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ABSTRACT

In this report, results of a study of Halphen distributions carried out between January and March 1987 at INRS-Eau are described. They are based mainly on a general approach to probability density functions in an attempt to classify and find their links with already known types of distributions. The effort resulted in the recognition that Halphen distributions in the general context create their own system of frequency functions. However, they do not seem to be as rich in a variety of forms and types as the Pearson system and they do not fulfill some particular requirements as ease of computation and facility of algebraic manipulation.

The related systems of frequency functions, together with the Pearson system, generalize the Pearson approach of generating frequency functions by a common differential equation. But unlike the Pearson concept, the Halphen p.d.f.'s, are largely based on empirical justification. They were in fact motivated by Halphen's desire to construct his distributions so as to meet some specific requirements of fitting hydrological variables (Halphen was a hydrologist). The common generating differential equation is used as a powerful and useful tool for investigating some of the properties of the system. Of course, the principle can be used in the Pearson sense, although such an approach will have probably little support by statistical theory.

The Halphen type B distribution has also been identified as Toranzos system described in [4] in 1949 and 1952.

In this report, the motives to support the idea of empirical justification of Halphen distributions are given and some of their properties are discussed.

1. INTRODUCTORY NOTES

As a main source of information on Halphen distributions article [1] has been used. G. Morlat (1956) introduced in it three types of Halphen probability density functions originated by Etienne Halphen during World War II (1941-1948). The author gave some historical remarks on them.

The p.d.f.'s of Halphen distributions are:

$$\underline{\text{Type A}} : f(x) = \frac{1}{2m^{\nu} K_{\nu}(2\alpha)} x^{\nu-1} e^{-\alpha\left(\frac{x}{m} + \frac{m}{x}\right)} \qquad (1)$$

with $x > 0$, $m > 0$, $\alpha > 0$ and any ν. $K_{\nu}(2\alpha)$ is a modified Bessel function of the second kind as described below in this report. The parameters of the distribution are α, ν and m; m being the scale parameter.

$$\underline{\text{Type B}} : f(x) = \frac{2}{m^{2\nu} \, ef_{\nu}(\alpha)} x^{2\nu-1} e^{\left(-\left(\frac{x}{m}\right)^2 + \alpha \frac{x}{m}\right)} \qquad (2)$$

with $x > 0$, $m > 0$, $\nu > 0$ and any α. $ef_{\nu}(\alpha)$ is the function described by E. Halphen in [3], or, more generally, Weber parabolic cylinder function as in [4] p. 469 and [5], paragraph 19.5.3. The author of [1] treats this function as a Hermite function.

$$\underline{\text{Type B}^{-1}} : \quad f(x) = \frac{2}{m^{-2\nu} ef_{\nu}(\alpha)} x^{-2\nu-1} e^{\left(-\left(\frac{m}{x}\right)^2 + \alpha \frac{m}{x}\right)} \qquad (3)$$

with $x > 0$, $m > 0$, $\nu > 0$, and α and $ef_\nu(\alpha)$ are as above. This type can be deduced from (2) by substituting $\frac{x}{m} = \frac{m}{y}$, i.e. $x = m^2/y$, $dx = (-m^2/y^2)dy$. This function was introduced in 1956 by M. Larcher.

The author of [1] also presented the moments of all three types of the Halphen distribution including the logarithm of the geometrical mean (moment of order "quasi zero"). He gave some recommendations on parameter estimation, but in a rather intuitive way. The author also set up the relationship between the ratio of the arithmetic and geometric mean and the ratio of the geometric and harmonic mean. He presented an example of a numerical table of distribution function for type A and B for particular values of parameters.

In this report, the classification of Halphen distributions follows [1] and the above remarks, although it does not have a clear systematical basis and it interferes with [4].

2. GENERATING DIFFERENTIAL EQUATIONS OF HALPHEN DISTRIBUTIONS

From (1), Type A can be written in the form
$\ln f(x) = \ln k + (\nu-1)\ln x - \alpha \frac{x}{m} - \alpha \frac{m}{x}$ where k is a constant satisfying the relation

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

We have, after differentiating both sides,

$$\frac{1}{f(x)}\, df(x) = \left(\frac{\nu-1}{x} - \frac{\alpha}{m} + \frac{\alpha m}{x^2} \right) dx$$

which can be denoted as

$$\frac{1}{f}\,\frac{df}{dx} = \frac{a_o + a_1 x + a_2 x^2}{x^2}, \qquad (4)$$

where

$$a_o = \alpha m;\; a_1 = \nu - 1;\; a_2 = -\frac{\alpha}{m}.$$

Similarly from (2) - (Type B) we have

$$\frac{1}{f}\,\frac{df}{dx} = \frac{a_o + a_1 x + a_2 x^2}{x}, \qquad (5)$$

where

$$a_o = 2\nu - 1;\; a_1 = \frac{\alpha}{m};\; a_2 = -\frac{2}{m^2}.$$

and from (3) - (Type B^{-1}) we have

$$\frac{1}{f}\,\frac{df}{dx} = \frac{a_o + a_1 x + a_2 x^2}{x^3} \qquad (6)$$

where $a_o = 2m^2$; $a_1 = -\alpha m$; $a_2 = -(2\nu + 1)$.

Note, that the substitution $x = m^2/y$ deduces (6) from (5), and that (5) is identical to Toranzos system [4].

3. MOMENTS OF THE HALPHEN DISTRIBUTIONS

The study of moments and their relations is important is interesting for studying hydrological samples because the samples are usually described by their moments.

Using (4), (5), (6), it is possible to derive recurrence relations between the moments about the origin for all three types of distributions. For example, for Type A, we have immediately from (4)

$$x^2 \, df = (a_o + a_1 x + a_2 x^2) \, fdx$$

and

$$x^{r+2} \frac{df}{dx} dx = (a_o x^r + a_1 x^{r+1} + a_2 x^{r+2}) \, fdx.$$

Integrating the left-hand side by parts over the range of the distribution $(0, +\infty)$ and assuming $x^{r+2} f(x) \to 0$ as $x \to 0$ and as $x \to \infty$, we obtain

$$a_o \mu'_r + (a_1 + r + 2) \, \mu'_{r+1} + a_2 \mu'_{r+2} = 0. \qquad (7)$$

Similarly, from (5) for Type B :

$$(a_o + r + 1) \, \mu'_r + a_1 \mu'_{r+1} + a_2 \mu'_{r+2} = 0, \qquad (8)$$

and from (6) for Type B^{-1} :

$$a_o \mu'_r + a_1 \mu'_{r+1} + (a_2 + r + 3) \, \mu'_{r+2} = 0 \qquad (9)$$

For the Pearson system, the similar relation is

$$r a_o \mu'_{r-1} + [(r+1)\, a_1 - a]\, \mu'_r + [(r+2)\, a_2 + 1]\, \mu'_{r+1} = 0, \tag{10}$$

where a is the fourth parameter of the system. Later in this study, similar generalization will be performed.

Now, posing r = 0, an important and principal difference of the Halphen system from the Pearson system can be shown, as from (7) for Type A

$$a_o + (a_1+2)\, \mu'_1 + a_2 \mu'_2 = 0, \tag{11}$$

from (8) for Type B

$$(a_o+1) + a_1 \mu'_1 + a_2 \mu'_2 = 0, \tag{12}$$

from (9) for Type B^{-1}

$$a_o + a_1 \mu'_1 + (a_2 + 3)\, \mu'_2 = 0, \tag{13}$$

and for Pearson system

$$(a_1 - a) + (2a_2 + 1)\, \mu'_1 = 0. \tag{14}$$

It is clear, that while the arithmetic mean of each member of the Pearson system follows immediately from (14), no similar relation can be derived neither from (7) - (9) nor from (11) - (13) for the Halphen system.

It also follows, that relations (7) - (9) are pure recurrence relations and that one of the moments must be found by an independent relation, if the relations (7) - (9) are to be used for derivation of the noncentral moments. This fact is also the reason, why algebraic manipulation is not so easy with the Halphen distributions, because the remaining moment has usually to be computed by direct integration of the frequency function. For instance, the arithmetic mean of (4)

$$\mu_1' = \sqrt{\left(-\frac{a_0}{a_2}\right)}\ \frac{K_{a_1+2}\,(2\sqrt{-a_0a_2}\ )}{K_{a_1+1}\,(2\sqrt{-a_0a_2}\ )}, \qquad (15)$$

where K(·) is the modified Bessel function of the second kind.

To allow more general analysis with central moments, the centralisation of (4) - (6) is necessary. This is done by substituting $t = x - \bar{x}$; $x = t + \bar{x}$; $dx = dt$, where $\bar{x} = \mu_1'(x)$ is the arithmetic mean of a random variable x.

For Type A, from (4)

$$\frac{f'}{f} = \frac{a_o+a_1x+a_2x^2}{x^2} = \frac{a_o+a_1(t+\bar{x}) + a_2(t+\bar{x})^2}{(t+\bar{x})^2} = \frac{b_o+b_1t+ b_2t^2}{(t+\bar{x})^2}, \qquad (16)$$

where

$$b_o = a_o+ a_1\bar{x} + a_2\bar{x}^2,$$

$b_1 = a_1 + 2a_2\bar{x}$,

$b_2 = a_2$.

By a similar procedure as above, we have

$$r\bar{x}^2 \mu'_{r-1}(t)+(b_o+(r+1)2\bar{x})\mu'_r(t)+(b_1+r+2)\mu'_{r+1}(t)+b_2\mu'_{r+2}(t) = 0.$$

As $\mu'_1(t)=0$, $\mu'_r(t)=\mu_r(x)$, while $\mu_1(x)=0$ and $\mu_o(x)=1$, we have then

$$r\bar{x}^2 \mu_{r-1}(x)+(b_o+(r+1)2\bar{x})\ \mu_r(x)+(b_1+r+2)\ \mu_{r+1}(x)+b_2\mu_{r+2}(x)=0. \qquad (17)$$

Relation (17) allows:

- the application of the generalized method of moments, i.e. the estimation of the unknown parameters by equating four arbitrary population central moments and population mean to the four corresponding sample central moments and sample arithmetic mean

- an analysis of the skewness-kurtosis relationship.

Concerning the generalized method of moments, we obtain by putting r=r, r+1, r+2 in (17) and using matrix notation

$$\begin{bmatrix} \mu_r & \mu_{r+1} & \mu_{r+2} \\ \mu_{r+1} & \mu_{r+2} & \mu_{r+3} \\ \mu_{r+2} & \mu_{r+3} & \mu_{r+4} \end{bmatrix} \begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} r\mu_{r-1} & (r+1)\ \mu_r & (r+2)\ \mu_{r+1} \\ (r+1)\mu_r & (r+2)\ \mu_{r+1} & (r+3)\ \mu_{r+2} \\ (r+2)\mu_{r+1} & (r+3)\ \mu_{r+2} & (r+4)\ \mu_{r+3} \end{bmatrix} \begin{bmatrix} -\bar{x}^2 \\ -2\bar{x} \\ -1 \end{bmatrix} \qquad (18)$$

As the relation between parameters **b** and **a** of (16) can be expressed as

$$\begin{bmatrix} b_o \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & \bar{x} & \bar{x}^2 \\ 0 & 1 & 2\bar{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} \qquad (19)$$

The system (18) becomes

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{x} & \bar{x}^2 \\ 0 & 1 & -2\bar{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_r & \mu_{r+1} & \mu_{r+2} \\ \mu_{r+1} & \mu_{r+2} & \mu_{r+3} \\ \mu_{r+2} & \mu_{r+3} & \mu_{r+4} \end{bmatrix}^{-1} \begin{bmatrix} r\mu_{r-1} & (r+1)\mu_r & (r+2)\mu_{r+1} \\ (r+1)\mu_r & (r+2)\mu_{r+1} & (r+3)\mu_{r+2} \\ (r+2)\mu_{r+1} & (r+3)\mu_{r+2} & (r+4)\mu_{r+3} \end{bmatrix} \begin{bmatrix} -\bar{x}^2 \\ -2\bar{x} \\ -1 \end{bmatrix}$$

(20)

Putting r= 0 in (20) we have ($\mu_o(x)=1$; $\mu_1(x)=0$)

$$\begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{x} & \bar{x}^2 \\ 0 & 1 & -2\bar{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \mu_2 \\ 0 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3\mu_2 \\ 0 & 3\mu_2 & 4\mu_3 \end{bmatrix} \begin{bmatrix} -\bar{x}^2 \\ -2\bar{x} \\ -1 \end{bmatrix} \tag{21}$$

i.e. $\mathbf{a} = \mathbf{M}_1^{-1}\,\mathbf{M}_2^{-1}\,\mathbf{M}_3\,\mathbf{C}.$

The product $\mathbf{M}_2^{-1}\mathbf{M}_3$ of (21) is

$$\begin{bmatrix} \dfrac{-\dfrac{\mu_2}{\mu_3}\beta_1}{\beta} & \dfrac{\beta_1 - \dfrac{\mu_2}{\mu_3}\beta_2 + 3}{\beta} & \dfrac{\dfrac{\mu_3}{\mu_2}}{\beta} \\ \dfrac{\dfrac{1}{\mu_2}(1-\beta_2)}{\beta} & \dfrac{2\dfrac{\mu_2}{\mu_3}\beta_1}{\beta} & \dfrac{3(1+\dfrac{4}{3}\beta_1 - \beta_2)}{\beta} \\ \dfrac{\dfrac{1}{\mu_3}\beta_1}{\beta} & \dfrac{-2\dfrac{1}{\mu_2}}{\beta} & \dfrac{-\dfrac{\mu_2}{\mu_3}\beta_1}{\beta} \end{bmatrix} \tag{22}$$

where $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ is an index of skewness,

$\beta_2 = \dfrac{\mu_4}{\mu_2^2}$ is an index of kurtosis, and

$\beta = 1 + \beta_1 - \beta_2$, coming from the determinant of $\mathbf{M}_2$, as

$$|\mathbf{M}|_2 = \mu_2\mu_4 - \mu_2^3 - \mu_3^2 = \beta\,(-\mu_2^3).$$

Matrix (22) can be expressed also as $(\ \mathbf{M}_3^{-1}\ \ \mathbf{M}_2\ )^{-1}$, i.e.

$$\begin{bmatrix} \dfrac{3\mu_2^2}{2\mu_3} & \dfrac{\mu_2}{4} & \dfrac{9\mu_2^3 + 4\mu_3^2 - 3\mu_2\mu_4}{4\mu_3} \\[2ex] 1 & 0 & \mu_2 \\[2ex] -\dfrac{\mu_2}{2\mu_3} & \dfrac{1}{4} & \dfrac{-3\mu_2^2 + \mu_4}{4\mu_3} \end{bmatrix}^{-1}$$

which can be of some interest.

Concerning the skewness-kurtosis relationship, relation (17) can be rewritten, putting r=0,1,2 into

$$\begin{bmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} b_2 & 0 & 0 \\ b_1+3 & b_2 & 0 \\ b_0+6\bar{x} & b_1+4 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & b_0+2\bar{x} & b_1+2 \\ 0 & \bar{x}^2 & b_0+4\bar{x} \\ 0 & 0 & 2\bar{x}^2 \end{bmatrix} \begin{bmatrix} -\mu_{-1} \\ -1 \\ 0 \end{bmatrix} \qquad (24)$$

(where $\bar{x} = f(a_0, a_1, a_2)$)

which gives

$$\beta_1 = \frac{2\bar{x}^2(b_1+3)}{(b_0+2\bar{x})^2} - \frac{(b_1+3)^2}{b_2(b_0+2\bar{x})} - \frac{\bar{x}^4 b_2}{(b_0+2\bar{x})^3},$$

and

$$\beta_2 = \frac{b_0+6\bar{x}}{b_0+2\bar{x}} - \frac{(b_1+3)\ (b_1+4)}{b_2(b_0+2\bar{x})} + \frac{\bar{x}^2(b_1+4)}{(b_0+2\bar{x}\ )^2},$$

i.e.

$$\beta_1 = \frac{2\bar{x}^2(a_1+3+2a_2\bar{x})}{(a_0+(a_1+2)\bar{x}+a_2\bar{x}^2)^2} - \frac{(a_1+3+2a_2\bar{x})^2}{a_2(a_0+(a_1+2)\bar{x}+a_2\bar{x}^2)} - \frac{a_2\bar{x}^4}{(a_0+(a_1+2)\bar{x}+a_2\bar{x}^2)^3}$$

and

$$\beta_2 = 1+\frac{4\bar{x}}{a_0+(a_1+2)\bar{x}+a_2\bar{x}^2} - \frac{(a_1+3+2a_2\bar{x})(a_1+4+2a_2\bar{x})}{a_2(a_0+(a_1+2)\bar{x}+a_2\bar{x}^2)} + \frac{(a_1+4+2a_2\bar{x})\bar{x}^2}{(a_0+(a_1+2)\bar{x}+a_2\bar{x}^2)}$$

which is, in the case of the Halphen type A distribution,

$$\beta_1 = \frac{2\bar{x}^2(\nu+2-2\alpha\bar{x})}{((\nu+1)\bar{x}+(1-\bar{x}^2)\alpha)^2} + \frac{(\nu+2-2\alpha\bar{x})^2}{\alpha((\nu+1)\bar{x}+(1-\bar{x}^2)\alpha)^2} + \frac{\alpha\bar{x}^4}{((\nu+1)\bar{x}+(1-\bar{x}^2)\alpha)^3}$$

and (25)

$$\beta_2 = 1 + \frac{4\bar{x}}{(\nu+1)\bar{x}+(1-\bar{x}^2)\alpha} + \frac{(\nu+2-2\alpha\bar{x})(\nu+3-2\alpha\bar{x})}{\alpha((\nu+1)\bar{x}+(1-\bar{x}^2)\alpha)} + \frac{\bar{x}^2(\nu+3-2\alpha\bar{x})}{((\nu+1)\bar{x}+(1-\bar{x}^2)\alpha)^2)}$$

putting m=1, as the (β_1, β_2) relationship does not depend on the value of the parameter m.

Unfortunately, relation (25) still employ the first non-central moment which has to be expressed by an integration of the frequency function. This makes an analysis of the relationship rather difficult from an analytical point of view, because

$$\bar{x} = m\,\frac{K_{\nu+1}(2\alpha)}{K_{\nu}(2\alpha)}$$

for Type A, where K(·) is the Bessel function as mentioned above. For a particular case of ν=0,5 we have $\bar{x}$ = m, because $K_{-0,5}(\cdot) = K_{0,5}(\cdot)$ and relations (25) should give

$$\beta_2 = 3 + \frac{5}{3}\beta_1. \qquad (26)$$

Equation (26) has been obtained by an independent numerical analysis, the results of which are presented in Appendix 1.

Of course, a similar analysis can be performed for Type B and Type B^{-1} - although such an analysis is not covered in this report.

The numerical analysis of the skewness-kurtosis relationship of type A has been performed by a computer program HALFTYA on a CYBER 830 (Quebec University) according to the following algorithm:

I. arithmetic mean

$$\mu_1' = m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}$$

II. moments about the origin

$$\mu_2' = \frac{\nu+1}{\alpha} m\mu_1' + m^2$$

$$\mu_3' = \frac{\nu+2}{\alpha} m\mu_2' + m^2\mu_1'$$

$$\mu_4' = \frac{\nu+3}{\alpha} m\mu_3' + m^2\mu_2^2$$

III. central moments

$$\mu_2 = \mu_2' - \mu_1^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

IV. Index of skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

V. Index of kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2}.$$

Because β_1 and β_2 do not depend on the parameter m, m is set to one in the computer program. The program and numerical results are presented in Appendix 2.

4. GENERALIZATION OF HALPHEN DISTRIBUTIONS AND RELATED SYSTEMS OF FREQUENCY FUNCTIONS.

The generating differential equation of type (16) for Type A Halphen distribution may be considered also from a slightly different standpoint. Taking $\bar{x}$ as a fourth independent parameter and t as a new variable, equation (16) becomes

$$\frac{f'}{f} = \frac{b_0+b_1t+b_2t^2}{(t+a)^2}, \tag{27}$$

where b_0, b_1, b_2 and a are four parameters of the distribution of the random variable t, parameter a being the location parameter. The range of t is $(-a, +\infty)$, and the parameter a is a lower (upper) boundary of the distribution. By centralisation and using a similar procedure as above, or by substituing $\bar{x}$ in (17) by $\bar{t}+a$, $(\bar{t}=\bar{x}-a)$, we have

$$r(\bar{t}+a)^2\mu'_{r-1}(Z)+(c_0+(r+1)2(\bar{t}+a))\mu'_r(Z)+(c_1+(r+2))\mu'_{r+1}(Z)+c_2\mu'_{r+2}(Z)=0,$$

or

$$r(\bar{t}+a)^2\mu_{r-1}(t)+(c_0+(r+1)2(\bar{t}+a))\mu_r(t)+(c_1+(r+2))\mu_{r+1}(t)+c_2\mu_{r+2}(t)=0, \tag{28}$$

where

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & \bar{t} & \bar{t}^2 \\ 0 & 1 & 2\bar{t} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix},$$

and $Z=t-\bar{t}$ is a centralized random variable with $\mu'_1(Z)=0$ and $\mu'_r(Z)=\mu_r(t)$, while $\mu_1(t)=0$ and $\mu_0(t)=1$ as in (17), and

$$\bar{t} = \mu'_1(t)= \sqrt{\left(-\frac{b_0}{b_2}\right)}\ \frac{K_{b_1+2}\ (2\sqrt{-b_0b_2})}{K_{b_1+1}\ (2\sqrt{-b_0b_2})} - a. \tag{29}$$

The recurrence relation for moments about origin for (27) is

$$ra^2\mu'_{r-1} +(b_0+(r+1)2a)\mu'_r+(b_1+r+2)\mu'_{r+1}+b_2\mu'_{r+2}= 0, \tag{30}$$

which can also be deduced from (28) by letting $\bar{t} = 0$ (formally). Comparison of (30) with (10) gives an interesting view on differences from the Pearson system, especially the employment of one additional moment in (30) as compared to the Pearson system.

