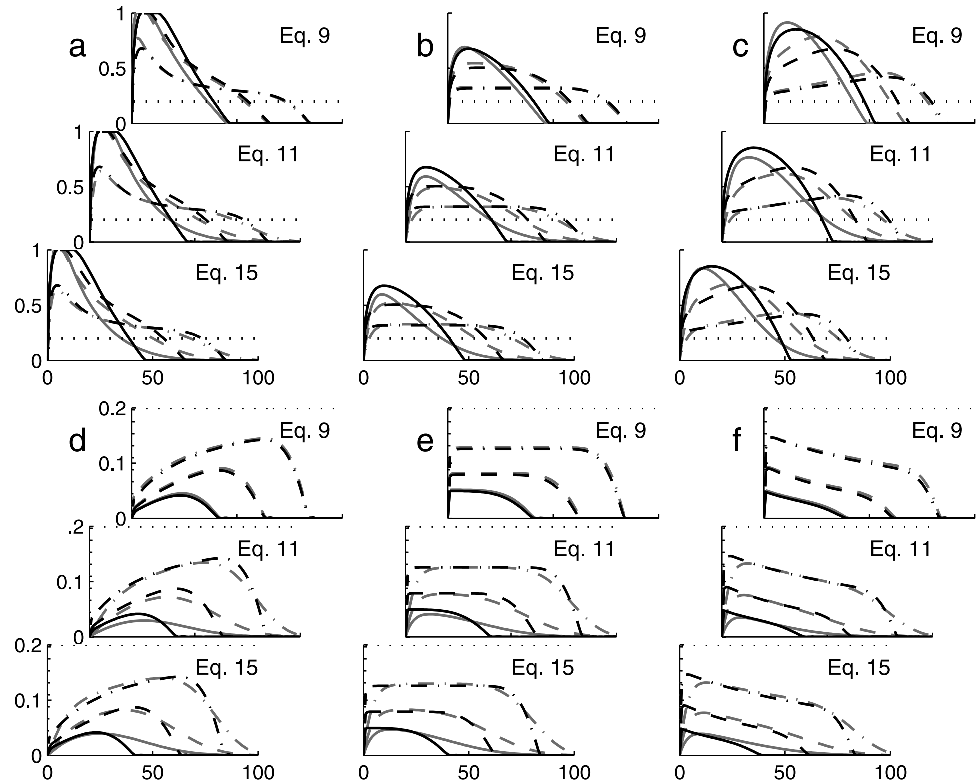

Figure 10. Same as Figure 9 but for 30% slope angle. Dotted line is initial time, $t = 0$; dash-dotted line is $t = 1$ day; dashed line is $t = 2$ days; and solid line is $t = 3$ days. See color version of this figure at back of this issue.