Summarising, we have the following generating differential equations

$$\underline{\text{Type A}}: \quad \frac{f'}{f} = \frac{b_0+b_1t+b_2t^2}{(t+a)^2}, \qquad \underline{\text{Type B}}: \quad \frac{f'}{f} = \frac{b_0+b_1t+b_2t^2}{t+a}, \qquad (31)$$

$$\underline{\text{Type B}}^{-1}: \quad \frac{f'}{f} = \frac{b_0+b_1t+b_2t^2}{(t+a)^3}, \qquad \underline{\text{Pearson System}}: \quad \frac{f'}{f} = \frac{t+a}{b_0+b_1t+b_2t^2}$$

Now, we could, as is done in the Pearson classification, search for all possible distributions generated by (31). Some of them could be interesting, some of them can be redundant, as for instance, Type B-III, which corresponds to the case $b_2=0$ ($b_1\neq0$) of Pearson classification, is (in the case a=0).

$$\frac{f'}{f} = \frac{b_0+b_1t}{t}, \text{ i.e. } f(t)=C.t^{b_0}.e^{b_1t}, \text{ which is a gamma distribution.}$$

Type A-III corresponds to

$$f(t)= C.t^{b_1}e^{-b_0\frac{1}{t}}, \quad (a=0), \text{ and}$$

Type B^{-1}-III corresponds to

$$f(t)=C.e^{-(b_0\frac{1}{2t^2} + b_1\frac{1}{t})}, \quad (a=0)$$

Note, that assigning particular values to the parameters b_0, b_1, b_2 according to the solution of equation $b_0+b_1t+b_2t^2=0$ is not entirely possible for distributions (4), (5), (6), because they have certain restrictions to their parameters values, as

TYPE OF HALPHEN DISTRIBUTIONS	RESTRICTIONS ON PARAMETERS VALUES	
	VALUES OF a_i	PARAMETERS
TYPE A	$a_0 = \alpha m$ $a_1 = \nu - 1$ $a_2 = -\frac{\alpha}{m}$	$\alpha = \sqrt{-a_0\ a_2}$ $\nu = a_1 + 1$ with $a_2 \neq 0$ $m = \sqrt{\frac{-a_0}{a_2}}$
TYPE B	$a_0 = -2\nu - 1$ $a_1 = \alpha/m$ $a_2 = \frac{-2}{m^2}$	$\nu = \frac{a_0 + 1}{2}$ $\alpha = a_1 \sqrt{\frac{-2}{a_2}}$ with $a_2 \neq 0$ $m = \sqrt{\frac{-2}{a_2}}$
TYPE B^{-1}	$a_0 = 2m^2$ $a_1 = -\alpha m$ $a_2 = -2\nu-1$	$m = \sqrt{a_0/2}$ $\alpha = -a_1 \sqrt{2/a_0}$ with $a_0 \neq 0$ $\nu = \frac{a_2 + 1}{2}$

The complete exploration of systems (31) has not been done within the framework of this study, but such exploration is felt as being an important part of possible future studies.

Looking at complex (31), we can easily imagine systems similar to the Types A, B, B^{-1}, but derived from the Pearson system. They could be of the form

$$\frac{f'}{f} = \frac{(t+a)^2}{b_0+b_1t+b_2t^2} \quad \text{and} \quad \frac{f'}{f} = \frac{(t+a)^3}{b_0+b_1t+b_2t^2},$$

the redundancy of which might be significant.

The common concept of the right-hand sides of (31) as a ratio of two different polynomials of a specific degree as suggested in [4] does not seem to be quite appropriate, because too many parameters are involved in such ratios. More practical concept would be

$$\frac{f'}{f} = \frac{BN(x)}{Qm(x)}, \text{ and } \frac{f'}{f} = \frac{Qm(x)}{BN(x)},$$

where Qm(x) is polynomial of degree m, and binomial B(x) = a+x. The latter concept has two advantages. One of having (m+2) parameters regardless and the value of the exponent n, and second of having an explicit location parameter a.

Although the above considerations are quite formal from the point of view of statistical theory, a fundamental question arises, <u>why</u> do so many distributions originated by empirical knowledge have the generating differential equation of the form (31). The answer that comes first to mind, seems to be connected to the principle of sampling distributions. This explanation gives much broader meaning to the fact that, for example, the (sampling) distribution of the sum of squares of n independent variables, each of which being normally distributed with zero mean and unit variance gives a Pearson type III distribution ([7] chap. 11) - a member of the system (31).

Analogical principle may very well correspond to the nature of "natural" random variables as dealt with in hydrology, as they may be, and they actually are, considered as a combination (convolution) of a number of individual random variables.To prove such as idea would mean to demonstrate, that at least some specific combinations of random variables tend to have generating differential equations of the form (31).

Carrying out the division of the polynomials of (31) as is done in [4] (where it is possible - i.e. for type A and B), gives another view on the origin of Type A and Type B distributions. The Type A p.d.f. can be considered as a product of an exponential distribution and a Pearson type V distribution. The origin of Type B p.d.f. can be considered as a product of Gaussian function and Pearson Type III distribution.

Similar, but a little different view on Type A and Type B Halphen distributions shows Type A as a product of a gamma distributions and a function $e^{-\alpha/t}$, and Type B as a product of a chi-square distribution and a function $e^{\alpha t}$ ($t = \frac{x}{m}$ in (1), (2)). This view gives some support to the idea of empirical justification of Halphen distributions, for, as we can see in the sketch in Appendix 3, the Type A inherently lowers the lower tail of the gamma distribution, while Type B raises the upper tail of chi-square distribution. This "behaviour" can be of some practical interest, as smart practitioners do something of this nature, but by simpler means, for instance by summing up the two different distributions (composition).

The relationship of the parameters and moments of (27) - for example, can be studied by an investigation of (28). It gives for four successive values of r=0,1,2,3 a system of four equations:

$$c_0+2(\bar{t}+a)+c_2\mu_2=0$$
$$(\bar{t}+a)^2+(c_1+3)\mu_2+c_2\mu_3=0$$
$$(c_0+6(\bar{t}+a))\mu_2+(c_1+4)\mu_3+c_2\mu_4=0 \qquad (32)$$
$$3(\bar{t}+a)^2\mu_2+(c_0+8(\bar{t}+a))\mu_3+(c_1+5)\mu_4+c_2\mu_5=0.$$

For comparison purposes, let us introduce analogical equations of the Pearson System. They are

$$c_1+\bar{t}+a=0$$
$$c_0+(3c_2+1)\mu_2=0$$
$$(3c_1+\bar{t}+a)\mu_2+(4c_2+1)\mu_3=0 \qquad (33)$$
$$3c_0\mu_2+(4c_1+\bar{t}+a)\mu_3+(5c_2+1)\mu_4=0$$

Now, we can adjust both systems by letting $\bar{t}=0$ (i.e. by putting the origin at the mean - note that this is not always possible in (32), as if $a=0$, the range of $t \in (0,\infty)$, which implies $\bar{t}>0$. It follows, that if $\bar{t}=0$, then $a<0$. The reason of that lies in the fact, that Halphen types in (31) have $t+a \neq 0$. After above adjusting,

$$b_0+2a+b_2\mu_2=0$$
$$a^2+(b_1+3)\mu_2+b_2\mu_3=0 \qquad \text{(Halphen System)} \qquad (34)$$
$$(b_0+6a)\mu_2+(b_1+4)\mu_3+b_2\mu_4=0$$

$$3a^{2}\mu_2+(b_0+8a)\mu_3+(b_1+5)\mu_4+b_2\mu_5=0, \text{ and}$$

$$b_1+a=0$$

$$b_0+(3b_2+1)\mu_2=0 \qquad \text{(Pearson System)} \qquad (35)$$

$$(3b_1+a)\mu_2+(4b_2+1)\mu_3=0$$

$$3b_0\mu_2+(4b_1+a)\mu_3+(5b_2+1)\mu_4=0$$

generally,

$$a = f_1(\mu_2,\mu_3,\mu_4,\mu_5),$$

$$b_0= f_2(\mu_2,\mu_3,\mu_4,\mu_5),$$

$$b_1= f_3(\mu_2,\mu_3,\mu_4,\mu_5),$$

$$b_2= f_4(\mu_2,\mu_3,\mu_4,\mu_5), \text{ for system (34), and}$$

$$a = g_1(\mu_2,\mu_3,\mu_4) = -b_1$$

$$b_0= g_2(\mu_2,\mu_3,\mu_4) = -\mu_2(4\mu_2\mu_4-3\mu_3^2)/D$$

$$b_1= g_3(\mu_2,\mu_3,\mu_4) = -\mu_3(3\mu_2^2+\mu_4)/D$$

$$b_2= g_4(\mu_2,\mu_3,\mu_4) = (6\mu_2^3+3\mu_3^2-2\mu_2\mu_4)/D, \text{ where}$$

$$D = 10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2 , \text{ for Pearson system (35).}$$

Now, it is quite clear, that while the Pearson (four parameters) systems can be determined by the three central moments μ_2,μ_3,μ_4, or by two moment-ratios (shape factors) - index of skewness $\beta_1 = \mu_3^2/\mu_2^3$ and index of kurtosis $\beta_2 = \mu_4/\mu_2^2$, the generalised -four parameters Halphen type A distribution needs for the same purpose four central moments, or three moment - ratios derived from them, i.e. β_1, β_2, $\beta_3 = \mu_5^2/\mu_2^5$, which holds probably also for generalised type B and B^{-1}. It means, that (β_1, β_2) chart relating the types of Pearson frequency curves is insufficient for generalised Halphen (four parameter) distributions and has to be extended by one more dimension (β_3), which also means, that the (β_1, β_2) plane might be covered entirely by the Halphen (generalised) distributions.

5. SOME PROPERTIES OF HALPHEN DISTRIBUTIONS

Results of this chapter are related to the forms (1), (2), (3).

5.1 CHARACTERISTIC FUNCTION

There are several ways to determine the characteristic function. One of them uses a Laplace transform as shown in [4] for Type B. For example, for Type A Halphen distribution it can be derived as follows.

Taking (4) as a reference form, we have

$$f(x) = k.x^{a_1} e^{a_2 x - a_0 \frac{1}{x}}, \text{ (k is an adjusting constant)}$$

and the characteristic function is defined as

$$\Phi(t) = E[e^{itx}] = k\int_0^\infty x^{a_1} e^{a_2 x - a_0\frac{1}{x} + itx}\, dX =$$

$$= k\int_0^\infty x^{a_1} e^{-a_0\frac{1}{x} + (a_2+it)x}\, dx = k\, L_{-(a_2+it)}\{x^{a_1} e^{-a_0\frac{1}{x}}\},$$

using $L_{-(a_2+it)}$ for the Laplace transform of linearly transformed complex variable. The only reference of such Laplace transform in [5] has been found in paragraph 29.3.120, which gives for specific values $a_1=-1$ (i.e. $\nu=0$: Halphen's "harmonic law"), $a_0 = c^2$, and $x = 2y$, the characteristic function $k = K_0(2\sqrt{(-a_0a_2-a_0it)})$, where $K_0(\cdot)$ is a modified Bessel function of zero order.

Another interesting relation can be derived from generating differential equations (4), (5), (6). As

$$\Phi(t) = \int_{-\infty}^{\infty} e^{\Theta x} f(x)dx,$$

$$\frac{d\Phi(t)}{d\Theta} = \int_{-\infty}^{\infty} xe^{\Theta x} f(x)dx,$$

$$\frac{d^2\Phi(t)}{d\Theta^2} = \int_{-\infty}^{\infty} x^2 e^{\Theta x} f(x)dx,$$

where $\Theta = it$, then after multiplying both sides of (4) by $e^{\Theta x}$ for Type A,

$$x^2 e^{\Theta x} f' = a_0 e^{\Theta x} f + a_1 x e^{\Theta x} f + a_2 x^2 e^{\Theta x} f,$$

and after integration over the domain of x (the left-hand side by parts), we have

$$(a_2+\Theta)\frac{d^2\Phi(t)}{d\Theta^2} + (a_1+2)\frac{d\Phi(t)}{d\Theta} + a_0\Phi(t) = 0 \qquad (36)$$

As for the cumulant generating function, we can write

$$\psi(t)= \ln\Phi(t),$$

$$\frac{d\psi(t)}{d\Theta} = \frac{1}{\Phi(t)}\frac{d\Phi(t)}{d\Theta},$$

$$\frac{d^2\psi(t)}{d\Theta^2} = -\frac{1}{\Phi^2(t)}\left(\frac{d\Phi(t)}{d\Theta}\right)^2 + \frac{1}{\Phi(t)}\frac{d^2\Phi(t)}{d\Theta^2},$$

and it follows from the above relations, that

$$\frac{1}{\Phi(t)}\frac{d^2\Phi(t)}{d\Theta^2} = \frac{d^2\psi(t)}{d\Theta^2} + \left(\frac{d\psi(t)}{d\Theta}\right)^2 .$$

Therefore, after dividing (36) by $\Phi(t)$, we have, in terms of the cumulant generating function, a relation

$$(a_2+\Theta)\left\{\frac{d^2\psi(t)}{d\Theta^2} + \left(\frac{d\psi(t)}{d\Theta}\right)^2\right\} + (a_1+2)\frac{d\psi(t)}{d\Theta} + a_0 = 0. \qquad (37)$$

Analogical relations for Type B Halphen distribution are

$$a_2\frac{d^2\Phi}{d\Theta^2} + (a_1+\Theta)\frac{d\Phi}{d\Theta} + (a_0+1)\Phi = 0, \text{ and} \qquad (38)$$

$$a_2 \left\{\frac{d^2\psi}{d\Theta^2} + \left(\frac{d\psi}{d\Theta}\right)^2\right\} + (a_1+\Theta)\frac{d\psi}{d\Theta} + a_0 + 1 = 0. \qquad (39)$$

For Type B^{-1}, we have

$$\Theta\frac{d^3\Phi}{d\Theta^3} + (a_2+3)\frac{d^2\Phi}{d\Theta^2} + a_1\frac{d\Phi}{d\Theta} + a_0\Phi = 0, \text{ and} \qquad (40)$$

$$\Theta\left\{\frac{d^3\psi}{d\Theta^3} + 3\frac{d\psi}{d\Theta}\cdot\frac{d^2\psi}{d\Theta^2} + \left(\frac{d\psi}{d\Theta}\right)^3\right\} + (a_2+3)\left\{\frac{d^2\psi}{d\Theta^2} + \left(\frac{d\psi}{d\Theta}\right)^2\right\} + a_1\frac{d\psi}{d\Theta} + a_0 = 0, \qquad (41)$$

as

$$\frac{1}{\Phi}\frac{d^3\Phi}{d\Theta^3} = \frac{d^3\psi}{d\Theta^3} + 3\frac{d\psi}{d\Theta}\frac{d^2\psi}{d\Theta^2} + \left(\frac{d\psi}{d\Theta}\right)^3$$

The Pearson distributions have

$$a_2\Theta\frac{d^2\Phi}{d\Theta^2} + (1+2a_2+a_1\Theta)\frac{d\Phi}{d\Theta} + (a+a_1+a_0\Theta)\Phi = 0, \qquad (42)$$

and

$$a_2\Theta\left\{\frac{d^2\psi}{d\Theta^2} + \left(\frac{d\psi}{d\Theta}\right)^2\right\} + (1+2a_2+a_1\Theta)\frac{d\psi}{d\Theta} + (a+a_1+a_0\Theta) = 0. \qquad (43)$$

Before proceeding with the derivation of a series expansion of the characteristic function for Type A by direct computation of $E[e^{itx}]$, let us derive the constant adjusting the p.d.f. to have $\mu_0' = 1$.

Inspecting (1), we have

$$f(x) = k\, x^{\nu-1} e^{-\alpha(\frac{x}{m} + \frac{m}{x})}, \quad x > 0,$$

$$\lim_{x\to 0+} f(x) = 0, \ \alpha > 0, \ m > 0,$$

$$\lim_{x\to\infty} f(x) = 0, \ \alpha > 0, \ m > 0, \text{ and}$$

$$k = \left(\int_0^\infty x^{\nu-1} e^{-\alpha(\frac{x}{m} + \frac{m}{x})} dx\right)^{-1},$$

The value of the integral $\int_0^\infty x^{\nu-1} e^{-\alpha(\frac{x}{m} + \frac{m}{x})} dx$ is

Substituing $\frac{x}{m} = e^t$; $t = \ln \frac{x}{m} \rightarrow dx = me^t dt$

$$= \int_{-\infty}^{\infty} m^{\nu-1} e^{(\nu-1)t} e^{-\alpha(e^t+e^{-t})} me^t dt = m^\nu \int_{-\infty}^{\infty} e^{\nu t-2\alpha\cosh t} dt$$

$$= m^\nu \left(\int_{-\infty}^{0} e^{\nu t-2\alpha\cosh t} dt + \int_0^\infty e^{\nu t-2\alpha\cosh t} dt\right)$$

$$= m^\nu \left(\int_0^\infty e^{-\nu t-2\alpha\cosh t} dt + \int_0^\infty e^{\nu t-2\alpha\cosh t} dt\right)$$

$$= m^\nu \left(\frac{1}{\sin(\nu\pi)}\right) \int_0^\pi e^{2\alpha\cos\Theta} \cos(\nu\Theta)d\Theta - \frac{\pi}{\sin(\nu\pi)} I_\nu\,(2\alpha) +$$

$$+ \frac{1}{\sin(-\nu\pi)} \int_0^\pi e^{2\alpha\cos\Theta} \cos(\nu\Theta) d\Theta - \frac{\pi}{\sin(-\nu\pi)} I_{-\nu}(2\alpha))$$

$$= m^\nu \frac{\pi}{\sin(\nu\pi)} \quad (I_{-\nu}(2\alpha) - I_\nu(2\alpha)) = m^\nu 2K_\nu(2\alpha),$$

where I,K are modified Bessel functions (see paragraphs 9.6.20, 9.6.2 of [5]), which confirms (1).

Sufficient treatment of above derivation for Type B is presented in [4], while Type B^{-1} can be always deduced from Type B by substitution shown in (3).

The relation

$$\int_{-\infty}^{0} e^{\nu t - 2\alpha\cosh t} dt = \int_{0}^{\infty} e^{-\nu t - 2\alpha \cosh t} dt,$$

which has been used above, can be demonstrated by substitution z=-t; dt=-dz.

The characteristic function of Type A expressed in a series expansion is:

$$\Phi(t) = E[e^{itx}] = k \int_0^\infty e^{itx} x^{\nu-1} e^{-\alpha(\frac{x}{m} + \frac{m}{x})} dx$$

$$= k(\int_0^\infty [1 + \frac{itx}{1!} + \frac{(itx)}{2!} + \ldots + \frac{(itx)^r}{r!} + \ldots] x^{\nu-1} e^{-\alpha(\frac{x}{m} + \frac{m}{x})} dx)$$

$$= 1 + \frac{it}{1!} m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)} + \frac{(it)^2}{2!} m^2 \frac{K_{\nu+2}(2\alpha)}{K_\nu(2\alpha)} + \ldots + \frac{(it)}{r!} m^r \frac{K_{\nu+r}(2\alpha)}{K_\nu(2\alpha)} + \ldots. \tag{44}$$

5.2 EXISTENCE OF MOMENTS ABOUT THE ORIGIN, MOMENT OF ORDER "QUASI ZERO"- TYPE A HALPHEN DISTRIBUTION.

The rth moment about origin

$$\mu_r' = m^r \frac{K_{\nu+r}(2\alpha)}{K_\nu(2\alpha)} \tag{45}$$

follows from (44). It is also evident from (44), that μ_r' exists for all $r \geq 0$. Existence of moments about the origin for $r < 0$ can be demonstrated by

$$\mu_r' = E[x^r] = k \int_0^\infty x^{\nu-1+r} e^{-\alpha(\frac{x}{m} + \frac{m}{x})} dx.$$

Since this integral exists for $\alpha > 0$, $m > 0$, regardless of the value of the exponent at x, the moments about the origin also exist for all $r < 0$, and have the form (45).

Using (45), the validity of the recurrence relation (7) can be also demonstrated. Since

$$\frac{K_{\nu+r+1}(2\alpha)}{K_\nu(2\alpha)} = \frac{\nu+r}{\alpha} \frac{K_{\nu+r}(2\alpha)}{K_\nu(2\alpha)} + \frac{K_{\nu+r-1}(2\alpha)}{K_\nu(2\alpha)}$$

(consult paragraph 9.6.26 in [5]),

then

$$\mu'_{r+1} = \frac{\nu+r}{\alpha} m \mu'_r + m^2\mu'_{r-1}$$

while (7) gives, after substituting $a_o = \alpha m$, $a_1 = \nu-1$, $a_2 = -\alpha/m$,

$$\mu'_{r+2} = \frac{v+r+1}{\alpha} m \mu'_{r+1} + m^2 \mu_r ,$$

i.e. the above formula.