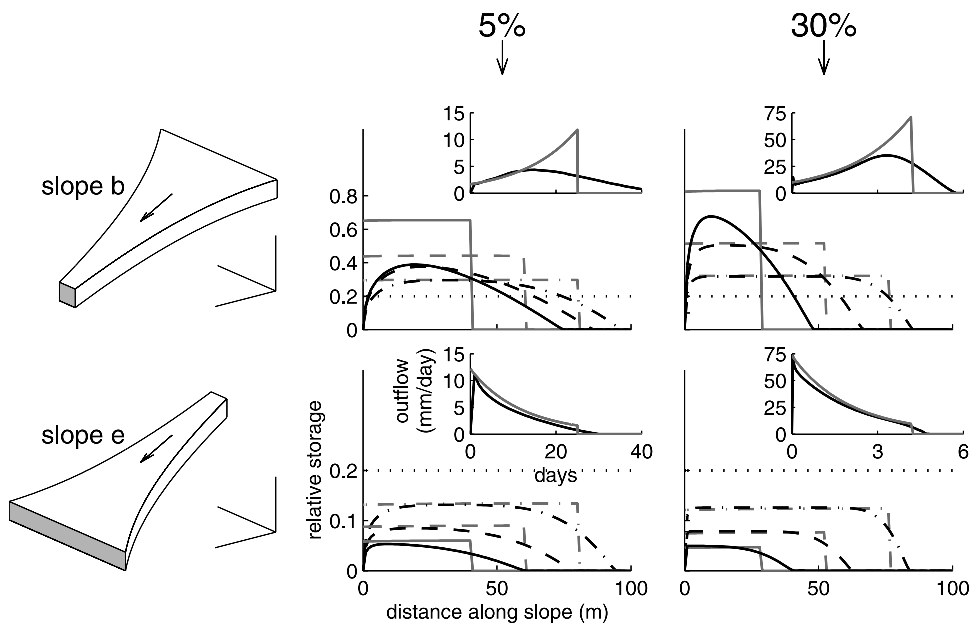

Figure 11. Relative storage profiles and normalized subsurface flow rates (mm/day) at the outlet computed by the hillslope-storage kinematic wave model of *Troch et al.* [2002] for hillslopes b and e of Figure 3 at (left) 5% and (right) 30% slope angle. The red lines are the kinematic wave model results and the black lines are the HSB results. For the relative storage profiles at 5% slope angle, dotted line is initial time, $t = 0$; dash-dotted line is $t = 5$ days; dashed line is $t = 10$ days; and solid line is $t = 15$ days. For the 30% slope angle, dotted line is initial time, $t = 0$; dash-dotted line is $t = 1$ day; dashed line is $t = 2$ days; and solid line is $t = 3$ days. See color version of this figure at back of this issue.

computed from the analytical solutions presented by *Troch et al.* [2002]. Since the kinematic wave solutions are based on an exact solution of the hillslope-storage kinematic wave equation, the computed relative storage profiles retain their sharp fronts. If the hillslope-storage kinematic wave equation was solved numerically, the resulting storage profiles would show smoother fronts, due to numerical dissipation. Remarkable conclusions can be drawn from this analysis. First, it is observed that the hydrographs computed with the kinematic wave model for the convergent hillslopes are quite different from those computed with the HSB model, even for the 30% slope angle. The hydrographs compare very well during the first one third of the drainage period, then the kinematic wave model overestimates the outflow by about half the volume during the next one third of the drainage period, and it subsequently underestimates the outflow volume by the same amount during the final one third of the flow duration. The storage profiles reflect these patterns in computed outflow. The early stage storage profiles match almost perfectly with the HSB profiles, but at later stages of drainage they are very different. For the divergent hillslopes, the kinematic wave model almost perfectly matches the outflow of the HSB model, for both the 5% and 30% slope angles. As a result, also the storage profiles match very well. It can be concluded therefore that for complex hillslopes, the plan shape rather than the bedrock slope angle determine the validity of the kinematic wave approximation. We can conclude from this that the diffusive terms of the HSB model become less important for fast draining hillslopes, making the kinematic wave approximation a reasonable one for divergent slopes. To our knowledge this has never been reported in earlier studies on kinematic wave dynamics for subsurface flow processes.

5.3. Numerical Testing of the HSB Equation

[27] A follow-up paper on the HSB equation [*Paniconi et al.*, 2003] will demonstrate the validity of this approach by comparing simulation results with model results based on the 3-D Richards equation. This type of numerical testing of simplified flow models is common practice [e.g., *Duffy*, 1996; *Brandes et al.*, 1998; *Sloan*, 2000] and should precede further validation of the modeling concepts based on laboratory and field experiments. *Paniconi et al.* [2003] show that subsurface simulations for complex hillslopes based on the HSB equation are generally in very close agreement with the simulation results based on the 3-D Richards equation.