Later, we will use also the moment of order "quasi zero", i.e. logarithm of the geometric mean, which is

$$\ln G = E(\ln x) = E(\lim_{r\to 0} \frac{x^r-1}{r}) = \lim_{r\to 0} E(\frac{x^r-1}{r}) = \lim_{r\to 0} \frac{E(x^r)-1}{r}$$

$$= \lim_{r\to 0} \frac{m^r \frac{K_{\nu+r}(2\alpha)}{K_\nu(2\alpha)} - 1}{r} = \lim_{r\to 0} (\frac{m^r}{r} \cdot \frac{K_{\nu+r}}{K_\nu} - \frac{K_\nu}{rK_\nu})$$

$$= \lim_{r\to 0} \{(\frac{m^r}{r} - \frac{1}{r} + \frac{1}{r}) \frac{K_{\nu+r}}{K_\nu} - \frac{K_\nu}{rK_\nu}\} = \lim_{r\to 0} \{ \frac{m^r-1}{r} \cdot \frac{K_{\nu+r}}{K_\nu} + \frac{K_{\nu+r}}{rK_\nu} - \frac{K_\nu}{rK_v}\}$$

$$= \lim_{r\to 0} \frac{m^r-1}{r} \frac{K_{\nu+r}}{K_\nu} + \frac{1}{K_\nu} \lim_{r\to 0} \frac{K_{\nu+r}-K_\nu}{r} = \ln m + \frac{1}{K_\nu} \frac{\partial}{\partial_\nu} K_\nu \tag{46}$$

using the partial differentiation formula defined as

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x,y,\dots) - f(x,y,\dots)}{\Delta x} = \frac{\partial}{\partial x} f(x,y,\dots)$$

(note an error in p. 33 of [1] - formula of m_0)

5.3 Limits, extremes, and inflection points of Halphen frequency functions

Summarising limits of all three types, we have

Type A : from (1): $x > 0$, $m > 0$

$$\lim_{x\to 0+} f(x) = 0, \quad \alpha > 0,\ m > 0$$

$$\lim_{x\to \infty} f(x) = 0, \quad \alpha > 0,\ m > 0$$

Type B : from (2): $x > 0$, $m > 0$, $\nu > 0$

$$\lim_{x\to \infty} f(x)=0, \qquad \lim_{x\to 0+} f(x)= \begin{cases} 0 & \text{if } \nu > \tfrac{1}{2} \\ 1 & \text{if } \nu = \tfrac{1}{2} \\ +\infty & \text{if } \nu < \tfrac{1}{2}.\ (\text{and } \nu > 0). \end{cases}$$

Type B^{-1} :from (3): $x > 0$, $m > 0$, $\nu > 0$

$$\lim_{x\to 0+} f(x) = 0, \qquad \lim_{x\to \infty} f(x)= \begin{cases} 0 & \text{if } \nu > -\tfrac{1}{2}\ (\text{i.e if } \nu > 0) \\ 1 & \text{if } \nu = -\tfrac{1}{2}\ (\text{i.e. never, as } \nu > 0) \\ \infty & \text{if } \nu < -\tfrac{1}{2}\ (\text{i.e. never, as } \nu > 0) \end{cases}$$

Putting the first derivatives of frequency functions in (4), (5), (6) to zero, we obtain a condition the mode

$$a_o + a_1 x + a_2 x^2 = 0,$$

$$\text{i.e. } x_{1,2} = -\frac{a_1}{2a_2} \pm \sqrt{\left(\frac{a_1}{2a_2}\right)^2 - \frac{a_0}{a_2}}. \qquad (47)$$

Assigning $c = \frac{a_1}{2a_2}$ and $a = \frac{a_0}{a_2}$, we have

$$x_{1,2} = -c \pm \sqrt{c^2 - a}\ , \text{ with } (c^2-a) > 0.$$

The optional sign in (47) is chosen in such a way, that the condition $x > 0$ is satisfied. There are four possibilities:

	c	a	optional sign in (47)	note
I.	> 0	$> 0 (c^2 > a)$	no possibility	no mode
II.	> 0	< 0	+	unimodal
III.	< 0	$> 0 (c^2 > a)$	+ and −	bimodal (mode and antimode)
IV.	< 0	< 0	+	unimodal

(48)

If the conditions listed in (48) are not satisfied, especially the condition $c^2 > a$, the particular frequency function does not have any mode.

Type A Halphen distribution have $c = -m \frac{\nu-1}{2\alpha}$ and $a = m^2$ (see (4), (5), (6)); [note that $c^2 > a$ always satisfies conditions II and IV of (48)]. The frequency curve is therefore exclusively unimodal with

$$x_M = m\left(\frac{\nu-1}{2\alpha} + \sqrt{\left(\frac{\nu-1}{2\alpha}\right)^2 + 1}\right) \qquad (49)$$

Type B with $c = -m\alpha/4$ and $a = -m^2(\nu-\frac{1}{2})$ can satisfy conditions I to IV of (48) and has therefore all possibilities of (48) with

$$x_M = m\left(\frac{\alpha}{4} \pm \sqrt{\left(\frac{\alpha}{4}\right)^2 + \nu - \frac{1}{2}}\right). \qquad (50)$$

Type B^{-1} with $c = m\dfrac{\alpha}{4(\nu+\frac{1}{2})}$ and $a = -m^2\dfrac{1}{\nu+\frac{1}{2}}$ $(\nu > 0)$

satisfies conditions II and IV, which means that it is unimodal with

$$x_M = m\left(-\frac{\alpha}{4(\nu+\frac{1}{2})} + \sqrt{\left(\frac{\alpha}{4(\nu+\frac{1}{2})}\right)^2 + \frac{1}{\nu+\frac{1}{2}}}\right) \qquad (51)$$

note, that the parameter m in (49), (50), (51) works as a scale parameter.

Inflection points can be computed from the following quartic equations.

Type A: $x^2 g' - 2xg + g^2 = 0,$

Type B: $xg' - g + g^2 = 0,$ (52)

Type B^{-1} $x^3 g' - 3x^2 g + g^2 = 0,$

where $g = a_0 + a_1 x + a_2 x^2$ and $g' = a_1 + 2a_2 x$.

5.4 SUFFICIENT STATISTICS AND PARAMETER ESTIMATES

As all of the three types of Halphen distributions belong to the exponential family of distributions, the necessary and sufficient condition for the distribution to possess sufficient statistics

$$f(x) = \exp\left\{ \sum_{j=1}^{\ell} A_j(\Theta)\, B_j(x) + C(x) + D(\Theta) \right\}$$

(see that eq. 17.86 of (7)) is satisfied. Indeed, it is possible to write

Type A : $f(x) = \exp\{-\ln(m^{\nu} 2K_{\nu}(2\alpha)) - \frac{\alpha}{m} x - \alpha m \frac{1}{x} + (\nu-1)\ln x\}$,

Type B : $f(x) = \exp\{\ln(2/m^{2\nu}\, ef_{\nu}(\alpha)) - \frac{1}{m^2} x^2 + \frac{\alpha}{m} x + (2\nu-1)\ln x\}$,

Type B^{-1}: $f(x) = \exp\{\ln(2/m^{-2\nu}\, ef_{\nu}(\alpha)) - m^2 \frac{1}{x^2} + \alpha m \frac{1}{x} - (2\nu+1)\ln x\}$

Compare to section 3.3 of [1].

The likelihood function of Type A, for instance, is:

$$L(x|\Theta) = \exp\{-n\ln(m^{\nu} 2K_{\nu}(2\alpha)) - \frac{\alpha}{m} \sum_{i=1}^{n} x_i - \alpha m \sum_{i=1}^{n} \frac{1}{x_i} + (\nu-1) \sum_{i=1}^{n} \ln x_i\}, \quad (53)$$

the sufficient statistics being $\sum x_i$, $\sum \frac{1}{x_i}$, $\sum \ln x_i$.

Analogical considerations give $\sum x_i^2$, $\sum x_i$, $\sum \ln x_i$ as sufficient statistics for Type B, and $\sum \frac{1}{x_i^2}$, $\sum \frac{1}{x_i}$, $\sum \ln x_i$ as sufficient statistics for Type B^{-1}.

Maximum likelihood estimation of Type A distribution parameters follows from (53)

$$-m \frac{\nu}{\alpha} + \frac{1}{n} \sum x_i - m^2 \frac{1}{n} \sum \frac{1}{x_i} = 0$$

$$-\ln m - \frac{\frac{\partial}{\partial \nu} K_\nu(2\alpha)}{K_\nu (2\alpha)} + \frac{1}{n} \sum \ln x_i = 0 \tag{54}$$

$$m \frac{\frac{\partial}{\partial \alpha} K_\nu (2\alpha)}{K_\nu(2\alpha)} + \frac{1}{n} \sum x_i + m^2 \frac{1}{n} \sum \frac{1}{x_i} = 0$$

The derivation of the Bessel function with respect to its argument can be expressed as

$$\frac{\frac{\partial}{\partial \alpha} K_\nu(2\alpha)}{K_\nu(2\alpha)} = \frac{\nu}{\alpha} - 2 \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}$$

(compare with 9.6.26 of [5]), so the third equation of (54) becomes

$$m \frac{\nu}{\alpha} - 2m \frac{K_{\nu+1} (2\alpha)}{K_\nu (2\alpha)} + \frac{1}{n} \sum x_i + m^2 \frac{1}{n} \sum \frac{1}{x_i} = 0.$$

Equating the three functional moments of order -1, quasi zero, and 1, i.e. harmonic mean, logarithm of geometric mean and arithmetic mean for Type A to those of the sample:

$$\frac{1}{n} \sum \frac{1}{x_i} = \frac{1}{m} \frac{K_{\nu-1}(2\alpha)}{K_\nu(2\alpha)}$$

$$\frac{1}{n} \sum \ln x_i = \frac{\frac{\partial}{\partial\nu} K_\nu(2\alpha)}{K_\nu} + \ln m \tag{55}$$

$$\frac{1}{n} \sum x_i = m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}$$

The system (55) can be shown to be equivalent to maximum likelihood estimation (54), because we are employing sufficient statistics. This can be demonstrated on the first equation of (54), which is, in fact,

$$\mu_1' = \frac{\nu}{\alpha} m + m^2 \mu_{-1}',$$

or on the last equation of (54), which gives, using the latter one,

$$\mu_1' = -m \frac{\nu}{\alpha} - m^2 \mu_{-1}' + 2m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}, \text{ i.e.}$$

$$2\mu_1' = 2m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}.$$

The equivalency of the middle equations of (54) and (55) is evident. Having sufficient statistics, it is clear, that estimators (55) are at the same time minimum variance estimators of the population harmonic mean, logarithm of geometric mean and arithmetic mean. This can be demonstrated by showing that the condition for minimum variance estimators is satisfied. This condition is (c.f. eq. 17.27 of [7])

$$\frac{\partial \ln L}{\partial \Theta} = A(\Theta)\ \{\ t-\tau\ (\Theta)\},\ \text{where}$$

L is a likelihood function, Θ is a population parameter, $A(\Theta)$ is a function independent of the observations, and t is the minimum variance bound (MVB) estimator of the function $\tau(\Theta)$.

Thus (Type A- using (53))

$$\frac{\partial \ln L}{\partial m} = n\alpha \frac{1}{m^2} \{\ \mu_1' - m^2 \mu_{-1}' - m \frac{\nu}{\alpha}\}, \tag{56}$$

$$\frac{\partial \ln L}{\partial \alpha} = -n \frac{1}{m} \{\ \mu_1' + m^2 \mu_{-1}' + m \frac{\nu}{\alpha} - 2m \frac{K_{\nu+1}(2\alpha)}{K_\nu(2\alpha)}\ \}, \tag{57}$$

$$\frac{\partial \ln L}{\partial \nu} = n\ \{\mu_0' - \ln m - \frac{\frac{\partial}{\partial \nu} K_\nu\ (2\alpha)}{K_\nu(2\alpha)}\} \tag{58}$$

If we now equate μ_1', μ_{-1}' to sample moments, we obtain (54) as $\partial \ln L/\partial \Theta$ becomes equal to zero.

Hence (55) and therefore $\mu_1' = \frac{\nu}{\alpha} m + m^2 \mu_{-1}'$, and from (57)

$$\frac{\partial \ln L}{\partial \alpha} = -n \frac{1}{m} \{ \mu_1' - \frac{mK_{\nu+1}(2\alpha)}{K_\nu(2\alpha)} \} \qquad (59)$$

i.e. μ_1' is a MVB estimator of population arithmetic mean.

In similar way, if we substitute μ'_1 in (56) by $m\, K_{\nu+1}(2\alpha)/\, K_\nu(2\alpha)$, we obtain:

$$\frac{\partial \ln L}{\partial m} = -n\alpha \{\mu_{-1}' - \frac{1}{m} \frac{K_{\nu-1}(2\alpha)}{K_\nu(2\alpha)}\}, \qquad (60)$$

i.e. μ_{-1}' is a MVB estimator of population harmonic mean.

It follows directly from (58), that $\underline{\mu_0'}$ is a MVB estimator of population logarithm of geometric mean.

Since the previous holds, we have also

$$\text{var } t = |\tau'(\Theta)/A(\Theta)| \qquad \text{(see eq. 17.29 of [7])},$$

thus

$$\text{var } \mu_{-1}' = \frac{\partial}{\partial m} (\frac{1}{m} \frac{K_{\nu-1}}{K_\nu}) \cdot \frac{1}{n\alpha} = \frac{1}{n} \frac{1}{m\alpha} (\frac{1}{m} \frac{K_{\nu-1}}{K_\nu}), \qquad (61)$$

$$\text{var } \underline{\mu_0'} = \frac{\partial}{\partial \nu} (\ln m + \frac{1}{K_\nu} \frac{\partial K_\nu}{\partial \nu}) \cdot \frac{1}{n} = \frac{1}{n} \{\frac{1}{K_\nu} \frac{\partial^2 K_\nu}{\partial \nu^2} - (\frac{1}{K_\nu} \frac{\partial K_\nu}{\partial \nu})^2\} \qquad (62)$$

$$\text{var } \mu_1' = \frac{m}{2n} \frac{\partial}{\partial\alpha} (m \frac{K_{\nu+1}}{K_\nu}) = \frac{m}{n} \{\frac{m\partial K_{\nu+1}}{K_\nu \partial_\alpha} + \frac{mK_{\nu+1}}{K_\nu^2} \frac{\partial K_k}{\partial_\alpha} \} \qquad (63)$$

Using (54) and the parameters in the order m, α, ν, we can obtain the inversion the symetric dispersion matrix for the Type A in using:

$$V^{-1} \begin{bmatrix} -\frac{\partial^2 \text{LogL}}{\partial m^2} & -\frac{\partial^2 \text{LogL}}{\partial m \partial\alpha} & -\frac{\partial^2 \text{LogL}}{\partial m \partial\alpha} \\ -\frac{\partial^2 \text{LogL}}{\partial m \partial\alpha} & -\frac{\partial^2 \text{LogL}}{\partial\alpha^2} & -\frac{\partial^2 \text{LogL}}{\partial\nu^2} \\ -\frac{\partial^2 \text{LogL}}{\partial m \partial\nu} & -\frac{\partial^2 \text{LogL}}{\partial\alpha \partial\nu} & -\frac{\partial^2 \text{LogL}}{\partial\nu^2} \end{bmatrix}^{-1}$$

Concerning the method of moments, which has already been introduced (for type A) above, see (20), for considerably easier algebraic manipulation as compared to the ML estimation (54), the efficiency of the method of moments would be worth the further considerations to allow its alternative utilisation.

Note, that the "classical" method of moments introduced by equation (20) is not exactly equivalent to (55), but we could also call (55) the method of mometns. The reason is, that in (20) we use different degrees of moments and we do not use the moment of order quasi zero. On the other hand, equation (20) has the incomparable advantage of ease of computation.

5.5 Distribution function - notes on computation - Type A

Three ways for computing the Type A distribution function were tested. Two of them were based on the principle of numerical integration, none of them was found completely satisfactory.

1) The first one was (see program NHALF 1 in appendix 4).

$$F(a) = \int_{o}^{a} f(x)\,dx = \frac{1}{2K_{\nu}(2\alpha)} \int_{-\infty}^{\ln(a/m)} e^{\nu t-2\alpha\cosh t}\,dt =$$

$$= \frac{1}{2K_{\nu}(2\alpha)} \left\{ \int_{o}^{\infty} e^{-\nu t-2\alpha\cosh t}\,dt + \int_{o}^{\ln(a/m)} e^{\nu t-2\alpha\cosh t}\,dt \right\} \underset{\alpha}{=}$$

$$= \frac{1}{2K_{\nu}(2\alpha)} \left\{ \frac{1}{\sin(\nu\pi)} \int_{o}^{\pi} e^{2\alpha\cos\Theta} \cos(\nu\Theta)\,d\Theta - \frac{\pi}{\sin(\nu\pi)} I_{\nu}(2\alpha) + \right.$$

$$\left. + \int_{o}^{\ln(a/m)} e^{\nu t-2\alpha\cosh t}\,dt \right\}.$$

From the point of view of numerical computation, the main obstacle is $\nu \neq 0,1,2 \ldots$ (otherwise it can be replaced by limiting value).

2) The second way was

$$F(b) = \frac{1}{2K_{\nu}(2\alpha)} \int_{o}^{b/m} t^{\nu-1}\, e^{-\alpha(t+\frac{1}{t})}\,dt.$$

Again, there is an obstacle from the numerical point of view: $t \neq 0$ $(t > 0)$, therefore the above has been used in the sense $p(x) = \{a < x \leq b\}$, i.e.

$$F(b) - F(a) = \frac{1}{2K_\nu(2\alpha)} \int_{a/m}^{b/m} t^{\nu-1} e^{-\alpha(t+\frac{1}{t})} dt$$

(see program NHALF2 in appendix 4).

The third way is, in principle, the same as the previous one, but instead of numerical integration it uses a series expansion of

$$t^{\nu-1} \exp\{-\alpha(t+1/t)\} = \sum_{k=-\infty}^{\infty} t^{k+\nu-1} I_k(-2\alpha).$$

(see for instance 9.6.33 of [5]). The main obstacle is again $t \neq 0$.

Having a procedure for computation of the distribution function, one of the possible ways to generate a random variable following the distribution may use an inverse transform. The inverse transform can always be computed from the equation.

$$F(a) = \int_0^a f(x)dx.$$

If the argument a follows an uniform distribution, the variable x will have p.d.f. $f(x)$.

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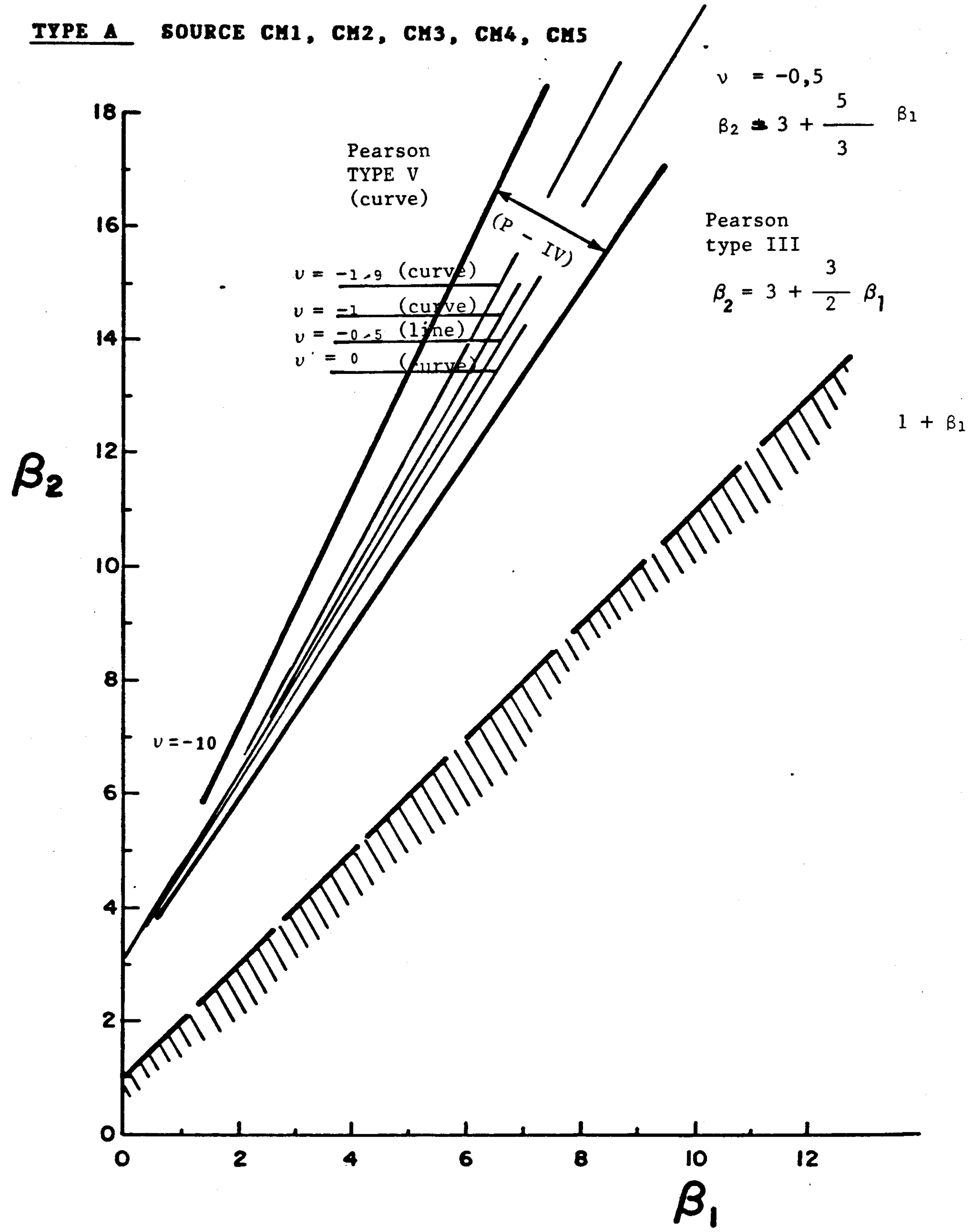
APPENDIX 1

β_1, β_2 RELATIONSHIP - TYPE A

TYPE A SOURCE CM1, CM2, CM3, CM4, CM5



APPENDIX 2

PROGRAM HALPTYA AND NUMERICAL RESULTS

```
                              BBBB    OOO   BBBB   EEEEE  EEEEE
                              B   B  O   O  B   B  E      E
                              B   B  O   O  B   B  E      E
                              BBBB   O   O  BBBB   EEEE   EEEE
                              B   B  O   O  B   B  E      E
                              B   B  O   O  B   B  E      E
                              BBBB    OOO   BBBB   EEEEE  EEEEE

        HH      HH    AAAAAA    LL          FFFFFFFFFF  TTTTTTTTTT  YY      YY    AAAAAA
        HH      HH    AAAAAA    LL          FFFFFFFFFF  TTTTTTTTTT  YY      YY    AAAAAA
        HH      HH  AA      AA  LL          FF              TT      YY      YY  AA      AA
        HH      HH  AA      AA  LL          FF              TT      YY      YY  AA      AA
        HH      HH  AA      AA  LL          FF              TT        YY  YY    AA      AA
        HH      HH  AA      AA  LL          FF              TT        YY  YY    AA      AA
        HHHHHHHHHH  AA      AA  LL          FFFFFFFF        TT          YY      AA      AA
        HHHHHHHHHH  AA      AA  LL          FFFFFFFF        TT          YY      AA      AA
        HH      HH  AAAAAAAAAA  LL          FF              TT          YY      AAAAAAAAAA
        HH      HH  AAAAAAAAAA  LL          FF              TT          YY      AAAAAAAAAA
        HH      HH  AA      AA  LL          FF              TT          YY      AA      AA
        HH      HH  AA      AA  LL          FF              TT          YY      AA      AA
        HH      HH  AA      AA  LLLLLLLLLL  FF              TT          YY      AA      AA
        HH      HH  AA      AA  LLLLLLLLLL  FF              TT          YY      AA      AA

                            CCCCCCCC  DDDDDDDD      CCCCCCCC      ;;;;      222222
                            CCCCCCCC  DDDDDDDD      CCCCCCCC      ;;;;      222222
                          CC          DD      DD  CC              ;;;;    22      22
                          CC          DD      DD  CC              ;;;;    22      22
                          CC          DD      DD  CC                              22
                          CC          DD      DD  CC                              22
                          CC          DD      DD  CC              ;;;;          22
                          CC          DD      DD  CC              ;;;;          22
                          CC          DD      DD  CC              ;;;;        22
                          CC          DD      DD  CC              ;;;;        22
                    ....  CC          DD      DD  CC                ;;      22
                    ....  CC          DD      DD  CC                ;;      22
                    ....    CCCCCCCC  DDDDDDDD      CCCCCCCC      ;;      2222222222
                    ....    CCCCCCCC  DDDDDDDD      CCCCCCCC      ;;      2222222222
```

File _DRC0:[BOBEE]HALFTYA.CDC;2 (7365,16,0), last revised on 5-MAR-1987 11:52, is a 9 block sequential file owned by UIC [BOBEE]. The records are variable length with implied (CR) carriage control. The longest record is 70 bytes.

Job HALFTYA (25) queued to SYS$PRINT on 5-MAR-1987 12:39 by user BOBEE, UIC [BOBEE], under account BOBEE at priority 4, started on printer _LPA0: on 5-MAR-1987 12:39 from queue LPA0.

```
C*******************************************************************
C*
C*    PROGRAM  :  HALFTYA
C*
C*    AUTHOR   :  MARCO LAVOIE
C*    DATE     :  MARCH 1987
C*
C*    PROGRAMMED FOR M. VACLAV DVORAK
C*
C*    DESCRIPTION  :  IT COMPUTES INDEX OF SKEWNESS-BETA1 AND
C*                    INDEX OF KURTOSIS-BETA2 FOR GIVEN VALUES
C*                    OF PARAMETERS V, ALPHA OF HALPHEN TYPE A
C*                    PROBABILITY DENSITY FUNCTION. AS THE
C*                    PURPOSE OF THIS COMPUTATION IS TO ESTIMATE
C*                    RELATIONSHIP OF BETA1 - BETA2 WHICH DOES
C*                    NOT DEPEND ON THE VALUES OF THE PARAMETER
C*                    M OF THE PROBABILITY DENSITY FUNCTION, THE
C*                    M PARAMETER IS SET TO 1 IMPLICITELY.
C*
C*    INPUT   :  PARAMETERS OF THE PROBABILITY DENSITY FUNCTION,
C*               WHICH ARE V AND ALPHA. FOR EACH PARAMETER,
C*               A LOWER BOUND, AN UPPER BOUND AND A STEP-COUNT
C*               VALUE ARE READ; THE PARAMETER IS THEN ASSIGNED
C*               ITERATIVELY RELATIVELY TO THESE ITERATION
C*               RESTRICTIONS.
C*
C*    OUTPUT  :  FOR EACH IMBRICATED ITERATION :
C*
C*                 - PARAMETERS OF THE PROBABILITY DENSITY
C*                   FUNCTION;
C*
C*                 - MODIFIED BESSEL FUNCTION OF THE SECOND
C*                   KIND FOR "V" AND "V + 1";
C*
C*                 - U1 : RATIO OF THE BESSEL FUNCTIONS OF
C*                        "V + 1" AND "V" (I.E. ARITH-
C*                        METIC MEAN FOR PARAMETER M = 1);
C*
C*                 - BETA1 AND BETA2.
C*
C*               THE RESULTS ARE PRINTED ON THE SCREEN AND ALSO
C*               SAVED IN A FILE NAMED "TAPE1".
C*
C*
C*******************************************************************

      PROGRAM HALFTYA (INPUT, OUTPUT, TAPE1)

      INTEGER    N, IER
      REAL       ALPHA, V, BSUB(2), BK(2)
      REAL       AMIN, AMAX, STPALP, OMIN, OMAX, STPV
      REAL       U(4), Q(4), B(2)
      CHARACTER  RESP*1

      DATA     M/1.0/, N/2/

      PRINT 901, 'SKEWNESS-KURTOSIS RELATIONSHIP'

      PRINT 902
      READ (UNIT=*,FMT=*) AMIN, AMAX, STPALP
      PRINT 903
      READ (UNIT=*,FMT=*) OMIN, OMAX, STPV
```

```
      WRITE (UNIT=1,FMT=909)
      PRINT 909
      WRITE (UNIT=1,FMT=907)
      PRINT 907

      DO 10 V = OMIN, OMAX, STPV
         DO 20 ALPHA = AMIN, AMAX, STPALP

            CALL MMBSKR(2*ALPHA, ABS(V), N, BSUB, IER)

            BK(1) = BSUB(1)
            BK(2) = BSUB(2)

            IF (IER .GT. 0) THEN
               WRITE (UNIT=1,FMT=906)
               PRINT 906
            ELSE
               U(1) = (M * (BK(2) / BK(1)))
               U(2) = ((V+1) / ALPHA) * M * U(1) + M**2
               U(3) = ((V+2) / ALPHA) * M * U(2) + M**2 * U(1)
               U(4) = ((V+3) / ALPHA) * M * U(3) + M**2 * U(2)

               Q(2) = U(2) - U(1)**2
               Q(3) = U(3) - (3 * U(2) * U(1)) + 2 * U(1)**3
               Q(4) = U(4) - (4 * U(3) * U(1)) + (6 * U(2) * U(1)**2)
     *                - 3 * U(1)**4

               B(1) = Q(3)**2 / Q(2)**3
               B(2) = Q(4) / Q(2)**2

               WRITE (UNIT=1,FMT=908) V, ALPHA, BK(2), BK(1),
     *                                U(1), B(1), B(2)
               PRINT 908, V, ALPHA, BK(2), BK(1),
     *                    U(1), B(1), B(2)

            ENDIF

 20      CONTINUE
 10   CONTINUE

      STOP

 901  FORMAT(//49X,A30///)
 902  FORMAT('ENTER DESIRED RANGE OF VALUES FOR ALPHA ',
     *       '[MIN MAX STEP] : ')
 903  FORMAT(/'ENTER DESIRED RANGE OF VALUES FOR V ',
     *       '[MIN MAX STEP] : ')
 904  FORMAT(/'*** ERROR : I/O ERROR. RETRY ***'/)
 905  FORMAT(/'*** ERROR : EOF ENCOUNTERED. RETRY ***'/)
 906  FORMAT(38X,'*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. ',
     *       'FATAL. ***')
 907  FORMAT(2X,127('-')/)
 908  FORMAT(2X,F8.2,6X,F8.2,5X,E16.7,5X,E16.7,9X,F12.7,
     *       9X,F12.7,9X,F12.7)
 909  FORMAT(/6X,'V',11X,'ALPHA',11X,'BESSEL(V+1)',11X,
     *       'BESSEL(V)',16X,'U1',19X,'B1',19X,'B2')

      END
```

ORDER	ALPHA	BESSEL(ORDER+1)	BESSEL(ORDER)	U1	B1	B2
.00	.50	1.636	1.144	1.4296254	6.3391500	13.1688584
.00	1.00	1.033	.842	1.2280369	3.7293043	9.0607236
.00	1.50	.807	.698	1.1559299	2.6444839	7.3257786
.00	2.00	.682	.609	1.1186256	2.0478978	6.3629617
.00	2.50	.600	.548	1.0957750	1.6704344	5.7501924
.00	3.00	.542	.502	1.0803262	1.4101714	5.3259479
.00	3.50	.498	.466	1.0691785	1.2199101	5.0148717
.00	4.00	.463	.437	1.0607528	1.0747866	4.7770460
.00	4.50	.435	.412	1.0541594	.9604591	4.5893460
.00	5.00	.411	.392	1.0488587	.8680761	4.4374498
.00	5.50	.390	.374	1.0445041	.7918793	4.3120143
.00	6.00	.373	.358	1.0408629	.7279616	4.2066854
.00	6.50	.357	.344	1.0377729	.6735790	4.1169915
.00	7.00	.344	.332	1.0351176	.6267478	4.0396947
.00	7.50	.332	.321	1.0328114	.5859986	3.9723927
.00	8.00	.321	.311	1.0307896	.5502197	3.9132663
.00	8.50	.311	.302	1.0290026	.5185546	3.8609119
.00	9.00	.301	.293	1.0274117	.4903329	3.8142299
.00	9.50	.293	.286	1.0259864	.4650223	3.7723464
.00	10.00	.285	.279	1.0247020	.4421946	3.7345581
1.00	.50	4.417	1.636	2.6994839	3.2555656	7.9973880
1.00	1.00	1.875	1.033	1.8143078	2.5147718	6.9377486
1.00	1.50	1.235	.807	1.5317710	2.0132586	6.1935526
1.00	2.00	.950	.682	1.3939542	1.6674778	5.6684520
1.00	2.50	.788	.600	1.3125961	1.4185663	5.2844453
1.00	3.00	.683	.542	1.2589796	1.2321802	4.9936032
1.00	3.50	.608	.498	1.2210118	1.0879615	4.7666199
1.00	4.00	.552	.463	1.1927267	.9733273	4.5849898
1.00	4.50	.509	.435	1.1708453	.8801631	4.4365881
1.00	5.00	.474	.411	1.1534173	.8030330	4.3131915
1.00	5.50	.445	.390	1.1392103	.7381724	4.2090489
1.00	6.00	.420	.373	1.1274080	.6828979	4.1200279
1.00	6.50	.399	.357	1.1174481	.6352491	4.0430888
1.00	7.00	.381	.344	1.1089309	.5937619	3.9759488
1.00	7.50	.365	.332	1.1015643	.5573218	3.9168618
1.00	8.00	.351	.321	1.0951301	.5250666	3.8644706
1.00	8.50	.338	.311	1.0894619	.4963183	3.8177050
1.00	9.00	.327	.301	1.0844307	.4705377	3.7757101
1.00	9.50	.317	.293	1.0799349	.4472897	3.7377950
1.00	10.00	.307	.285	1.0758935	.4262204	3.7033955
2.00	.50	19.303	4.417	4.3704412	1.9333591	5.9191699
2.00	1.00	4.784	1.875	2.5511744	1.7398850	5.6651852
2.00	1.50	2.454	1.235	1.9861724	1.5303486	5.3750191
2.00	2.00	1.632	.950	1.7173837	1.3461589	5.1109550
2.00	2.50	1.231	.788	1.5618490	1.1927898	4.8856861
2.00	3.00	.997	.683	1.4609607	1.0664301	4.6967491
2.00	3.50	.846	.608	1.3904215	.9619065	4.5383162
2.00	4.00	.739	.552	1.3384150	.8746647	4.4046507
2.00	4.50	.661	.509	1.2985282	.8010868	4.2909403
2.00	5.00	.600	.474	1.2669889	.7383858	4.1933481
2.00	5.50	.552	.445	1.2414374	.6844278	4.1088642
2.00	6.00	.513	.420	1.2203237	.6375723	4.0351315
2.00	6.50	.480	.399	1.2025884	.5965478	3.9702962
2.00	7.00	.453	.381	1.1874837	.5603587	3.9128894
2.00	7.50	.429	.365	1.1744666	.5282175	3.8617380
2.00	8.00	.408	.351	1.1631335	.4994948	3.8158960
2.00	8.50	.390	.338	1.1531784	.4736827	3.7745946

2.00	9.00	.374	.327	1.1443650	.4503871	3.7372033
2.00	9.50	.360	.317	1.1365080	.4292077	3.7032010
2.00	10.00	.347	.307	1.1294601	.4099226	3.6721538
3.00	.50	120.236	19.303	6.2288099	1.3239708	4.9895574
3.00	1.00	16.226	4.784	3.3919763	1.2712220	4.9253876
3.00	1.50	6.143	2.454	2.5034810	1.1865345	4.8154412
3.00	2.00	3.398	1.632	2.0822811	1.0928195	4.6882568
3.00	2.50	2.265	1.231	1.8402668	1.0024319	4.5616458
3.00	3.00	1.680	.997	1.6844811	.9201106	4.4435781
3.00	3.50	1.333	.846	1.5763492	.8469455	4.3367085
3.00	4.00	1.107	.739	1.4971524	.7825256	4.2412371
3.00	4.50	.949	.661	1.4367693	.7259220	4.1563553
3.00	5.00	.834	.600	1.3892729	.6761061	4.0809205
3.00	5.50	.746	.552	1.3509724	.6321126	4.0137555
3.00	6.00	.677	.513	1.3194547	.5930936	3.9537695
3.00	6.50	.621	.480	1.2930781	.5583255	3.8999985
3.00	7.00	.575	.453	1.2706882	.5271998	3.8516104
3.00	7.50	.537	.429	1.2514504	.4992069	3.8078944
3.00	8.00	.504	.408	1.2347465	.4739203	3.7682466
3.00	8.50	.476	.390	1.2201097	.4509828	3.7321539
3.00	9.00	.452	.374	1.2071804	.4300940	3.6991803
3.00	9.50	.430	.360	1.1956777	.4110005	3.6689544
3.00	10.00	.411	.347	1.1853788	.3934871	3.6411584
4.00	.50	981.193	120.236	8.1605443	.9980786	4.4979632
4.00	1.00	69.687	16.226	4.2948134	.9819392	4.4794777
4.00	1.50	18.836	6.143	3.0661105	.9470628	4.4367227
4.00	2.00	8.427	3.398	2.4802426	.8999696	4.3759947
4.00	2.50	4.854	2.265	2.1433995	.8478870	4.3062660
4.00	3.00	3.237	1.680	1.9269880	.7955774	4.2342039
4.00	3.50	2.369	1.333	1.7772343	.7455972	4.1637923
4.00	4.00	1.846	1.107	1.6679347	.6990991	4.0970989
4.00	4.50	1.505	.949	1.5848949	.6564494	4.0350198
4.00	5.00	1.267	.834	1.5198010	.6176082	3.9777887
4.00	5.50	1.095	.746	1.4674803	.5823428	3.9252876
4.00	6.00	.964	.677	1.4245555	.5503433	3.8772275
4.00	6.50	.862	.621	1.3887331	.5212827	3.8332484
4.00	7.00	.781	.575	1.3584037	.4948455	3.7929740
4.00	7.50	.715	.537	1.3324062	.4707413	3.7560402
4.00	8.00	.660	.504	1.3098828	.4487095	3.7221081
4.00	8.50	.614	.476	1.2901867	.4285188	3.6908694
4.00	9.00	.575	.452	1.2728211	.4099663	3.6620477
4.00	9.50	.541	.430	1.2573984	.3928740	3.6353967
4.00	10.00	.511	.411	1.2436122	.3770863	3.6106981
5.00	.50	9932.162	981.193	10.1225408	.7994604	4.1994426
5.00	1.00	364.659	69.687	5.2328390	.7937804	4.1932091
5.00	1.50	68.929	18.836	3.6594794	.7786593	4.1755316
5.00	2.00	24.465	8.427	2.9031864	.7546857	4.1459965
5.00	2.50	11.973	4.854	2.4665486	.7247971	4.1076283
5.00	3.00	7.075	3.237	2.1856113	.6919407	4.0640648
5.00	3.50	4.717	2.369	1.9912434	.6582936	4.0182870
5.00	4.00	3.415	1.846	1.8495439	.6252466	3.9723729
5.00	4.50	2.621	1.505	1.7420678	.5935982	3.9276337
5.00	5.00	2.101	1.267	1.6579809	.5637523	3.8848262
5.00	5.50	1.741	1.095	1.5905311	.5358675	3.8443359
5.00	6.00	1.480	.964	1.5353067	.5099565	3.8063123
5.00	6.50	1.285	.862	1.4893116	.4859502	3.7707603
5.00	7.00	1.133	.781	1.4504439	.4637376	3.7376009
5.00	7.50	1.014	.715	1.4171886	.4431888	3.7067089
5.00	8.00	.917	.660	1.3884271	.4241692	3.6779372
5.00	8.50	.837	.614	1.3633169	.4065475	3.6511317

5.00	9.00	.771	.575	1.3412119	[illegible]	[illegible]
5.00	9.50	.715	.541	1.3216087	.3750101	3.6028143
5.00	10.00	.667	.511	1.3041092	.3608744	3.5810188
6.00	.50	120167.140	9932.162	12.0987894	.6664751	3.9998046
6.00	1.00	2257.638	364.659	6.1911009	.6641883	3.9973652
6.00	1.50	294.551	68.929	4.2732629	.6571807	3.9894700
6.00	2.00	81.821	24.465	3.3444491	.6446110	3.9745761
6.00	2.50	33.589	11.973	2.8054248	.6272888	3.9531647
6.00	3.00	17.387	7.075	2.4575379	.6066627	3.9267723
6.00	3.50	10.455	4.717	2.2164845	.5841476	3.8971356
6.00	4.00	6.968	3.415	2.0406738	.5608667	3.8657652
6.00	4.50	5.000	2.621	1.9073638	.5376172	3.8338184
6.00	5.00	3.789	2.101	1.8031433	.5149228	3.8021145
6.00	5.50	2.994	1.741	1.7196299	.4931025	3.7711982
6.00	6.00	2.444	1.480	1.6513357	.4723322	3.7414090
6.00	6.50	2.048	1.285	1.5945281	.4526914	3.7129397
6.00	7.00	1.753	1.133	1.5465870	.4341976	3.6858829
6.00	7.50	1.526	1.014	1.5056224	.4168293	3.6602633
6.00	8.00	1.348	.917	1.4702395	.4005416	3.6360622
6.00	8.50	1.205	.837	1.4393875	.3852773	3.6132335
6.00	9.00	1.089	.771	1.4122610	.3709729	3.5917149
6.00	9.50	.992	.715	1.3882326	.3575633	3.5714357
6.00	10.00	.911	.667	1.3668069	.3449846	3.5523216
7.00	.50	1692272.123	120167.140	14.0826529	.5713480	3.8570612
7.00	1.00	16168.121	2257.638	7.1615222	.5703137	3.8559780
7.00	1.50	1443.500	294.551	4.9006799	.5668323	3.8521626
7.00	2.00	310.837	81.821	3.7990029	.5599842	3.8443039
7.00	2.50	106.021	33.589	3.1564523	.5497498	3.8320640
7.00	3.00	47.644	17.387	2.7402447	.5367000	3.8158959
7.00	3.50	25.628	10.455	2.4511649	.5216194	3.7966462
7.00	4.00	15.610	6.968	2.2400342	.5052689	3.7752438
7.00	4.50	10.399	5.000	2.0798394	.4882817	3.7525284
7.00	5.00	7.406	3.789	1.9545871	.4711403	3.7291852
7.00	5.50	5.552	2.994	1.8542477	.4541904	3.7057374
7.00	6.00	4.332	2.444	1.7722371	.4376663	3.6825648
7.00	6.50	3.490	2.048	1.7040679	.4217180	3.6599315
7.00	7.00	2.886	1.753	1.6465850	.4064337	3.6380114
7.00	7.50	2.438	1.526	1.5975105	.3918577	3.6169120
7.00	8.00	2.096	1.348	1.5551613	.3780045	3.5966918
7.00	8.50	1.830	1.205	1.5182694	.3648678	3.5773746
7.00	9.00	1.618	1.089	1.4858622	.3524285	3.5589602
7.00	9.50	1.446	.992	1.4571825	.3406592	3.5414318
7.00	10.00	1.305	.911	1.4316322	.3295276	3.5247619
8.00	.50	27196521.110	1692272.123	16.0710093	.4999616	3.7499613
8.00	1.00	131602.609	16168.121	8.1396351	.4994466	3.7494283
8.00	1.50	7993.216	1443.500	5.5373867	.4975997	3.7474450
8.00	2.00	1325.169	310.837	4.2632270	.4937108	3.7430951
8.00	2.50	372.855	106.021	3.5168114	.4875126	3.7358874
8.00	3.00	144.436	47.644	3.0315976	.4791431	3.7258104
8.00	3.50	69.034	25.628	2.6936836	.4689782	3.7131946
8.00	4.00	38.188	15.610	2.4464218	.4574766	3.6985410
8.00	4.50	23.486	10.399	2.2585841	.4450826	3.6823893
8.00	5.00	15.639	7.406	2.1116170	.4321788	3.6652410
8.00	5.50	11.070	5.552	1.9938477	.4190715	3.6475234
8.00	6.00	8.220	4.332	1.8975919	.4059939	3.6295806
8.00	6.50	6.344	3.490	1.8176003	.3931157	3.6116785
8.00	7.00	5.051	2.886	1.7501747	.3805557	3.5940151
8.00	7.50	4.127	2.438	1.6926406	.3683927	3.5767328
8.00	8.00	3.444	2.096	1.6430201	.3566762	3.5599302
8.00	8.50	2.928	1.830	1.5998211	.3454323	3.5436713