6. Summary and Conclusions

[28] In this paper we have presented a new mathematical formulation of subsurface flow along complex hillslopes, the hillslope-storage Boussinesq (HSB) equation. It is derived by reformulating Boussinesq's equation in terms of storage instead of groundwater table. The method proposed by *Fan and Bras* [1998] is used to collapse the three-dimensional soil mantle of complex hillslopes into a one-dimensional drainable pore space. In this way a closed-form mathematical model is obtained that allows for the computation of subsurface flow and seepage. The model implicitly accounts for plan shape through the hillslope's width function, and profile curvature through the slope's soil depth function. The behavior of the HSB equation is studied by comparing subsurface flow simulations for distinct complex hillslopes. In order to further investigate the relative importance of the different terms in the HSB equation, simplified and linearized expressions of the HSB equation were derived. Subsurface flow simulations based on these simplified and linearized equations were then compared with the full version of the HSB equation. Finally, simulation results from the HSB equation were compared with analytical solutions of the hillslope-storage kinematic wave equation presented by *Troch et al.* [2002].

[29] The main conclusion of this paper is that the dynamic response of complex hillslopes during drainage and recharge events depends very much on the plan shape and bedrock slope. Convergent hillslopes drain much slower than divergent hillslopes, due to the reduced flow domain near the outlet. The characteristic hydrographs of convergent hillslopes during free drainage show a typical bell-shaped form, whereas the corresponding hydrographs for the divergent hillslopes show high peakedness in the early stages of drainage. It is further found that for moderate bedrock slopes (on the order of 5%), none of the linearized approximations to the HSB equation are able to capture the full dynamics of subsurface flow along complex hillslopes. When bedrock slope increases (for values on the order of 30%), the linearized solutions start to be able to reproduce the flow processes predicted by the HSB equation. Another important conclusion from our study is that hillslope plan shape rather than bedrock slope angle determines the validity of the kinematic wave approximation to describe subsurface flow processes along complex hillslopes. These observations have important consequences for the use of these simplified assumptions (linearization, kinematic wave approximation) to model subsurface flow processes along complex hillslopes.

[30] **Acknowledgments.** This work has been supported in part by the European Commission (contract EVK1-CT-2000-00082) and the Italian Ministry of the University (project ISR8, C11-B).

References

Anderson, M., and S. Brooks, *Advances in Hillslope Processes*, 2 vols., 1306 pp., John Wiley, Hoboken, N. J., 1996.

Anderson, M. G., and T. P. Burt, The role of topography in controlling throughflow generation, *Earth Surf. Processes*, *3*, 331–344, 1978.

Bear, J., *Dynamics of Fluids in Porous Media*, Dover, Mineola, N. Y., 1972.

Betson, R. P., and J. B. Marius, Source area of storm runoff, *Water Resour. Res.*, *5*(3), 574–582, 1969.

Beven, K. J., and M. J. Kirkby, A physically-based variable contributing area model of basin hydrology, *Hydrol. Sci. Bull.*, *24*, 43–69, 1979.

Brandes, D., C. J. Duffy, and J. P. Cusumano, Stability and damping in a dynamical model of hillslope hydrology, *Water Resour. Res.*, *34*, 3303–3313, 1998.

Brutsaert, W., The unit reponse of groundwater outflow from a hillslope, *Water Resour. Res.*, *30*, 2759–2763, 1994.

Childs, E., Drainage of groundwater resting on a sloping bed, *Water Resour. Res.*, *7*, 1256–1263, 1971.

Dikau, R., The application of a digital relief model to landform analysis in geomorphology, in *Three Dimensional Applications in Geographical Information Systems*, edited by J. Raper, pp. 51–77, Taylor and Francis, Philadelphia, Pa., 1989.

Duffy, C., A two-state integral-balance model for soil moisture and groundwater dynamics in complex terrain, *Water Resour. Res.*, *32*, 2421–2434, 1996.

Dunne, T., and R. D. Black, Partial area contributions to storm runoff in a small New England watershed, *Water Resour. Res.*, *6*, 1296–1311, 1970.

Fan, Y., and R. Bras, Analytical solutions to hillslope subsurface storm flow and saturation overland flow, *Water Resour. Res.*, *34*, 921–927, 1998.