8.00	9.00	2.527	1.618	1.5618988	.3346720	3.5279949
8.00	9.50	2.210	1.446	1.5283611	.3243946	3.5129200
8.00	10.00	1.955	1.305	1.4985034	.3145915	3.4984519
9.00	.50	491229652.099	27196521.110	18.0622238	.4444243	3.6666464
9.00	1.00	1200591.598	131602.609	9.1228556	.4441469	3.6663617
9.00	1.50	49402.797	7993.216	6.1805906	.4431079	3.6652623
9.00	2.00	6274.096	1325.169	4.7345641	.4408069	3.6627402
9.00	2.50	1448.299	372.855	3.8843485	.4369495	3.6583584
9.00	3.00	480.953	144.436	3.3298591	.4314887	3.6519452
9.00	3.50	203.144	69.034	2.9426674	.4245673	3.6435691
9.00	4.00	101.532	38.188	2.6587603	.4164344	3.6334618
9.00	4.50	57.372	23.486	2.4427553	.4073751	3.6219372
9.00	5.00	35.556	15.639	2.2735707	.3976660	3.6093298
9.00	5.50	23.666	11.070	2.1379065	.3875509	3.5959562
9.00	6.00	16.663	8.220	2.0269837	.3772326	3.5820950
9.00	6.50	12.274	6.344	1.9347913	.3668724	3.5679797
9.00	7.00	9.380	5.051	1.8570858	.3565938	3.5537987
9.00	7.50	7.390	4.127	1.7907929	.3464888	3.5396997
9.00	8.00	5.971	3.444	1.7336353	.3366232	3.5257952
9.00	8.50	4.930	2.928	1.6838934	.3270423	3.5121687
9.00	9.00	4.145	2.527	1.6402463	.3177753	3.4988798
9.00	9.50	3.540	2.210	1.6016641	.3088395	3.4859695
9.00	10.00	3.064	1.955	1.5673325	.3002425	3.4734639
10.00	.50	9851789563.090	491229652.099	20.0553642	.3999887	3.5999886
10.00	1.00	12137518.589	1200591.598	10.1096148	.3998293	3.5998260
10.00	1.50	337345.195	49402.797	6.8284635	.3992139	3.5991820
10.00	2.00	32695.651	6274.096	5.2112127	.3977989	3.5976554
10.00	2.50	6166.052	1448.299	4.2574434	.3953309	3.5949056
10.00	3.00	1747.613	480.953	3.6336463	.3916991	3.5907308
10.00	3.50	649.446	203.144	3.1969706	.3869264	3.5850828
10.00	4.00	292.016	101.532	2.8761151	.3811303	3.5780403
10.00	4.50	150.979	57.372	2.6315960	.3744795	3.5697662
10.00	5.00	86.752	35.556	2.4398368	.3671606	3.5604669
10.00	5.50	54.098	23.666	2.2859291	.3593541	3.5503603
10.00	6.00	35.992	16.663	2.1600105	.3512228	3.5396557
10.00	6.50	25.227	12.274	2.0553131	.3429056	3.5285415
10.00	7.00	18.451	9.380	1.9670495	.3345168	3.5171807
10.00	7.50	13.979	7.390	1.8917452	.3261475	3.5057091
10.00	8.00	10.909	5.971	1.8268226	.3178684	3.4942376
10.00	8.50	8.727	4.930	1.7703324	.3097331	3.4828540
10.00	9.00	7.132	4.145	1.7207756	.3017808	3.4716268
10.00	9.50	5.936	3.540	1.6769822	.2940392	3.4606078
10.00	10.00	5.020	3.064	1.6380267	.2865268	3.4498355

ORDER	ALPHA	BESSEL (ORDER+1)	BESSEL (ORDER)	U1	B1	B2
-1.90	.10	.4887660E+01	.4615354E+02	.1059000	433.0091704	1308.0794925
-1.90	.20	.2883502E+01	.1465034E+02	.1968216	94.3183163	233.5000908
-1.90	.30	.2158516E+01	.7899912E+01	.2732330	43.4479910	98.7610975
-1.90	.40	.1773494E+01	.5254035E+01	.3375489	26.3225172	57.5894410
-1.90	.50	.1530559E+01	.3903659E+01	.3920831	18.3049586	39.4810630
-1.90	.60	.1361308E+01	.3102797E+01	.4387357	13.8103577	29.7666081
-1.90	.70	.1235534E+01	.2579377E+01	.4790049	10.9886156	23.8645131
-1.90	.80	.1137735E+01	.2213230E+01	.5140611	9.0743915	19.9610581
-1.90	.90	.1059094E+01	.1943930E+01	.5448208	7.7007759	17.2162140
-1.90	1.00	.9942009E+00	.1738095E+01	.5720060	6.6723098	15.1948129
-1.90	1.10	.9395452E+00	.1575913E+01	.5961909	5.8762795	13.6516590
-1.90	1.20	.8927398E+00	.1444946E+01	.6178363	5.2435445	12.4392610
-1.90	1.30	.8521019E+00	.1337018E+01	.6373151	4.7295396	11.4641339
-1.90	1.40	.8164084E+00	.1246554E+01	.6549320	4.3043502	10.6644330
-1.90	1.50	.7847474E+00	.1169627E+01	.6709382	3.9472050	9.9977606
-1.90	1.60	.7564238E+00	.1103395E+01	.6855422	3.6432619	9.4341620
-1.90	1.70	.7308979E+00	.1045756E+01	.6989184	3.3816523	8.9519209
-1.90	1.80	.7077436E+00	.9951206E+00	.7112139	3.1542455	8.5349382
-1.90	1.90	.6866196E+00	.9502684E+00	.7225534	2.9548451	8.1710440
-1.80	.10	.4152802E+01	.3545980E+02	.1171130	417.9999650	1178.3688197
-1.80	.20	.2579392E+01	.1202824E+02	.2144447	91.0684005	216.9467086
-1.80	.30	.1980223E+01	.6728040E+01	.2943239	42.0630479	93.2547936
-1.80	.40	.1652377E+01	.4584963E+01	.3603905	25.5540681	54.9473300
-1.80	.50	.1441211E+01	.3467242E+01	.4156650	17.8144267	37.9483035
-1.80	.60	.1291808E+01	.2793354E+01	.4624576	13.4689088	28.7694894
-1.80	.70	.1179433E+01	.2347087E+01	.5025093	10.7365816	23.1646424
-1.80	.80	.1091192E+01	.2031496E+01	.5371373	8.8803199	19.4426873
-1.80	.90	.1019656E+01	.1797232E+01	.5673481	7.5464910	16.8166462
-1.80	1.00	.9602193E+00	.1616749E+01	.5939200	6.5465601	14.8772191
-1.80	1.10	.9098619E+00	.1473550E+01	.6174626	5.7717178	13.3930112
-1.80	1.20	.8665153E+00	.1357199E+01	.6384587	5.1551640	12.2244294
-1.80	1.30	.8287104E+00	.1260788E+01	.6572953	4.6538069	11.2827696
-1.80	1.40	.7953723E+00	.1179578E+01	.6742857	4.2386986	10.5092177
-1.80	1.50	.7656949E+00	.1110208E+01	.6896860	3.8897233	9.8633722
-1.80	1.60	.7390607E+00	.1050239E+01	.7037075	3.5924969	9.3166364
-1.80	1.70	.7149878E+00	.9978532E+00	.7165260	3.3364786	8.8482444
-1.80	1.80	.6930940E+00	.9516745E+00	.7282889	3.1137777	8.4427780
-1.80	1.90	.6730720E+00	.9106390E+00	.7391206	2.9183768	8.0885659
-1.70	.10	.3582157E+01	.2743803E+02	.1305545	392.4335143	1036.7905006
-1.70	.20	.2333870E+01	.9940794E+01	.2347770	86.7250374	199.1499487
-1.70	.30	.1833513E+01	.5764811E+01	.3180525	40.3744121	87.4017685
-1.70	.40	.1551535E+01	.4023366E+01	.3856310	24.6609068	52.1620215
-1.70	.50	.1366212E+01	.3095356E+01	.4413747	17.2607582	36.3425671
-1.70	.60	.1233117E+01	.2526617E+01	.4880508	13.0910602	27.7300123
-1.70	.70	.1131836E+01	.2145022E+01	.5276572	10.4616236	22.4379207
-1.70	.80	.1051558E+01	.1872232E+01	.5616598	8.6708588	18.9061804
-1.70	.90	.9859702E+00	.1667877E+01	.5911528	7.3813632	16.4042285
-1.70	1.00	.9311191E+00	.1509187E+01	.6169673	6.4128768	14.5501746
-1.70	1.10	.8843874E+00	.1382408E+01	.6397443	5.6611723	13.1272019
-1.70	1.20	.8439672E+00	.1278766E+01	.6599856	5.0621563	12.0040362
-1.70	1.30	.8085653E+00	.1192418E+01	.6780891	4.5744209	11.0969971
-1.70	1.40	.7772298E+00	.1119324E+01	.6943743	4.1701109	10.3504473
-1.70	1.50	.7492424E+00	.1056610E+01	.7091004	3.8298459	9.7260735
-1.70	1.60	.7240501E+00	.1002173E+01	.7224799	3.5397512	9.1966977
-1.70	1.70	.7012193E+00	.9544445E+00	.7346885	3.2896483	8.7425441
-1.70	1.80	.6804047E+00	.9122263E+00	.7458728	3.0719102	8.3489035

-1.70	1.90	.6613275E+00	.8745914E+00	.7561560	2.8807150	8.0046227
-1.60	.10	.3140766E+01	.2139254E+02	.1468159	358.7833263	892.4407305
-1.60	.20	.2137164E+01	.8272995E+01	.2583302	81.4974740	180.7607916
-1.60	.30	.1713953E+01	.4970857E+01	.3448004	38.4332011	81.3456678
-1.60	.40	.1468490E+01	.3550957E+01	.4135478	23.6620104	49.2833871
-1.60	.50	.1304002E+01	.2777934E+01	.4694145	16.6527741	34.6859682
-1.60	.60	.1184176E+01	.2296408E+01	.5156644	12.6815308	26.6596018
-1.60	.70	.1091983E+01	.1969097E+01	.5545601	10.1665209	21.6908999
-1.60	.80	.1018261E+01	.1732582E+01	.5877134	8.4477638	18.3555904
-1.60	.90	.9575950E+00	.1553779E+01	.6163005	7.2065625	15.9816161
-1.60	1.00	.9065518E+00	.1413838E+01	.6411991	6.2720728	14.2154992
-1.60	1.10	.8628399E+00	.1301267E+01	.6630766	5.5452272	12.8555240
-1.60	1.20	.8248637E+00	.1208681E+01	.6824494	4.9649532	11.7790279
-1.60	1.30	.7914734E+00	.1131124E+01	.6997227	4.4917081	10.9075264
-1.60	1.40	.7618176E+00	.1065152E+01	.7152194	4.0988395	10.1886658
-1.60	1.50	.7352503E+00	.1008298E+01	.7291992	3.7677710	9.5862895
-1.60	1.60	.7112717E+00	.9587498E+00	.7418741	3.4851829	9.0746825
-1.60	1.70	.6894879E+00	.9151463E+00	.7534182	3.2412892	8.6350907
-1.60	1.80	.6695841E+00	.8764466E+00	.7639760	3.0287473	8.2535350
-1.60	1.90	.6513052E+00	.8418402E+00	.7736685	2.8419458	7.9193956
-1.50	.10	.2802496E+01	.1681497E+02	.1666667	320.0000000	753.0000000
-1.50	.20	.1981664E+01	.6935823E+01	.2857143	75.6250000	162.3750000
-1.50	.30	.1618022E+01	.4314724E+01	.3750000	36.2962963	75.2222222
-1.50	.40	.1401248E+01	.3152808E+01	.4444444	22.5781250	46.3593750
-1.50	.50	.1253314E+01	.2506628E+01	.5000000	16.0000000	33.0000000
-1.50	.60	.1144114E+01	.2097542E+01	.5454545	12.2453704	25.5694444
-1.50	.70	.1059244E+01	.1815846E+01	.5833333	9.8542274	20.9300292
-1.50	.80	.9908318E+00	.1610102E+01	.6153846	8.2128906	17.7949219
-1.50	.90	.9341652E+00	.1453146E+01	.6428571	7.0233196	15.5514403
-1.50	1.00	.8862269E+00	.1329340E+01	.6666667	6.1250000	13.8750000
-1.50	1.10	.8449842E+00	.1229068E+01	.6875000	5.4244929	12.5792637
-1.50	1.20	.8090108E+00	.1146099E+01	.7058824	4.8640046	11.5503472
-1.50	1.30	.7772724E+00	.1076223E+01	.7222222	4.4060082	10.7150660
-1.50	1.40	.7489985E+00	.1016498E+01	.7368421	4.0251458	10.0244169
-1.50	1.50	.7236013E+00	.9648017E+00	.7500000	3.7037037	9.4444444
-1.50	1.60	.7006239E+00	.9195689E+00	.7619048	3.4289551	8.9509277
-1.50	1.70	.6797050E+00	.8796183E+00	.7727273	3.1915327	8.5261551
-1.50	1.80	.6605545E+00	.8440419E+00	.7826087	2.9843964	8.1568930
-1.50	1.90	.6429366E+00	.8121304E+00	.7916667	2.8021577	7.8330660
-1.40	.10	.2547946E+01	.1333255E+02	.1911072	279.0330117	624.1495390
-1.40	.20	.1861503E+01	.5860170E+01	.3176534	69.3547430	144.5002390
-1.40	.30	.1542950E+01	.3771220E+01	.4091382	34.0227642	69.1536151
-1.40	.40	.1348221E+01	.2816712E+01	.4786507	21.4308020	43.4344458
-1.40	.50	.1213131E+01	.2274507E+01	.5333599	15.3123216	31.3049648
-1.40	.60	.1112232E+01	.1925664E+01	.5775837	11.7878125	24.4702403
-1.40	.70	.1033112E+01	.1682332E+01	.6140950	9.5277996	20.1615328
-1.40	.80	.9688856E+00	.1502704E+01	.6447614	7.9681567	17.2280664
-1.40	.90	.9153828E+00	.1364432E+01	.6708894	6.8329047	15.1162713
-1.40	1.00	.8699072E+00	.1254515E+01	.6934213	5.9725365	13.5304494
-1.40	1.10	.8306273E+00	.1164886E+01	.7130545	5.2995977	12.2996868
-1.40	1.20	.7962493E+00	.1090280E+01	.7303162	4.7597733	11.3189245
-1.40	1.30	.7658289E+00	.1027113E+01	.7456131	4.3176701	10.5203162
-1.40	1.40	.7386591E+00	.9728631E+00	.7592632	3.9492983	9.8582386
-1.40	1.50	.7141981E+00	.9257032E+00	.7715196	3.6378541	9.3009594
-1.40	1.60	.6920229E+00	.8842774E+00	.7825858	3.3712349	8.8257677
-1.40	1.70	.6717977E+00	.8475579E+00	.7926274	3.1405135	8.4160065
-1.40	1.80	.6532518E+00	.8147512E+00	.8017808	2.9389673	8.0591969
-1.40	1.90	.6361649E+00	.7852347E+00	.8101589	2.7614414	7.7458148
-1.30	.10	.2362930E+01	.1067075E+02	.2214358	238.4513937	509.4307653
-1.30	.20	.1772249E+01	.4992244E+01	.3550006	62.9217972	127.5371228

-1.30	.30	.1486614E+01	.3320127E+01	.4477583	31.6705139	63.2445295
-1.30	.40	.1308180E+01	.2532670E+01	.5165222	20.2414772	40.5483660
-1.30	.50	.1182659E+01	.2075807E+01	.5697346	14.5996507	29.6195089
-1.30	.60	.1087980E+01	.1777107E+01	.6122196	11.3141304	23.3719941
-1.30	.70	.1013186E+01	.1566059E+01	.6469653	9.1903267	19.3913056
-1.30	.80	.9521194E+00	.1408602E+01	.6759320	7.7155040	16.6587443
-1.30	.90	.9010112E+00	.1286307E+01	.7004635	6.6366057	14.6785841
-1.30	1.00	.8574037E+00	.1188340E+01	.7215137	5.8155736	13.1835641
-1.30	1.10	.8196156E+00	.1107919E+01	.7397794	5.1711797	12.0180248
-1.30	1.20	.7864519E+00	.1040580E+01	.7557821	4.6527290	11.0856681
-1.30	1.30	.7570362E+00	.9832660E+00	.7699201	4.2270481	10.3239630
-1.30	1.40	.7307091E+00	.9338103E+00	.7825027	3.8715697	9.6906597
-1.30	1.50	.7069633E+00	.8906351E+00	.7937744	3.5704358	9.1562487
-1.30	1.60	.6854015E+00	.8525629E+00	.8039307	3.3121921	8.6995322
-1.30	1.70	.6657071E+00	.8186971E+00	.8131299	3.0883685	8.3049110
-1.30	1.80	.6476244E+00	.7883421E+00	.8215017	2.8925716	7.9606638
-1.30	1.90	.6309444E+00	.7609503E+00	.8291532	2.7198888	7.6578209
-1.20	.10	.2237382E+01	.8627566E+01	.2593294	200.2196588	410.4485178
-1.20	.20	.1710673E+01	.4290065E+01	.3987522	56.5338950	111.7735603
-1.20	.30	.1447445E+01	.2945186E+01	.4914612	29.2934712	57.5799358
-1.20	.40	.1280209E+01	.2292481E+01	.5584381	19.0306548	37.7353902
-1.20	.50	.1161304E+01	.1905733E+01	.6093742	13.8716213	27.9602747
-1.20	.60	.1070944E+01	.1648789E+01	.6495335	10.8295025	22.2838482
-1.20	.70	.9991633E+00	.1464908E+01	.6820654	8.8448643	18.6248250
-1.20	.80	.9403036E+00	.1326268E+01	.7089845	7.4568628	16.0904558
-1.20	.90	.8908712E+00	.1217628E+01	.7316449	6.4357077	14.2407298
-1.20	1.00	.8485731E+00	.1129934E+01	.7509935	5.6550027	12.8359869
-1.20	1.10	.8118321E+00	.1057468E+01	.7677133	5.0398787	11.7354633
-1.20	1.20	.7795218E+00	.9964356E+00	.7823102	4.5433437	10.8514570
-1.20	1.30	.7508129E+00	.9442200E+00	.7951673	4.1344986	10.1266730
-1.20	1.40	.7250791E+00	.8989550E+00	.8065800	3.7922346	9.5221953
-1.20	1.50	.7018374E+00	.8592732E+00	.8167803	3.5016638	9.0107176
-1.20	1.60	.6807082E+00	.8241492E+00	.8259526	3.2519980	8.5725444
-1.20	1.70	.6613884E+00	.7927982E+00	.8342456	3.0352356	8.1931296
-1.20	1.80	.6436327E+00	.7646087E+00	.8417805	2.8453217	7.8615072
-1.20	1.90	.6272402E+00	.7390973E+00	.8486571	2.6775925	7.5692606
-1.10	.10	.2164602E+01	.7052261E+01	.3069372	165.6298906	327.2751105
-1.10	.20	.1674575E+01	.3720790E+01	.4500591	50.3616586	97.3900400
-1.10	.30	.1424363E+01	.2633304E+01	.5409032	26.9394456	52.2244863
-1.10	.40	.1263674E+01	.2089412E+01	.6047988	17.8172392	35.0238242
-1.10	.50	.1148653E+01	.1760290E+01	.6525365	13.1373272	26.3416727
-1.10	.60	.1060835E+01	.1538113E+01	.6896990	10.3388937	21.2139617
-1.10	.70	.9908330E+00	.1377082E+01	.7195166	8.4943740	17.8670835
-1.10	.80	.9332775E+00	.1254395E+01	.7440060	7.1941192	15.5264411
-1.10	.90	.8848368E+00	.1157409E+01	.7644980	6.2314738	13.8049102
-1.10	1.00	.8433145E+00	.1078532E+01	.7819093	5.4917044	12.4892707
-1.10	1.10	.8071945E+00	.1012927E+01	.7968934	4.9063286	11.4531313
-1.10	1.20	.7753907E+00	.9573557E+00	.8099296	4.4320865	10.6171333
-1.10	1.30	.7471016E+00	.9095713E+00	.8213777	4.0403765	9.9290879
-1.10	1.40	.7217205E+00	.8679599E+00	.8315136	3.7115668	9.3533436
-1.10	1.50	.6987785E+00	.8313327E+00	.8405522	3.4317533	8.8647590
-1.10	1.60	.6779066E+00	.7987930E+00	.8486636	3.1908236	8.4451189
-1.10	1.70	.6588097E+00	.7696515E+00	.8559845	2.9812530	8.0809172
-1.10	1.80	.6412488E+00	.7433686E+00	.8626256	2.7973305	7.7619359
-1.10	1.90	.6250275E+00	.7195158E+00	.8686779	2.6346456	7.4803068
-1.00	.10	.2140757E+01	.5833386E+01	.3669837	135.3535988	258.9201901
-1.00	.20	.1662682E+01	[illegible]	.5102327	44.5343780	84.4726047
-1.00	.30	.1416738E+01	[illegible]	.5967925	24.6487577	47.2232528
-1.00	.40	.1250203E+01	[illegible]	.6560213	16.6180382	32.4359347
-1.00	.50	.1144463E+01	[illegible]	.6994839	12.4051127	24.7757675

-1.00	.60	.1057485E+01	.1442898E+01	.7328895	9.8469552	20.1694336
-1.00	.70	.9880700E+00	.1301054E+01	.7594383	8.1416706	17.1225401
-1.00	.80	.9309460E+00	.1191868E+01	.7810817	6.9290850	14.9696494
-1.00	.90	.8828335E+00	.1104805E+01	.7990851	6.0251273	13.3731588
-1.00	1.00	.8415682E+00	.1033477E+01	.8143078	5.3265365	12.1448658
-1.00	1.10	.8056540E+00	.9737702E+00	.8273554	4.7711502	11.1720928
-1.00	1.20	.7740181E+00	.9229137E+00	.8386680	4.3194184	10.3834961
-1.00	1.30	.7458682E+00	.8789673E+00	.8485734	3.9450321	9.7318203
-1.00	1.40	.7206041E+00	.8405301E+00	.8573211	3.6298373	9.1845828
-1.00	1.50	.6977616E+00	.8065635E+00	.8651044	3.3609173	8.7187516
-1.00	1.60	.6769751E+00	.7762803E+00	.8720756	3.1288391	8.3175598
-1.00	1.70	.6579523E+00	.7490721E+00	.8783564	2.9265582	7.9685212
-1.00	1.80	.6404560E+00	.7244607E+00	.8840452	2.7487104	7.6621530
-1.00	1.90	.6242916E+00	.7020647E+00	.8892223	2.5911410	7.3911280
-.90	.10	.2164602E+01	.4887660E+01	.4428708	109.5631128	203.7686796
-.90	.20	.1674575E+01	.2883502E+01	.5807435	39.1404654	73.0301465
-.90	.30	.1424363E+01	.2158516E+01	.6598804	22.4536002	42.6034681
-.90	.40	.1263674E+01	.1773494E+01	.7125337	15.4474453	29.9881504
-.90	.50	.1148653E+01	.1530559E+01	.7504796	11.6824187	23.2722672
-.90	.60	.1060835E+01	.1361308E+01	.7792767	9.3579453	19.1562683
-.90	.70	.9908330E+00	.1235534E+01	.8019472	7.7893793	16.3950917
-.90	.80	.9332775E+00	.1137735E+01	.8202940	6.6634727	14.4227185
-.90	.90	.8848368E+00	.1059094E+01	.8354660	5.8178361	12.9473254
-.90	1.00	.8433145E+00	.9942009E+00	.8482335	5.1603250	11.8041091
-.90	1.10	.8071945E+00	.9395452E+00	.8591332	4.6349451	10.8933388
-.90	1.20	.7753907E+00	.8927398E+00	.8685517	4.2057884	10.1512964
-.90	1.30	.7471016E+00	.8521019E+00	.8767749	3.8488076	9.5354500
-.90	1.40	.7217205E+00	.8164084E+00	.8840189	3.5473120	9.0163678
-.90	1.50	.6987785E+00	.7847474E+00	.8904502	3.2893656	8.5730580
-.90	1.60	.6779066E+00	.7564238E+00	.8961994	3.0662123	8.1901594
-.90	1.70	.6588097E+00	.7308979E+00	.9013703	2.8712870	7.8561798
-.90	1.80	.6412488E+00	.7077436E+00	.9060467	2.6995727	7.5623550
-.90	1.90	.6250275E+00	.6866196E+00	.9102966	2.5471706	7.3018876
-.80	.10	.2237382E+01	.4152802E+01	.5387643	88.0767224	159.9334161
-.80	.20	.1710673E+01	.2579392E+01	.6632076	34.2313142	63.0130841
-.80	.30	.1447445E+01	.1980223E+01	.7309503	20.3780318	38.3769173
-.80	.40	.1280209E+01	.1652377E+01	.7747682	14.3172893	27.6914872
-.80	.50	.1161304E+01	.1441211E+01	.8057837	10.9756850	21.8385987
-.80	.60	.1070944E+01	.1291808E+01	.8290271	8.8756723	18.1793780
-.80	.70	.9991633E+00	.1179433E+01	.8471556	7.4399017	15.6880613
-.80	.80	.9403036E+00	.1091192E+01	.8617212	6.3988752	13.8879622
-.80	.90	.8908712E+00	.1019656E+01	.8736974	5.6107004	12.5290670
-.80	1.00	.8485731E+00	.9602193E+00	.8837283	4.9938552	11.4682169
-.80	1.10	.8118321E+00	.9098619E+00	.8922585	4.4982895	10.6177823
-.80	1.20	.7795218E+00	.8665153E+00	.8996053	4.0916296	9.9212327
-.80	1.30	.7508129E+00	.8287104E+00	.9060015	3.7520349	9.3405208
-.80	1.40	.7250791E+00	.7953723E+00	.9116224	3.4642496	8.8491283
-.80	1.50	.7018374E+00	.7656949E+00	.9166019	3.2173030	8.4280221
-.80	1.60	.6807082E+00	.7390607E+00	.9210450	3.0031077	8.0631966
-.80	1.70	.6613884E+00	.7149878E+00	.9250345	2.8155731	7.7441217
-.80	1.80	.6436327E+00	.6930940E+00	.9286370	2.6500269	7.4627311
-.80	1.90	.6272402E+00	.6730720E+00	.9319065	2.5028254	7.2127434
-.70	.10	.2362930E+01	.3582157E+01	.6596388	70.4953696	125.5060283
-.70	.20	.1772249E+01	.2333870E+01	.7593608	29.8272016	54.3312173
-.70	.30	.1486614E+01	.1833513E+01	.8108013	18.4384685	34.5426533
-.70	.40	.1308180E+01	.1551535E+01	.8431524	13.2368315	25.5521304
-.70	.50	.1182659E+01	.1366212E+01	.8656485	10.2903057	20.4800523
-.70	.60	.1087980E+01	.1233117E+01	.8823003	8.4034593	17.2426178
-.70	.70	.1013186E+01	.1131836E+01	.8951697	7.0953917	15.0042026
-.70	.80	.9521194E+00	.1051558E+01	.9054372	6.1367493	13.3673673

-.70	.90	.9010112E+00	.9859702E+00	.9138321	5.4047414	12.1198421
-.70	1.00	.8574037E+00	.9311191E+00	.9208314	4.8278646	11.1382804
-.70	1.10	.8196156E+00	.8843874E+00	.9267608	4.3617297	10.3462541
-.70	1.20	.7864519E+00	.8439672E+00	.9318513	3.9773554	9.6939482
-.70	1.30	.7570362E+00	.8085653E+00	.9362709	3.6550328	9.1475381
-.70	1.40	.7307091E+00	.7772298E+00	.9401455	3.3809001	8.6832668
-.70	1.50	.7069633E+00	.7492424E+00	.9435709	3.1449284	8.2839684
-.70	1.60	.6854015E+00	.7240501E+00	.9466216	2.9396852	7.9369358
-.70	1.70	.6657071E+00	.7012193E+00	.9493564	2.7595468	7.6325645
-.70	1.80	.6476244E+00	.6804047E+00	.9518223	2.6001804	7.3634619
-.70	1.90	.6309444E+00	.6613275E+00	.9540574	2.4581945	7.1238470
-.60	.10	.2547946E+01	.3140766E+01	.8112500	56.3136753	98.7135613
-.60	.20	.1861503E+01	.2137164E+01	.8710153	25.9240316	46.8693512
-.60	.30	.1542950E+01	.1713953E+01	.9002287	16.6445162	31.0897655
-.60	.40	.1348221E+01	.1468490E+01	.9181000	12.2128813	23.5721082
-.60	.50	.1213131E+01	.1304002E+01	.9303134	9.6306298	19.1999757
-.60	.60	.1112232E+01	.1184176E+01	.9392458	7.9441279	16.3488453
-.60	.70	.1033112E+01	.1091983E+01	.9460881	6.7577406	14.3457176
-.60	.80	.9688856E+00	.1018261E+01	.9515098	5.8784053	12.8625968
-.60	.90	.9153828E+00	.9575950E+00	.9559185	5.2008937	11.7209099
-.60	1.00	.8699072E+00	.9065518E+00	.9595780	4.6630373	10.8152629
-.60	1.10	.8306273E+00	.8628399E+00	.9626668	4.2257779	10.0795003
-.60	1.20	.7962493E+00	.8248637E+00	.9653101	3.8633568	9.4700285
-.60	1.30	.7658289E+00	.7914734E+00	.9675991	3.5581051	8.9569669
-.60	1.40	.7386591E+00	.7618176E+00	.9696010	3.2975029	8.5191571
-.60	1.50	.7141981E+00	.7352503E+00	.9713673	3.0724331	8.1412005
-.60	1.60	.6920229E+00	.7112717E+00	.9729376	2.8760998	7.8116261
-.60	1.70	.6717977E+00	.6894879E+00	.9743430	2.7033347	7.5217142
-.60	1.80	.6532518E+00	.6695841E+00	.9756084	2.5501377	7.2647196
-.60	1.90	.6361649E+00	.6513052E+00	.9767538	2.4133648	7.0353428
-.50	.10	.2802496E+01	.2802496E+01	1.0000000	45.0000000	78.0000000
-.50	.20	.1981664E+01	.1981664E+01	1.0000000	22.5000000	40.5000000
-.50	.30	.1618022E+01	.1618022E+01	1.0000000	15.0000000	28.0000000
-.50	.40	.1401248E+01	.1401248E+01	1.0000000	11.2500000	21.7500000
-.50	.50	.1253314E+01	.1253314E+01	1.0000000	9.0000000	18.0000000
-.50	.60	.1144114E+01	.1144114E+01	1.0000000	7.5000000	15.5000000
-.50	.70	.1059244E+01	.1059244E+01	1.0000000	6.4285714	13.7142857
-.50	.80	.9908318E+00	.9908318E+00	1.0000000	5.6250000	12.3750000
-.50	.90	.9341652E+00	.9341652E+00	1.0000000	5.0000000	11.3333333
-.50	1.00	.8862269E+00	.8862269E+00	1.0000000	4.5000000	10.5000000
-.50	1.10	.8449842E+00	.8449842E+00	1.0000000	4.0909091	9.8181818
-.50	1.20	.8090108E+00	.8090108E+00	1.0000000	3.7500000	9.2500000
-.50	1.30	.7772724E+00	.7772724E+00	1.0000000	3.4615385	8.7692308
-.50	1.40	.7489985E+00	.7489985E+00	1.0000000	3.2142857	8.3571429
-.50	1.50	.7236013E+00	.7236013E+00	1.0000000	3.0000000	8.0000000
-.50	1.60	.7006239E+00	.7006239E+00	1.0000000	2.8125000	7.6875000
-.50	1.70	.6797050E+00	.6797050E+00	1.0000000	2.6470588	7.4117647
-.50	1.80	.6605545E+00	.6605545E+00	1.0000000	2.5000000	7.1666667
-.50	1.90	.6429366E+00	.6429366E+00	1.0000000	2.3684211	6.9473684
-.40	.10	.3140766E+01	.2547946E+01	1.2326656	36.0477231	62.0545927
-.40	.20	.2137164E+01	.1861503E+01	1.1480855	19.5215751	35.0930261
-.40	.30	.1713953E+01	.1542950E+01	1.1108288	13.5040665	25.2500984
-.40	.40	.1468490E+01	.1348221E+01	1.0892060	10.3507620	20.0816352
-.40	.50	.1304002E+01	.1213131E+01	1.0749066	8.4008212	16.8802813
-.40	.60	.1184176E+01	.1112232E+01	1.0646841	7.0729144	14.6971953
-.40	.70	.1091983E+01	.1033112E+01	1.0569840	6.1092404	13.1111007
-.40	.80	.1018261E+01	.9688856E+00	1.0509614	5.3775342	11.9056282
-.40	.90	.9575950E+00	.9153828E+00	1.0461142	4.8028074	10.9579849
-.40	1.00	.9065518E+00	.8699072E+00	1.0421247	4.3393187	10.1932020
-.40	1.10	.8628399E+00	.8306273E+00	1.0387811	3.9575584	9.5628740

-.40	1.20	.8248637E+00	.7962493E+00	1.0359365	3.6376245	9.0343302
-.40	1.30	.7914734E+00	.7658289E+00	1.0334859	3.3656012	8.5847112
-.40	1.40	.7618176E+00	.7386591E+00	1.0313521	3.1314631	8.1975376
-.40	1.50	.7352503E+00	.7141981E+00	1.0294767	2.9278028	7.8606261
-.40	1.60	.7112717E+00	.6920229E+00	1.0278152	2.7490279	7.5647735
-.40	1.70	.6894879E+00	.6717977E+00	1.0263327	2.5908363	7.3028970
-.40	1.80	.6695841E+00	.6532518E+00	1.0250015	2.4498647	7.0694559
-.40	1.90	.6513052E+00	.6361649E+00	1.0237994	2.3234449	6.8600535
-.30	.10	.3582157E+01	.2362930E+01	1.5159812	29.0034670	49.8065587
-.30	.20	.2333870E+01	.1772249E+01	1.3168970	16.9484785	30.5224399
-.30	.30	.1833513E+01	.1486614E+01	1.2333478	12.1522632	22.8137829
-.30	.40	.1551535E+01	.1308180E+01	1.1860252	9.5160458	18.5607478
-.30	.50	.1366212E+01	.1182659E+01	1.1552033	7.8346489	15.8397450
-.30	.60	.1233117E+01	.1087980E+01	1.1334009	6.6642546	13.9408180
-.30	.70	.1131836E+01	.1013186E+01	1.1171066	5.8008447	12.5369137
-.30	.80	.1051558E+01	.9521194E+00	1.1044388	5.1368541	11.4552534
-.30	.90	.9859702E+00	.9010112E+00	1.0942929	4.6099670	10.5955555
-.30	1.00	.9311191E+00	.8574037E+00	1.0859752	4.1814975	9.8954574
-.30	1.10	.8843874E+00	.8196156E+00	1.0790270	3.8261200	9.3140687
-.30	1.20	.8439672E+00	.7864519E+00	1.0731326	3.5265416	8.8234275
-.30	1.30	.8085653E+00	.7570362E+00	1.0680669	3.2705420	8.4037474
-.30	1.40	.7772298E+00	.7307091E+00	1.0636651	3.0492358	8.0406237
-.30	1.50	.7492424E+00	.7069633E+00	1.0598037	2.8560052	7.7233151
-.30	1.60	.7240501E+00	.6854015E+00	1.0563883	2.6858179	7.4436453
-.30	1.70	.7012193E+00	.6657071E+00	1.0533452	2.5347788	7.1952790
-.30	1.80	.6804047E+00	.6476244E+00	1.0506163	2.3998251	6.9732301
-.30	1.90	.6613275E+00	.6309444E+00	1.0481550	2.2785148	6.7735200
-.20	.10	.4152802E+01	.2237382E+01	1.8560992	23.4788173	40.4013599
-.20	.20	.2579392E+01	.1710673E+01	1.5078234	14.7375676	26.6707837
-.20	.30	.1980223E+01	.1447445E+01	1.3680822	10.9375286	20.6633649
-.20	.40	.1652377E+01	.1280209E+01	1.2907087	8.7453332	17.1795657
-.20	.50	.1441211E+01	.1161304E+01	1.2410278	7.3022909	14.8763213
-.20	.60	.1291808E+01	.1070944E+01	1.2062332	6.2749853	13.2306305
-.20	.70	.1179433E+01	.9991633E+00	1.1804207	5.5042361	11.9920802
-.20	.80	.1091192E+01	.9403036E+00	1.1604681	4.9036559	11.0243913
-.20	.90	.1019656E+01	.8908712E+00	1.1445610	4.4220344	10.2465652
-.20	1.00	.9602193E+00	.8485731E+00	1.1315694	4.0269780	9.6072383
-.20	1.10	.9098619E+00	.8118321E+00	1.1207514	3.6969460	9.0721768
-.20	1.20	.8665153E+00	.7795218E+00	1.1115986	3.4170339	8.6176429
-.20	1.30	.8287104E+00	.7508129E+00	1.1037509	3.1765893	8.2266369
-.20	1.40	.7953723E+00	.7250791E+00	1.0969455	2.9677902	7.8866531
-.20	1.50	.7656949E+00	.7018374E+00	1.0909861	2.7847601	7.5882803
-.20	1.60	.7390607E+00	.6807082E+00	1.0857233	2.6229969	7.3242965
-.20	1.70	.7149878E+00	.6613884E+00	1.0810408	2.4789920	7.0890651
-.20	1.80	.6930940E+00	.6436327E+00	1.0768471	2.3499700	6.8781214
-.20	1.90	.6730720E+00	.6272402E+00	1.0730690	2.2337060	6.6878816
-.10	.10	.4887660E+01	.2164602E+01	2.2579951	19.1514011	33.1690914
-.10	.20	.2883502E+01	.1674575E+01	1.7219306	12.8456700	23.4315982
-.10	.30	.2158516E+01	.1424363E+01	1.5154262	9.8510556	18.7709888
-.10	.40	.1773494E+01	.1263674E+01	1.4034424	8.0369984	15.9293218
-.10	.50	.1530559E+01	.1148653E+01	1.3324812	6.8039138	13.9871659
-.10	.60	.1361308E+01	.1060835E+01	1.2832413	5.9056948	12.5658713
-.10	.70	.1235534E+01	.9908330E+00	1.2469649	5.2200378	11.4766081
-.10	.80	.1137735E+01	.9332775E+00	1.2190752	4.6784923	10.6133248
-.10	.90	.1059094E+01	.8848368E+00	1.1969368	4.2394728	9.9113760
-.10	1.00	.9942009E+00	.8433145E+00	1.1789207	3.8761403	9.3289070
-.10	1.10	.9395452E+00	.8071945E+00	1.1639639	3.5703464	8.8375313
-.10	1.20	.8927398E+00	.7753907E+00	1.1513419	3.3093547	8.4172714
-.10	1.30	.8521049E+00	.7471016E+00	1.1405436	3.0839508	8.0536364
-.10	1.40	.8164084E+00	.7217205E+00	1.1311976	2.8872972	7.7358472

-.10	1.50	.7847474E+00	.6987785E+00	1.1230275	2.7142095	7.4557122
-.10	1.60	.7564238E+00	.6779066E+00	1.1158231	2.5606833	7.2068909
-.10	1.70	.7308979E+00	.6588097E+00	1.1094219	2.4235754	6.9843966
-.10	1.80	.7077436E+00	.6412488E+00	1.1036959	2.3003835	6.7842516
-.10	1.90	.6866196E+00	.6250275E+00	1.0985431	2.1890899	6.6032437

ORDER	ALPHA	BESSEL(ORDER+1)	BESSEL(ORDER)	U1	B1	B2
10.00	.10	.2213898E+18	.2213652E+16	100.0111096	.4000000	3.6000000
10.00	.20	.1316389E+15	.2631608E+13	50.0222099	.3999997	3.5999997
10.00	.30	.1849563E+13	.5543154E+11	33.3666251	.3999985	3.5999985
10.00	.40	.9474715E+11	.3783175E+10	25.0443462	.3999953	3.5999953

*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. FATAL. ***

20.00	.20	.8636026E+33	.8635118E+31	100.0105251	.2000000	3.3000000
20.00	.30	.2109458E+30	.3163438E+28	66.6824520	.2000000	3.3000000
20.00	.40	.6106562E+27	.1220799E+26	50.0210428	.1999999	3.2999999

*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. FATAL. ***

*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. FATAL. ***

30.00	.30	.3900723E+49	.3900319E+47	100.0103437	.1333333	3.2000000
30.00	.40	.6366341E+45	.8486894E+43	75.0137904	.1333333	3.2000000

*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. FATAL. ***

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ORDER	ALPHA	BESSEL(ORDER+1)	BESSEL(ORDER)	U1	B1	B2
-10.00	3.00	.1444364E+03	.4809528E+03	.3003130	1.6441359	6.1892892
-10.00	3.10	.1228417E+03	.3982139E+03	.3084816	1.6096194	6.1131229
-10.00	3.20	.1052797E+03	.3326199E+03	.3165165	1.5759444	6.0392621
-10.00	3.30	.9087333E+02	.2801116E+03	.3244183	1.5431145	5.9676745
-10.00	3.40	.7896008E+02	.2376967E+03	.3321884	1.5111292	5.8983202
-10.00	3.50	.6903405E+02	.2031443E+03	.3398277	1.4799839	5.8311531
-10.00	3.60	.6070532E+02	.1747732E+03	.3473378	1.4496712	5.7661227
-10.00	3.70	.5367056E+02	.1513040E+03	.3547200	1.4201807	5.7031748
-10.00	3.80	.4769179E+02	.1317541E+03	.3619759	1.3914999	5.6422527
-10.00	3.90	.4258079E+02	.1153616E+03	.3691070	1.3636146	5.5832979
-10.00	4.00	.3818755E+02	.1015315E+03	.3761151	1.3365089	5.5262507
-20.00	3.00	.6787264E+09	.4408587E+10	.1539555	.9047592	4.7582653
-20.00	3.10	.4302178E+09	.2708728E+10	.1588265	.8993259	4.7463157
-20.00	3.20	.2778634E+09	.1697667E+10	.1636737	.8938041	4.7341920
-20.00	3.30	.1826406E+09	.1083942E+10	.1684967	.8882000	4.7219081
-20.00	3.40	.1220414E+09	.7042410E+09	.1732950	.8825196	4.7094785
-20.00	3.50	.8281788E+08	.4650910E+09	.1780681	.8767690	4.6969165
-20.00	3.60	.5702273E+08	.3119138E+09	.1828157	.8709539	4.6842358
-20.00	3.70	.3980267E+08	.2122387E+09	.1875373	.8650800	4.6714492
-20.00	3.80	.2814359E+08	.1464039E+09	.1922325	.8591529	4.6585694
-20.00	3.90	.2014370E+08	.1023037E+09	.1969010	.8531780	4.6456085
-20.00	4.00	.1458494E+08	.7236659E+08	.2015425	.8471606	4.6325783
-30.00	3.00	.6510048E+18	.6361987E+19	.1023273	.5912611	4.1379648
-30.00	3.10	.3006925E+18	.2845808E+19	.1056615	.5897653	4.1347598
-30.00	3.20	.1430456E+18	.1312483E+19	.1089885	.5882308	4.1314785
-30.00	3.30	.6995739E+17	.6229067E+18	.1123080	.5866583	4.1281164
-30.00	3.40	.3511283E+17	.3036923E+18	.1156198	.5850489	4.1246770
-30.00	3.50	.1805930E+17	.1518562E+18	.1189237	.5834034	4.1211618
-30.00	3.60	.9504422E+16	.7776516E+17	.1222195	.5817226	4.1175761
-30.00	3.70	.5111838E+16	.4072945E+17	.1255071	.5800076	4.1139159
-30.00	3.80	.2806320E+16	.2179051E+17	.1287863	.5782593	4.1101889
-30.00	3.90	.1570829E+16	.1189509E+17	.1320569	.5764785	4.1063942
-30.00	4.00	.8955935E+15	.6618400E+16	.1353187	.5746661	4.1025350
-40.00	3.00	.2055870E+29	.2688759E+30	.0764617	.4351209	3.8323415
-40.00	3.10	.6879412E+28	.8710487E+29	.0789785	.4345278	3.8311094
-40.00	3.20	.2396167E+28	.2940362E+29	.0814922	.4339175	3.8297979
-40.00	3.30	.8665781E+27	.1031606E+29	.0840028	.4332902	3.8284662
-40.00	3.40	.3246649E+27	.3752911E+28	.0865101	.4326459	3.8271449
-40.00	3.50	.1257471E+27	.1412665E+28	.0890141	.4319850	3.8257448
-40.00	3.60	.5025394E+26	.5491355E+27	.0915146	.4313076	3.8243278
-40.00	3.70	.2068683E+26	.2200453E+27	.0940117	.4306140	3.8228677
-40.00	3.80	.8757338E+25	.9074487E+26	.0965051	.4299045	3.8213656
-40.00	3.90	.3806819E+25	.3845476E+26	.0989948	.4291790	3.8198585
-40.00	4.00	.1696955E+25	.1672196E+26	.1014807	.4284380	3.8182973
-50.00	3.00	.8678318E+40	.1422861E+42	.0609920	.3433075	3.6539731
-50.00	3.10	.2099019E+40	.3331303E+41	.0630089	.3430167	3.6537635
-50.00	3.20	.5340403E+39	.8212935E+40	.0650243	.3427174	3.6528875
-50.00	3.30	.1424771E+39	.2125317E+40	.0670380	.3424093	3.6521494
-50.00	3.40	.3974549E+38	.5756037E+39	.0690501	.3420922	3.6516016
-50.00	3.50	.1156296E+38	.1627201E+39	.0710605	.3417666	3.6508740
-50.00	3.60	.3499864E+37	.4789804E+38	.0730690	.3414324	3.6501118
-50.00	3.70	.1099719E+37	.1464811E+38	.0750758	.3410895	3.6495361
-50.00	3.80	.3580015E+36	.4644498E+37	.0770808	.3407382	3.6488271
-50.00	3.90	.1205182E+36	.1523930E+37	.0790838	.3403785	3.6480800
-50.00	4.00	.4188309E+35	.5165337E+36	.0810849	.3400105	3.6472975