Freeze, R. A., Three-dimensional, transient, saturated-unsaturated flow in a groundwater basin, *Water Resour. Res.*, *7*(2), 347–366, 1971.
Hilberts, A. G. J., E. E. van Loon, P. A. Troch, and C. Paniconi, The hillslope-Storage Boussinesq model for non-constanet bedrock slope, *J. Hydrol.*, in press, 2003.
Huff, D. D., R. V. O'Neill, W. R. Emanuel, J. W. Elwood, and J. D. Newbold, Flow variability and hillslope hydrology, *Earth Surf. Processes Landforms*, *7*, 91–94, 1982.
Kirkby, M., *Hillslope Hydrology*, John Wiley, Hoboken, N. J., 1978.
McDonnell, J. J., A rationale for old water discharge through macropores in a steep, humid catchment, *Water Resour. Res.*, *26*, 2821–2832, 1990.
Paniconi, C., and E. Wood, A detailed model for simulation of catchment scale subsurface hydrologic processes, *Water Resour. Res.*, *29*, 1601–1620, 1993.
Paniconi, C., P. Troch, E. van Loon, and A. Hilberts, Hillslope-storage Boussinesq model for subsurface flow and variable source areas along complex hillslopes: 2. Intercomparison with a three-dimensional Richards equation model, *Water Resour. Res.*, *39*, doi:10.1029/2002WR001730, in press, 2003.
Reggiani, P., M. Sivapalan, and S. Hassanizadeh, A unifying framework for watershed thermodynamics: Balance equations for mass, momentum, energy and entropy, and the second law of thermodynamics, *Adv. Water Resour.*, *22*, 367–398, 1998.
Reggiani, P., S. Hassanizadeh, M. Sivapalan, and W. Gray, A unifying framework for watershed thermodynamics: Constitutive relationships, *Adv. Water Resour.*, *23*, 15–39, 1999.
Salvucci, G., and D. Entekhabi, Hillslope and climatic controls on hydrologic fluxes, *Water Resour. Res.*, *31*, 1725–1739, 1995.
Sloan, W. T., A physics-based function for modeling transient groundwater discharge at the watershed scale, *Water Resour. Res.*, *36*(1), 225–241, 2000.
Troch, P., F. De Troch, and W. Brutsaert, Effective water table depth to describe initial conditions prior to storm rainfall in humid regions, *Water Resour. Res.*, *29*, 427–434, 1993.
Troch, P., E. van Loon, and A. Hilberts, Analytical solutions to a hillslope-storage kinematic wave equation for subsurface flow, *Adv. Water Resour.*, *25*, 637–649, 2002.
Verhoest, N., and P. Troch, Some analytical solutions of the linearized Boussinesq equation with recharge for a sloping aquifer, *Water Resour. Res.*, *36*, 793–800, 2000.
Western, A. W., R. B. Grayson, G. Blöschl, G. R. Willgoose, and T. A. McMahon, Observed spatial organization of soil moisture and its relation to terrain indices, *Water Resour. Res.*, *35*, 797–810, 1999.
Woods, R., and L. Rowe, The changing spatial variability of subsurface flow across a hillside, *J. Hydrol. N. Z.*, *35*, 51–86, 1996.
Woods, R., M. Sivapalan, and J. Robinson, Modeling the spatial variability of subsurface runoff using a topographic index, *Water Resour. Res.*, *33*, 1061–1073, 1997.

C. Paniconi, INRS-ETE, University of Quebec, Sainte-Foy, Quebec, Canada G1V 4C7. (claudio_paniconi@inrs-ete.uquebec.ca)

P. A. Troch and E. Emiel van Loon, Hydrology and Quantitative Water Management Group, Department of Environmental Sciences, Wageningen University, Nieuwe Kanaal 11, 6709 PA Wageningen, Netherlands. (peter.troch@wur.nl; emiel.vanloon@wur.nl)