ORDER	ALPHA	BESSEL (ORDER+1)	BESSEL (ORDER)	U1	B1	B2
-10.00	.50	.2719652E+08	.4912297E+09	.0553642	2.5612666	8.4300120
-10.00	.60	.6350408E+07	.9572938E+08	.0663371	2.5397395	8.3710561
-10.00	.70	.1906052E+07	.2467173E+08	.0772565	2.5149636	8.3037318
-10.00	.80	.6870738E+06	.7797522E+07	.0881144	2.4872267	8.2289981
-10.00	.90	.2846930E+06	.2878510E+07	.0989029	2.4568320	8.1478436
-10.00	1.00	.1316026E+06	.1200592E+07	.1096148	2.4240896	8.0612511
-10.00	1.10	.6643611E+05	.5525151E+06	.1202431	2.3893098	7.9701714
-10.00	1.20	.3605457E+05	.2756861E+06	.1307812	2.3527969	7.8755077
-10.00	1.30	.2078477E+05	.1471767E+06	.1412233	2.3148441	7.7781014
-10.00	1.40	.1261041E+05	.8320218E+05	.1515635	2.2757293	7.6787245
-10.00	1.50	.7993216E+04	.4940280E+05	.1617968	2.2357131	7.5780753
-10.00	1.60	.5262016E+04	.3060760E+05	.1719186	2.1950360	7.4767765
-10.00	1.70	.3580323E+04	.1968026E+05	.1819246	2.1539181	7.3753769
-10.00	1.80	.2507815E+04	.1307440E+05	.1918111	2.1125582	7.2743540
-10.00	1.90	.1802272E+04	.8940963E+04	.2015748	2.0711344	7.1741180
-10.00	2.00	.1325169E+04	.6274096E+04	.2112127	2.0298043	7.0750172
-10.00	2.10	.9944980E+03	.4505650E+04	.2207224	1.9887063	6.9773430
-10.00	2.20	.7601909E+03	.3303714E+04	.2301019	1.9479605	6.8813358
-10.00	2.30	.5908173E+03	.2468430E+04	.2393494	1.9076702	6.7871899
-10.00	2.40	.4661461E+03	.1876114E+04	.2484636	1.8679230	6.6950593
-10.00	2.50	.3728551E+03	.1448299E+04	.2574434	1.8287924	6.6050624
-10.00	2.60	.3019878E+03	.1134064E+04	.2662883	1.7903390	6.5172868
-10.00	2.70	.2474072E+03	.8996704E+03	.2749976	1.7526122	6.4317932
-10.00	2.80	.2048344E+03	.7223378E+03	.2835715	1.7156507	6.3486195
-10.00	2.90	.1712375E+03	.5864100E+03	.2920098	1.6794846	6.2677841
-20.00	.50	.4499311E+22	.1710987E+24	.0262966	.9936543	4.9463512
-20.00	.60	.1709660E+21	.5419617E+22	.0315458	.9923863	4.9537269
-20.00	.70	.1108275E+20	.3012478E+21	.0367895	.9908982	4.9497342
-20.00	.80	.1061810E+19	.2526509E+20	.0420268	.9891879	4.9460565
-20.00	.90	.1370616E+18	.2900359E+19	.0472568	.9872587	4.9421994
-20.00	1.00	.2237751E+17	.4264119E+18	.0524786	.9851149	4.9372937
-20.00	1.10	.4417342E+16	.7656843E+17	.0576914	.9827595	4.9320619
-20.00	1.20	.1019792E+16	.1621438E+17	.0628943	.9801971	4.9262676
-20.00	1.30	.2684660E+15	.3943017E+16	.0680864	.9774318	4.9200546
-20.00	1.40	.7902475E+14	.1078586E+16	.0732670	.9744686	4.9133968
-20.00	1.50	.2560774E+14	.3264833E+15	.0784351	.9713123	4.9062927
-20.00	1.60	.9021167E+13	.1079216E+15	.0835900	.9679682	4.8987779
-20.00	1.70	.3419632E+13	.3853939E+14	.0887308	.9644418	4.8908592
-20.00	1.80	.1383020E+13	.1473541E+14	.0938569	.9607386	4.8825615
-20.00	1.90	.5925389E+12	.5987215E+13	.0989674	.9568647	4.8738903
-20.00	2.00	.2673182E+12	.2568847E+13	.1040615	.9528259	4.8648589
-20.00	2.10	.1263370E+12	.1157582E+13	.1091387	.9486284	4.8554869
-20.00	2.20	.6227337E+11	.5453101E+12	.1141981	.9442785	4.8457877
-20.00	2.30	.3189184E+11	.2674614E+12	.1192390	.9397824	4.8357748
-20.00	2.40	.1691263E+11	.1361058E+12	.1242609	.9351466	4.8254670
-20.00	2.50	.9260274E+10	.7163904E+11	.1292630	.9303775	4.8148783
-20.00	2.60	.5221493E+10	.3889535E+11	.1342446	.9254817	4.8040239
-20.00	2.70	.3025020E+10	.2173063E+11	.1392053	.9204654	4.7929201
-20.00	2.80	.1796958E+10	.1246637E+11	.1441444	.9153353	4.7815826
-20.00	2.90	.1092523E+10	.7329354E+10	.1490613	.9100978	4.7700259
-30.00	.50	.2204947E+39	.1279263E+41	.0172361	.6138757	4.0351318
-30.00	.60	.1356112E+37	.6557445E+38	.0206805	.6135631	4.1875784
-30.00	.70	.1886611E+35	.7820671E+36	.0241234	.6132130	4.1764606
-30.00	.80	.4769247E+33	.1730214E+35	.0275645	.6128066	4.1820310
-30.00	.90	.1902179E+32	.6135350E+33	.0310036	.6123478	4.1821599
-30.00	1.00	.1086925E+31	.3155959E+32	.0344404	.6118359	4.1823019

-30.00	1.10	.8306500E+29	.2193154E+31	.0378747	.6112725	4.1799215
-30.00	1.20	.8069517E+28	.1953585E+30	.0413062	.6106565	4.1783787
-30.00	1.30	.9588169E+27	.2143341E+29	.0447347	.6099884	4.1774111
-30.00	1.40	.1352262E+27	.2807859E+28	.0481599	.6092689	4.1763077
-30.00	1.50	.2210552E+26	.4285542E+27	.0515816	.6084988	4.1747142
-30.00	1.60	.4108977E+25	.7470921E+26	.0549996	.6076782	4.1732345
-30.00	1.70	.8549745E+24	.1463657E+26	.0584136	.6068081	4.1712443
-30.00	1.80	.1965718E+24	.3179574E+25	.0618233	.6058888	4.1692744
-30.00	1.90	.4939867E+23	.7573159E+24	.0652286	.6049207	4.1672871
-30.00	2.00	.1344435E+23	.1958983E+24	.0686292	.6039048	4.1650783
-30.00	2.10	.3931739E+22	.5458862E+23	.0720249	.6028415	4.1627924
-30.00	2.20	.1227202E+22	.1627257E+23	.0754154	.6017315	4.1604220
-30.00	2.30	.4064357E+21	.5157780E+22	.0788005	.6005756	4.1579347
-30.00	2.40	.1421002E+21	.1729132E+22	.0821800	.5993745	4.1553589
-30.00	2.50	.5221298E+20	.6102945E+21	.0855537	.5981289	4.1526882
-30.00	2.60	.2008293E+20	.2258504E+21	.0889214	.5968396	4.1499267
-30.00	2.70	.8057832E+19	.8731675E+20	.0922828	.5955074	4.1470590
-30.00	2.80	.3361988E+19	.3515338E+20	.0956377	.5941330	4.1441137
-30.00	2.90	.1454615E+19	.1469517E+20	.0989859	.5927173	4.1410830
-40.00	.50	.3882378E+57	.3028766E+59	.0128184	.4438940	3.0315955
-40.00	.60	.3860391E+54	.2509863E+56	.0153809	.4438167	2.6245716
-40.00	.70	.1151013E+52	.6414904E+53	.0179428	.4436357	3.6021373
-40.00	.80	.7665480E+49	.3738535E+51	.0205040	.4434618	3.8733340
-40.00	.90	.9429920E+47	.4088531E+49	.0230643	.4432961	3.8008818
-40.00	1.00	.1882152E+46	.7345342E+47	.0256237	.4430991	3.8301393
-40.00	1.10	.5556492E+44	.1971636E+46	.0281821	.4428847	3.8340074
-40.00	1.20	.2266103E+43	.7371982E+44	.0307394	.4426479	3.8477442
-40.00	1.30	.1212167E+42	.3640642E+43	.0332954	.4423951	3.8414189
-40.00	1.40	.8168396E+40	.2278487E+42	.0358501	.4421194	3.8468862
-40.00	1.50	.6716152E+39	.1748847E+41	.0384033	.4418255	3.8458169
-40.00	1.60	.6566324E+38	.1603301E+40	.0409550	.4415116	3.8456792
-40.00	1.70	.7474607E+37	.1718100E+39	.0435051	.4411788	3.8438033
-40.00	1.80	.9735004E+36	.2113852E+38	.0460534	.4408257	3.8436187
-40.00	1.90	.1429598E+36	.2941567E+37	.0485999	.4404533	3.8431984
-40.00	2.00	.2338020E+35	.4571409E+36	.0511444	.4400618	3.8425768
-40.00	2.10	.4213592E+34	.7848456E+35	.0536869	.4396511	3.8419175
-40.00	2.20	.8292415E+33	.1474803E+35	.0562273	.4392217	3.8406940
-40.00	2.30	.1768130E+33	.3008796E+34	.0587654	.4387734	3.8397767
-40.00	2.40	.4056671E+32	.6617607E+33	.0613012	.4383062	3.8389673
-40.00	2.50	.9954749E+31	.1559461E+33	.0638345	.4378205	3.8380570
-40.00	2.60	.2598885E+31	.3916024E+32	.0663654	.4373166	3.8369896
-40.00	2.70	.7184431E+30	.1042830E+32	.0688936	.4367945	3.8358524
-40.00	2.80	.2094221E+30	.2932296E+31	.0714191	.4362545	3.8346629
-40.00	2.90	.6412767E+29	.8672716E+30	.0739419	.4356965	3.8335581

*** ERROR : ERROR IN MMBSKR SUBROUTINE CALL. FATAL. ***

-50.00	.60	.1518322E+73	.1240153E+75	.0122430	.3476230	-1.2845790
-50.00	.70	.9697374E+69	.6789576E+71	.0142827	.3474774	2.2848365
-50.00	.80	.1700399E+67	.1041778E+69	.0163221	.3473551	3.6350570
-50.00	.90	.6447606E+64	.3511572E+66	.0183610	.3472819	3.4684270
-50.00	1.00	.4491816E+62	.2201925E+64	.0203995	.3471858	3.5436866
-50.00	1.10	.5118470E+60	.2281218E+62	.0224374	.3470722	3.6830824
-50.00	1.20	.8755477E+58	.3577341E+60	.0244748	.3469644	3.6328631
-50.00	1.30	.2106372E+57	.7945103E+58	.0265116	.3468455	3.6085025
-50.00	1.40	.6774911E+55	.2373193E+57	.0285477	.3467100	3.6392699
-50.00	1.50	.2798615E+54	.9150879E+55	.0305830	.3465666	3.6514992
-50.00	1.60	.1437450E+53	.4406977E+54	.0326176	.3464150	3.6535100
-50.00	1.70	.8940287E+51	.2580069E+53	.0346514	.3462555	3.6472528
-50.00	1.80	.6587070E+50	.1795614E+52	.0366842	.3460835	3.6551415
-50.00	1.90	.5644622E+49	.1457949E+51	.0387162	.3459028	3.6577911
-50.00	2.00	.5538954E+48	.1359347E+50	.0407472	.3457132	3.6582061

-50.00	2.10	.6141978E+47	.1435810E+49	.0427771	.3455146	3.6568561
-50.00	2.20	.7608843E+46	.1698177E+48	.0448060	.3453065	3.6562851
-50.00	2.30	.1042708E+46	.2226405E+47	.0468337	.3450881	3.6580850
-50.00	2.40	.1567085E+45	.3207280E+46	.0488602	.3448616	3.6565920
-50.00	2.50	.2563422E+44	.5037621E+45	.0508856	.3446252	3.6566909
-50.00	2.60	.4533610E+43	.8568595E+44	.0529096	.3443799	3.6560848
-50.00	2.70	.8617716E+42	.1568788E+44	.0549323	.3441256	3.6551798
-50.00	2.80	.1751347E+42	.3075039E+43	.0569537	.3438618	3.6549398
-50.00	2.90	.3787337E+41	.6422091E+42	.0589736	.3435891	3.6545524

APPENDIX 3

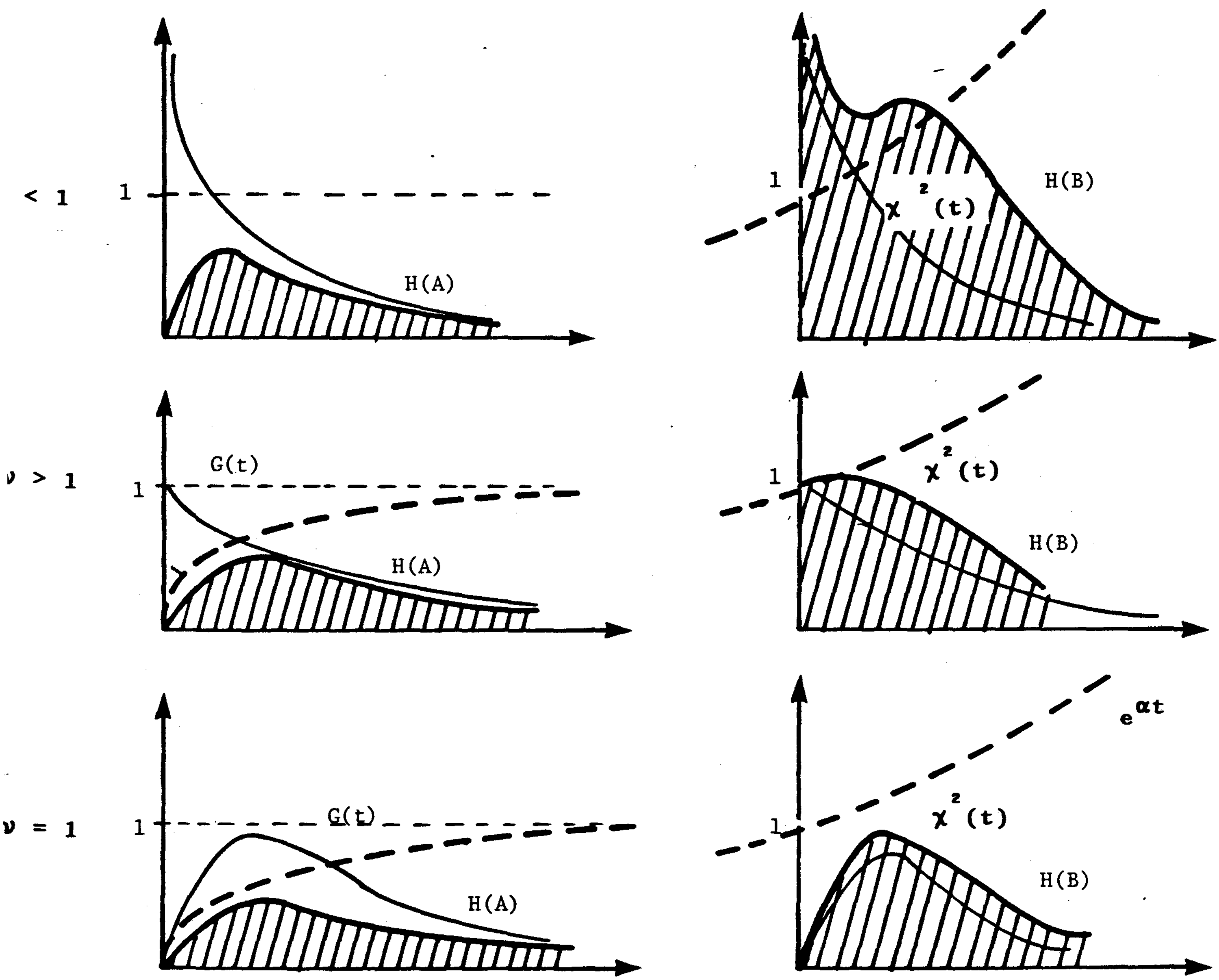
RELATION OF TYPE A AND TYPE B HALPHEN DISTRIBUTIONS WITH THE GAMMA AND χ^2 DISTRIBUTIONS

TYPE A: $K \cdot t^{\nu-1} e^{-\alpha t} e^{-\alpha \frac{1}{t}}$

G(t)

TYPE B: $K \cdot t^{2\nu-1} e^{-t^2} e^{\alpha t}$

$\chi^2(t)$



LOWER TAIL CORRECTION

DECREASE

UPPER TAIL CORRECTION

INCREASE

APPENDIX 4

NHACL 1, NHALF 2 COMPUTER PROGRAMS LISTINGS

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                              BBBB    OOO   BBBB   EEEEE  EEEEE
                              B   B  O   O  B   B  E      E
                              B   B  O   O  B   B  E      E
                              BBBB   O   O  BBBB   EEEE   EEEE
                              B   B  O   O  B   B  E      E
                              B   B  O   O  B   B  E      E
                              BBBB    OOO   BBBB   EEEEE  EEEEE

                NN      NN  HH      HH    AAAAAA    LL          FFFFFFFFFF      11
                NN      NN  HH      HH    AAAAAA    LL          FFFFFFFFFF      11
                NN      NN  HH      HH  AA      AA  LL          FF            1111
                NN      NN  HH      HH  AA      AA  LL          FF            1111
                NNNN    NN  HH      HH  AA      AA  LL          FF              11
                NNNN    NN  HH      HH  AA      AA  LL          FF              11
                NN  NN  NN  HHHHHHHHHH  AA      AA  LL          FFFFFFFF        11
                NN  NN  NN  HHHHHHHHHH  AA      AA  LL          FFFFFFFF        11
                NN    NNNN  HH      HH  AAAAAAAAAA  LL          FF              11
                NN    NNNN  HH      HH  AAAAAAAAAA  LL          FF              11
                NN      NN  HH      HH  AA      AA  LL          FF              11
                NN      NN  HH      HH  AA      AA  LL          FF              11
                NN      NN  HH      HH  AA      AA  LLLLLLLLLL  FF            111111
                NN      NN  HH      HH  AA      AA  LLLLLLLLLL  FF            111111

                      CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;;;        11
                      CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;      1111
                    CC          DD      DD  CC          CC              ;;;;      1111
                    CC          DD      DD  CC          CC                          11
                    CC          DD      DD  CC          CC                          11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
              ....  CC          DD      DD  CC          CC                ;;        11
              ....  CC          DD      DD  CC          CC                ;;        11
              ....    CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;        111111
              ....    CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;        111111
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File _DRCO:[BOBEE]NHALF1.CDCC;1 (2945,122,0), last revised on 5-MAR-1987 12:22, is a 19 block sequential file owned by UIC [BOBEE]. The records are variable length with implied (CR) carriage control. The longest record is 70 bytes.

Job NHALF1 (23) queued to SYS$PRINT on 5-MAR-1987 12:39 by user BOBEE, UIC [BOBEE], under account BOBEE at priority 4, started on printer _LPA0: on 5-MAR-1987 12:39 from queue LPA0.


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FFFFFFFFFF   999999999999999999999999  Digital Equipment Corporation - VAX/VMS Version V4.2  999999999999999999999999   FFFFFFFFFF
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```
C*********************************************************************
C*
C*   PROGRAM  :  NHALF1
C*
C*   AUTHOR   :  MARCO LAVOIE
C*   DATE     :  MARCH 1987
C*
C*   PROGRAMMED FOR M. VACLAV DVORAK
C*
C*   DESCRIPTION :   THIS PROGRAM CALCULATES THE DISTRIBUTION
C*                   FUNCTION FOR THE HALPHEN - TYPE A PROBABI-
C*                   LITY DENSITY FUNCTION BY NUMERICAL INTEGRATION.
C*                   IT USES IMSL ROUTINES TO CALCULATES INTERMEDIATE
C*                   RESULTS.
C*
C*   INPUT   :  ALPHA, NHI, M AND X ITERATION INTERVALES
C*              CORRESPONDING TO THEIR DEFINITION IN THE
C*              HALPHEN - TYPE A DISTRIBUTION.
C*
C*   OUTPUT  :  INTERMEDIATES RESULTS OF CALCULATIONS AND
C*              F(X) FOR EACH X IN THE SPECIFIED BOUNDS
C*              ACCORDING TO THE ITERATION STEP.
C*              THE RESULTS ARE PRINTED ON THE SCREEN AND IN
C*              IN A LOCAL FILE NAMED "TAPE1".
C*
C*   CONSTRAINTS  :  PARAMETER V (ORDER OF THE MODIFIED BESSEL
C*                   FUNCTIONS) MUST DIFFER FROM INTEGERS!
C*                   (V <> + OR - (0, 1, 2, 3, ...))
C*
C*
C*   THE PROGRAM USES THE FOLLOWING IMSL ROUTINES TO ACHIEVE
C*   THE REQUIRED CALCULATIONS :
C*
C*        MMBSKR  :  MODIFIED BESSEL FUNCTION OF THE SECOND
C*                   KIND OF NONNEGATIVE REAL FRACTIONAL
C*                   ORDER FOR REAL POSITIVE ARGUMENTS SCALED
C*                   BY EXP(ARG).
C*
C*        MMBSIN  :  MODIFIED BESSEL FUNCTION OF THE FIRST
C*                   KIND OF NONNEGATIVE INTEGER ORDER FOR
C*                   REAL ARGUMENTS.
C*
C*        MMBSIR  :  MODIFIED BESSEL FUNCTION OF THE FIRST
C*                   KIND OF NONNEGATIVE REAL ORDER FOR REAL
C*                   POSITIVE ARGUMENTS WITH EXPONENTIAL
C*                   SCALING OPTION.
C*
C*        DCADRE : NUMERICAL INTEGRATION OF A FUNCTION USING
C*                 CAUTIOUS ADAPTIVE ROMBERG EXTRAPOLATION.
C*
C*
C*   HERE IS THE FORMULA USED TO ESTIMATE THE DISTRIBUTION
C*   FUNCTION :
C*
C*     F(A) = COEFMLT * (COEFADD + NUMINTEG1)
C*
C*     WHERE :
C*
C*        COEFMLT = 1/(2*[BESSEL SECOND KIND ORDER V](2*ALPHA))
C*
C*        COEFADD = 1/SIN(V*PI) * (NUMINTEG2 - PI *
C*                       [MODIFIED BESSEL FIRST KIND ORDER V](2*ALPHA)
C*
C*            NUMINTEG2 = EXP(2*ALPHA*COS(T)) * COS(V*T) DT
C*                             OVER [0,PI]
C*
C*         NUMINTEG1 = EXP(2*T - 2*ALPHA*COSH(T)) DT OVER
C*                                  [0,LN(X/M)]
```

```
C*
C*   TO RUN THE PROGRAM, YOU MUST DO :
C*
C*              /GET,NHALF1
C*              /FTN5,I=NHALF1,L=0
C*              /RUN,SC,IMSL
C*              /LDSET(LIB=IMSL)
C*              /LDR)? LGO
C*
C********************************************************************

      PROGRAM NHALF1 (INPUT, OUTPUT, TAPE1)

      COMMON   /CONST/ V, ALPHA

      INTEGER  IER
      REAL     DCADRE, G, H, A, B, AERR, RERR, ERROR, RESULT, M
      REAL     V, ALPHA, IRES(1), PI, FRES, FSIR, IRES2(51)
      REAL     COEFMLT, COEFADD, COEFIND, X, XMIN, XMAX, XSTEP

C*
C*    FUNCTIONS TO BE INTEGRATED (DEFINED LATER).
C*

      EXTERNAL G, H

      DATA     PI/3.1415926535898/,IER/0/,AERR/0.0/,RERR/1.0E-5/

      PRINT 901
      WRITE (UNIT=1, FMT=901)

C*
C*    READING THE INPUT PARAMETERS.
C*

      PRINT 902
      READ (UNIT=*, FMT=*) V

      PRINT 903
      READ (UNIT=*, FMT=*) ALPHA

      PRINT 904
      READ (UNIT=*, FMT=*) M

      PRINT 906
      READ (UNIT=*, FMT=*) XMIN, XMAX, XSTEP

C*
C*    CALCULATING THE MULTIPLICATIVE COEFFICIENT OF THE
C*    FORMULA (CALLED COEFMLT).
C*

      CALL MMBSKR (2*ALPHA, ABS(V), 1, IRES, IER)
      IF (IER .GT. 0) THEN
         PRINT 915, IER
         WRITE (UNIT=1, FMT=915) IER
         STOP
      ENDIF

      IRES(1) = IRES(1) / EXP(2*ALPHA)
      PRINT *,'RETURNED FROM MMBSKR CALL : ',IRES(1)

      COEFMLT = 1 / (2*IRES(1))
      PRINT *, 'CALCULATED COEFMLT : ', COEFMLT
```

```
C*
C*    CALCULATING THE FIRST COEFFICIENT INSIDE THE
C*    PARENTHISIS (CALLED COEFADD). WE MUST FIRST CALCULATE
C*    THE MODIFIED BESSEL FUNCTION OF THE FIRST KIND FOR
C*    PARAMETERS 2*ALPHA AND V. NOTE THAT V MUST BE REAL,
C*    BECAUSE IF NOT, THE DEFINED FUNCTION COMPUTES A THEORICAL
C*    DIVISION BY ZERO. FOR V REAL, IT USES THE IMSL "MMBSIR"
C*    FUNCTION. IF V )= 1, THEN WE MUST USE AN ITERATIVE
C*    TECHNIQUE TO CALCULATE THE MODIFIED BESSEL FUNCTION BECAUSE
C*    THE "MMBSIR" ROUTINE RESTRICT V TO [0,1[
C*

      IF (FLOAT(INT(V)) .EQ. V) THEN
         CALL MMBSIN (2*ALPHA, INT(V)+1, IRES2, IER)
         IRES(1) = IRES2(INT(V)+1)
      ELSE
         IRES(1) = FSIR (ALPHA, V, IER)
      ENDIF

      IF (IER .GT. 0) THEN
         PRINT 916, IER
         WRITE (UNIT=1, FMT=916) IER
         STOP
      ELSE
         PRINT *,'RETURNED FOR I(ORDER) CALL : ', IRES(1)
      ENDIF

C*
C*    THE FIRST INTEGRAL (CALLED NUMINTEG2 IN THE HEADINGS)
C*    IS CALCULATED USING THE IMSL "DCADRE" ROUTINE. THE
C*    FUNCTION IS DEFINED IN THE H ROUTINE, DEFINED LATER.
C*

      RESULT = DCADRE (H, 0, PI, AERR, RERR, ERROR, IER)
      IF ((IER .NE. 0) .AND. (IER .NE. 65)) THEN
         PRINT 917, IER
         WRITE (UNIT=1, FMT=917) IER
         STOP
      ENDIF

C*
C*    THE COEFFICIENT "COEFADD" DEFINED IN THE HEADINGS IS
C*    CALCULATED HERE, AS DESCRIBED BEFORE.
C*

      COEFADD = (1 / SIN(V * PI)) * (RESULT - PI * IRES(1))
      PRINT *,'CALCULATED COEFADD : ', COEFADD

      COEFIND = COEFMLT * COEFADD

      PRINT 908
      WRITE (UNIT=1, FMT=908)
      PRINT 909
      WRITE (UNIT=1, FMT=909)

C*
C*    F(X) IS CALCULATED FOR EACH X IN THE INTERVAL [XMIN, XMAX]
C*    ACCORDING TO THE ITERATION STEP. THE RESULTS ARE PRINTED
C*    ON THE SCREEN AND IN A LOCAL FILE NAMED "TAPE1".
C*

      DO 10 X = XMIN, XMAX+0.0009, XSTEP

         RESULT = DCADRE (G, 0, LOG(X/M), AERR, RERR, ERROR, IER)
```

```
      IF ((IER .NE. 0) .AND. (IER .NE. 65)) THEN
         PRINT 917, IER
         WRITE (UNIT=1, FMT=917) IER
      ELSE
         FRES = COEFIND + COEFMLT * RESULT
         PRINT 918, X, RESULT, FRES
         WRITE (UNIT=1, FMT=918) X, RESULT, FRES
      ENDIF

 10   CONTINUE

      STOP

 901  FORMAT(//20X,'DISTRIBUTION FUNCTION - HALPHEN TYPE A'//)
 902  FORMAT('WHAT IS THE V PARAMETER OF THE DISTRIBUTION FUNCTION?')
 903  FORMAT(/'WHAT IS THE ALPHA PARAMETER OF THE DISTRIBUTION',
     *        ' FUNCTION?')
 904  FORMAT(/'WHAT IS THE M PARAMETER OF THE DISTRIBUTION',
     *         ' FUNCTION?')
 906  FORMAT(/'WHAT ARE THE ITERATIVE BOUNDS OF THE X ',
     *         'PARAMETER [MIN, MAX, STEP] ?')
 908  FORMAT(//6X,'X',15X,'RESULT OF INTEGRATION',25X,
     *        'F(X)')
 909  FORMAT(3X,74('-'))
 915  FORMAT(/'*** FATAL ERROR IN MMBSKR CALL ***'/
     *         '*** ERROR CODE : ', I3)
 916  FORMAT(/'*** FATAL ERROR IN MMBSIR CALL ***'/
     *         '*** ERROR CODE : ', I3)
 917  FORMAT('*** ERROR IN NUMERICAL INTEGRATION ***'/
     *        '*** ERROR CODE (DCADRE CALL) : ', I3)
 918  FORMAT(1X,F10.3,16X,F12.5,25X,E12.5)

      END

C****************************************************************
C*
C*  FUNCTION WHICH CALCULATES THE MODIFIED BESSEL FUNCTION
C*  OF THE FIRST KIND FOR V PARAMETER VALUE >= 1.0 . THE
C*  FUNCTION USES THE IMSL "MMBSIR" ROUTINE TO CALCULATE
C*  INITIAL VALUE AND THEN ITERATE, CALCULATING SUCCESSIVE
C*  MODIFIED BESSEL VALUES ACCORDING TO THE RECURRENT
C*  DEFINITION OF THE FUNCTION (SEE LITTERATURE).
C*
C****************************************************************

      REAL FUNCTION FSIR (ALPHA, V, IER)

      REAL  ALPHA, V, VPOS, ORDSTR, IRES(1), IVEC(50), PI
      INTEGER  IER, K

      PI = 3.1415926535898

      ORDSTR = V - FLOAT (INT (V))

C*
C*    CALCULATING THE INITIAL VALUE OF THE MODIFIED BESSEL
C*    FUNCTION FOR V < 1.0 . SUCCESSIVE VALUES ARE THEN
C*    EVALUATED.
C*

      CALL MMBSIR (2*ALPHA, ORDSTR, 1, 1, IRES, IER)
      IF (IER .GT. 0) RETURN

      IVEC(2) = IRES(1)

      CALL MMBSKR (2*ALPHA, ORDSTR, 1, IRES, IER)
      IF (IER .GT. 0) RETURN
      IRES(1) = IRES(1) / EXP(2*ALPHA)
```

```
      IVEC(1) = IVEC(2) + (2 * SIN(ORDSTR*PI) * IRES(1)) / PI
      DO 10 K = 3, INT(V) + 2
         IVEC(K) = (-1*((K-3+ORDSTR)/ALPHA) * IVEC(K-1)) + IVEC(K-2)
 10   CONTINUE

      FSIR = IVEC(INT(V)+2)

      RETURN
      END


C****************************************************
C*
C*  FUNCTION WHICH IS TO BE INTEGRATED BY THE
C*      DCADRE FUNCTION.
C*
C****************************************************

      REAL FUNCTION G (X)

      COMMON /CONST/ V, ALPHA
      REAL X, V, ALPHA

      G = EXP(V*X - 2*ALPHA*COSH(X))

      RETURN
      END


C****************************************************
C*
C*   FUNCTION WHICH IS TO BE INTEGRATED BY THE
C*        DCADRE FUNCTION.
C*
C****************************************************

      REAL FUNCTION H (X)

      COMMON /CONST/ V, ALPHA
      REAL X, V, ALPHA

      H = EXP(2 * ALPHA * COS(X)) * COS(V * X)

      RETURN
      END
```

```
                              BBBB     OOO    BBBB   EEEEE  EEEEE
                              B   B   O   O   B   B  E      E
                              B   B   O   O   B   B  E      E
                              BBBB    O   O   BBBB   EEEE   EEEE
                              B   B   O   O   B   B  E      E
                              B   B   O   O   B   B  E      E
                              BBBB     OOO    BBBB   EEEEE  EEEEE


              NN      NN  HH      HH    AAAAAA    LL          FFFFFFFFFF    222222
              NN      NN  HH      HH    AAAAAA    LL          FFFFFFFFFF    222222
              NN      NN  HH      HH  AA      AA  LL          FF          22      22
              NN      NN  HH      HH  AA      AA  LL          FF          22      22
              NNNN    NN  HH      HH  AA      AA  LL          FF                  22
              NNNN    NN  HH      HH  AA      AA  LL          FF                  22
              NN  NN  NN  HHHHHHHHHH  AA      AA  LL          FFFFFFFF          22
              NN  NN  NN  HHHHHHHHHH  AA      AA  LL          FFFFFFFF          22
              NN    NNNN  HH      HH  AAAAAAAAAA  LL          FF              22
              NN    NNNN  HH      HH  AAAAAAAAAA  LL          FF              22
              NN      NN  HH      HH  AA      AA  LL          FF            22
              NN      NN  HH      HH  AA      AA  LL          FF            22
              NN      NN  HH      HH  AA      AA  LLLLLLLLLL  FF          2222222222
              NN      NN  HH      HH  AA      AA  LLLLLLLLLL  FF          2222222222


                      CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;;;        11
                      CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;      1111
                    CC          DD      DD  CC          CC              ;;;;      1111
                    CC          DD      DD  CC          CC                          11
                    CC          DD      DD  CC          CC                          11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
                    CC          DD      DD  CC          CC              ;;;;        11
             ....   CC          DD      DD  CC          CC                ;;        11
             ....   CC          DD      DD  CC          CC                ;;        11
             ....     CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;        111111
             ....     CCCCCCCC  DDDDDDDD      CCCCCCCC    CCCCCCCC      ;;        111111
```

File _DRC0:[BOBEE]NHALF2.CDC;1 (7549,13,0), last revised on 5-MAR-1987 12:07, is a 10 block sequential file owned by UIC [BOBEE]. The records are variable length with implied (CR) carriage control. The longest record is 70 bytes.

Job NHALF2 (24) queued to SYS$PRINT on 5-MAR-1987 12:39 by user BOBEE, UIC [BOBEE], under account BOBEE at priority 4, started on printer _LPA0: on 5-MAR-1987 12:40 from queue LPA0.

```
C*********************************************************************
C*
C*   PROGRAM   :  NHALF2
C*
C*   AUTHOR    :  MARCO LAVOIE
C*   DATE      :  MARCH 1987
C*
C*   PROGRAMMED FOR M. VACLAV DVORAK
C*
C*   DESCRIPTION :    THIS PROGRAM CALCULATES THE PROBABILITY OF
C*                    [A < X <= B] FOR THE HALPHEN - TYPE A PROBABILITY
C*                    DENSITY FUNCTION BY NUMERICAL INTEGRATION. IT
C*                    USES IMSL ROUTINES TO CALCULATES INTERMEDIATE
C*                    RESULTS.
C*
C*   INPUT    :  ALPHA, V, M, THE VALUES A AND B (BOUNDS)
C*               CORRESPONDING TO THEIR DEFINITION IN THE
C*               HALPHEN - TYPE A DISTRIBUTION.
C*
C*   OUTPUT   :  INTERMEDIATE RESULT OF INTEGRATION AND
C*               THE FINAL RESULT FOR F(B) - F(A).
C*               THE RESULTS ARE PRINTED ON THE SCREEN AND IN
C*               IN A LOCAL FILE NAMED "TAPE1".
C*
C*   CONSTRAINTS  :  A <>0 AND B <> 0.
C*
C*
C*   THE PROGRAM USES THE FOLLOWING IMSL ROUTINES TO ACHIEVE
C*   THE REQUIRED CALCULATIONS :
C*
C*        MMBSKR  :  MODIFIED BESSEL FUNCTION OF THE SECOND
C*                   KIND OF NONNEGATIVE REAL FRACTIONAL
C*                   ORDER FOR REAL POSITIVE ARGUMENTS SCALED
C*                   BY EXP(ARG).
C*
C*        DCADRE : NUMERICAL INTEGRATION OF A FUNCTION USING
C*                 CAUTIOUS ADAPTIVE ROMBERG EXTRAPOLATION.
C*
C*
C*   HERE IS THE FORMULA USED :
C*
C*     F(B) - F(A) = COEFMLT * NUMINTEG
C*
C*     WHERE :
C*
C*        COEFMLT = 1/(2*[BESSEL SECOND KIND ORDER V](2*ALPHA))
C*
C*        NUMINTEG = EXP(-1*ALPHA*(T+(I/T))) * T**(V-1) DT OVER
C*                                        [A/M,B/M]
C*
C*
C*   TO RUN THE PROGRAM, YOU MUST DO :
C*
C*              /GET,NHALF2
C*              /FTN5,I=NHALF2,L=0
C*              /RUN,SC,IMSL
C*              /LDSET(LIB=IMSL)
C*              /LDR)? LGO
C*
C*********************************************************************

      PROGRAM NHALF2 (INPUT, OUTPUT, TAPE1)
```

```
      COMMON   /CONST/ V, ALPHA

      INTEGER  IER
      REAL     DCADRE, G, A, B, AERR, RERR, ERROR, RESULT, M
      REAL     V, ALPHA, IRES(1), PI, FRES
      REAL     COEFMLT, INFB, SUPB

C*
C*    FUNCTIONS TO BE INTEGRATED (DEFINED LATER).
C*

      EXTERNAL G

      DATA     PI/3.1415926535898/,IER/0/,AERR/0.0/,RERR/1.0E-5/

      PRINT 901
      WRITE (UNIT=1, FMT=901)

C*
C*    READING THE INPUT PARAMETERS.
C*

      PRINT 902
      READ (UNIT=*, FMT=*) V

      PRINT 903
      READ (UNIT=*, FMT=*) ALPHA

      PRINT 904
      READ (UNIT=*, FMT=*) M

      PRINT 906
      READ (UNIT=*, FMT=*) INFB, SUPB

C*
C*    CALCULATING THE MULTIPLICATIVE COEFFICIENT OF THE
C*    FORMULA (CALLED COEFMLT).
C*

      CALL MMBSKR (2*ALPHA, ABS(V), 1, IRES, IER)
      IF (IER .GT. 0) THEN
         PRINT 915, IER
         WRITE (UNIT=1, FMT=915) IER
         STOP
      ENDIF

      IRES(1) = IRES(1) / EXP(2*ALPHA)
      PRINT *,'RETURNED FROM MMBSKR CALL : ',IRES(1)

      COEFMLT = 1 / (2*IRES(1))
      PRINT *, 'CALCULATED COEFMLT : ', COEFMLT

      RESULT = DCADRE (G, INFB/M, SUPB/M, AERR, RERR, ERROR, IER)

      IF ((IER .NE. 0) .AND. (IER .NE. 65)) THEN
         PRINT 917, IER
         WRITE (UNIT=1, FMT=917) IER
      ELSE
         FRES = COEFMLT * RESULT
         PRINT 918, RESULT, FRES
         WRITE (UNIT=1, FMT=918) RESULT, FRES
      ENDIF

      STOP
```

```
 901  FORMAT(//20X,'DISTRIBUTION FUNCTION - HALPHEN TYPE A'//)
 902  FORMAT('WHAT IS THE V PARAMETER OF THE DISTRIBUTION FUNCTION?')
 903  FORMAT(/'WHAT IS THE ALPHA PARAMETER OF THE DISTRIBUTION',
     *        ' FUNCTION?')
 904  FORMAT(/'WHAT IS THE M PARAMETER OF THE DISTRIBUTION',
     *        ' FUNCTION?')
 906  FORMAT(/'WHAT ARE THE TWO VALUES TO BE DIFFERENTIATED ',
     *        ' [A, B] ?')
 915  FORMAT(/'*** FATAL ERROR IN MMBSKR CALL ***'/
     *        '*** ERROR CODE : ', I3)
 916  FORMAT(/'*** FATAL ERROR IN MMBSIR CALL ***'/
     *        '*** ERROR CODE : ', I3)
 917  FORMAT('*** ERROR IN NUMERICAL INTEGRATION ***'/
     *       '*** ERROR CODE (DCADRE CALL) : ', I3)

 918  FORMAT(/2X,'RESULT OF INTEGRATION : ',E16.7/
     *        2X,'F(B) - F(A) : ',E16.7)

      END

C****************************************************
C*
C*  FUNCTION WHICH IS TO BE INTEGRATED BY THE
C*      DCADRE FUNCTION.
C*
C****************************************************

      REAL FUNCTION G (X)

      COMMON /CONST/ V, ALPHA
      REAL X, V, ALPHA

      G = X ** (V - 1) * EXP(-1 * ALPHA * (X + (1/X)))

      RETURN
      END
```