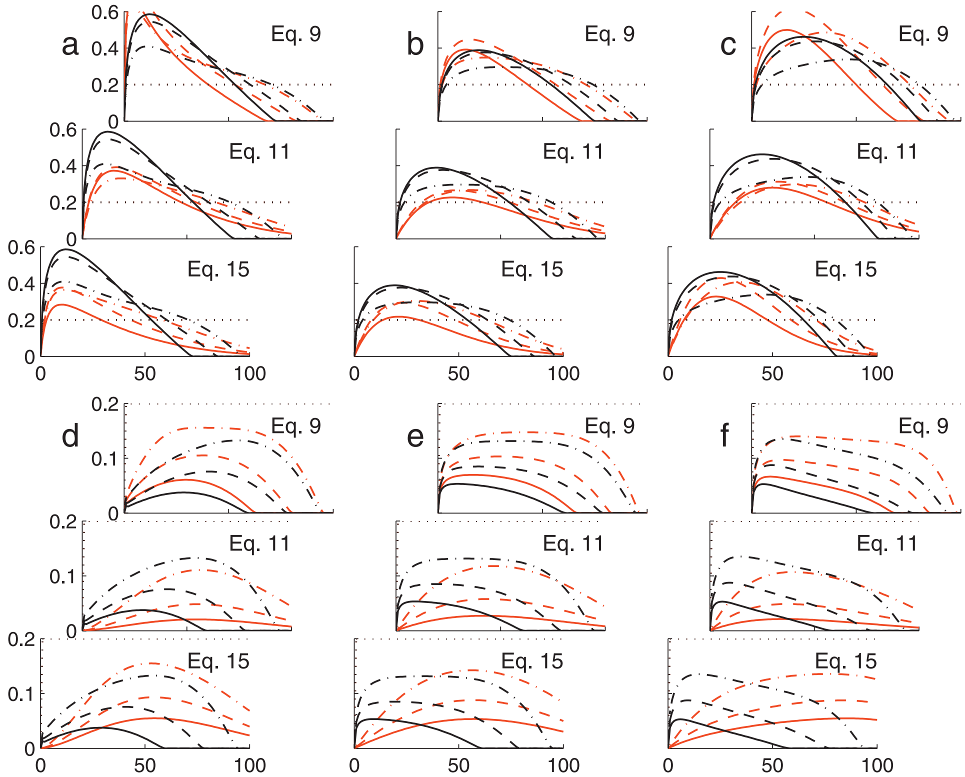

Figure 9. Relative storage profiles along the hillslope for the six hillslopes of Figure 3 computed from equations (9), (11), and (15) during the drainage run at a 5% slope angle (the labels "a" to "f" refer to those in Figure 3). In all plots the red lines are the simplified/linearized model results, and the black lines are the HSB results. Dotted line is initial time, $t = 0$; dash-dotted line is $t = 5$ days; dashed line is $t = 10$ days; and solid line is $t = 15$ days.

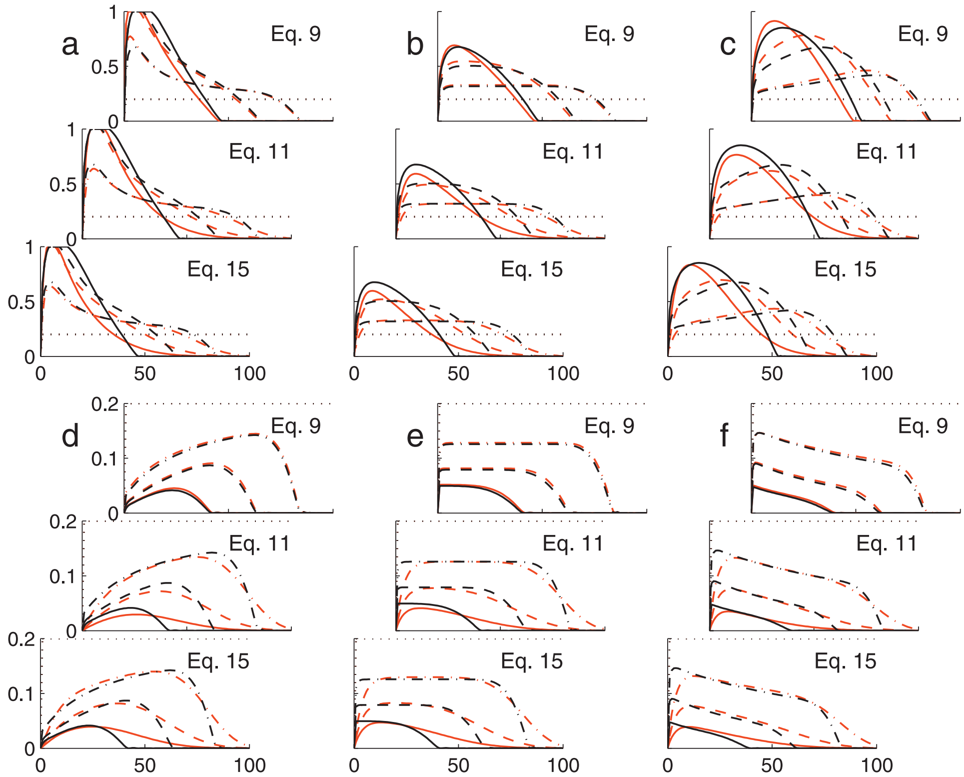

Figure 10. Same as Figure 9 but for 30% slope angle. Dotted line is initial time, $t = 0$; dash-dotted line is $t = 1$ day; dashed line is $t = 2$ days; and solid line is $t = 3$ days.

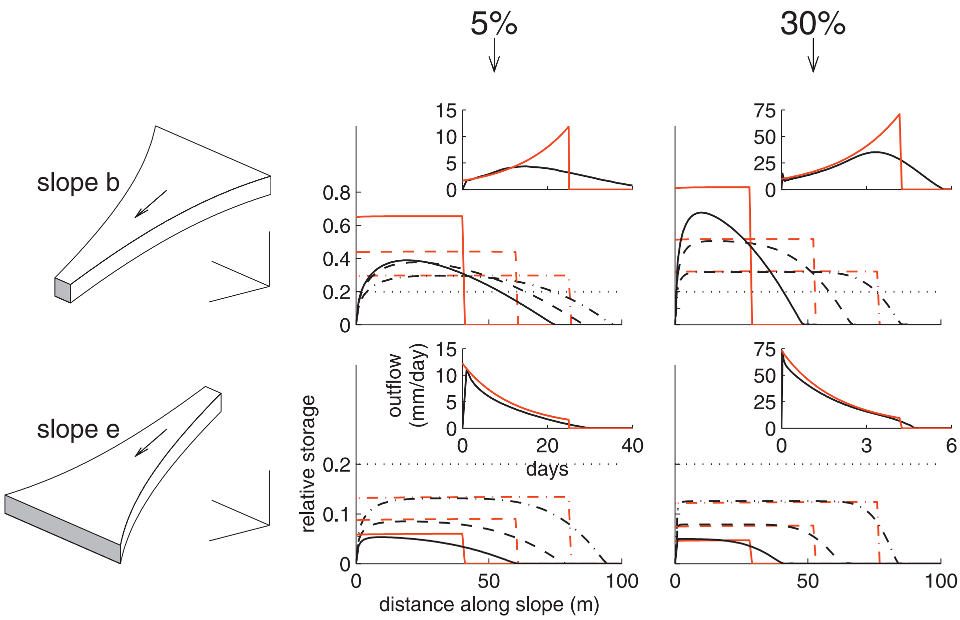

Figure 11. Relative storage profiles and normalized subsurface flow rates (mm/day) at the outlet computed by the hillslope-storage kinematic wave model of *Troch et al.* [2002] for hillslopes b and e of Figure 3 at (left) 5% and (right) 30% slope angle. The red lines are the kinematic wave model results and the black lines are the HSB results. For the relative storage profiles at 5% slope angle, dotted line is initial time, $t = 0$; dash-dotted line is $t = 5$ days; dashed line is $t = 10$ days; and solid line is $t = 15$ days. For the 30% slope angle, dotted line is initial time, $t = 0$; dash-dotted line is $t = 1$ day; dashed line is $t = 2$ days; and solid line is $t = 3$ days.