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ON THE CONSTRUCTION OF CONFIDENCE INTERVALS FOR THE QUANTILES OF THE GAMMA DISTRIBUTION

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We present an approximate method for constructing confidence intervals for the quantiles of the 2-parameter gamma distribution when both scale and shape parameters θ and κ, respectively, of the distribution are unknown. There is a relationship between these confidence intervals and one-sided tolerance limits for the distribution. Simulation shows that the method is highly accurate for many practical applications. The method is also quite general and might be useful for other distributions as well.

KEYS WORDS: Confidence bounds; percentiles; tolerance limits; approximate methods; small and moderate sample sizes; simulation.

1. INTRODUCTION

A well known problem in statistics is how to calculate a lower γ probability tolerance limit for proportion 1-p of a given statistical distribution. This can also be regarded as a lower γ level confidence limit for the pth quantile x_p of the distribution because if $L(x_p;\gamma)$ is a lower γ LCL for x_p and R(t) is the reliability at time t, then

$$\Pr\left\{ R(L(x_p;\gamma)) \geq 1-p \right\} = \gamma$$

where by definition $R(t) = 1-F(t)$ and $F(\cdot)$ is the cdf of the distribution.

Ways of obtaining exact confidence limits for quantiles of such distributions as the normal/log normal, extreme value type 1 /Weibull, and exponential, are well known and well documented in the statistical literature. One distribution for which a method for constructing exact confidence limits for its quantiles is not yet known, is the gamma distribution in the case where the two parameters θ and κ of the distribution are unknown. This because unlike the other distributions mentioned above, the parameters of the gamma distribution are not of the location-scale type. Approximate methods for the gamma distribution have been sought for quite a long time and it is only recently that Bain, Engelhardt and Shiue (1984) have had some success in developing a usefully approximate method for this distribution. The tolerance limits obtained by Bain *et al.* for the gamma distribution are calculated by first assuming the distribution mean known and the shape parameter (κ) unknown, and then replacing the distribution mean by the sample mean. This method is shown to be satisfactory for quite a broad range of values of the shape parameter, and for moderate sample sizes, but not for all probability levels p.

In fact, for values of p greater than .20, the method of Bain *et al.* fails to produce good results. Therefore, in situations where one is interested in constructing confidence intervals for quantiles situated at the right tail of the distribution (values of p close to unity), this method does not provide an adequate solution. This is exactly the kind of situation which we will be interested in, in the present study. To handle this kind of problem, we shall develop a new approximate method which we shall discuss and test using simulation. We start by presenting one area of application where quantiles of the gamma distribution play an important role.

2. DESCRIPTION OF THE PROBLEM

The gamma distribution (also known as the Pearson type 3 distribution) is a widely used distribution in hydrology. For example, in the area of flood frequency analysis the 3-parameter gamma distribution is very frequently used to fit annual maximum flood series which consists of the maximum discharge value recorded each year at a given gaging station over an n-year period. See Bobée (1975), United States Water Resources Council (1981) or Rao (1981) for more detail on this subject. In these references, either the annual maximum flood discharge or its logarithm is assumed follow a 3-parameter gamma distribution.

The probability density function of the 3-parameter gamma distribution is given by:

$$f(x;\ \Theta,\kappa,\eta) = \frac{(y-\eta)^{\kappa-1} \exp\left[-(y-\eta)/\Theta\right]}{\Theta^{\kappa}\Gamma(\kappa)}$$

$$\eta<y<\infty;\ \Theta,\kappa>0,\ -\infty<\eta<\infty$$

where $\Gamma(\kappa) = \int_0^\infty t^{\kappa-1}e^{-t}dt$ is the gamma function. In the present study, we shall restrict our attention the case where the location parameter η is equal to zero but we mention that in flood frequency analysis η signifies the lower bound of flood flows and for this reason it is frequently assumed to be greater than zero (the value of η depends on the hydrogeographical conditions of the watershed and has to be estimated from the recorded flood sample but on certain rivers, especially those in arid or semi-arid regions η is sometimes taken to be equal to zero and the 2-parameter gamma distribution is obtained).

There are different ways of describing the shape of the gamma distribution one of which is by using the coefficient of skewness $\gamma = 2/\sqrt{\kappa}$. The mean and variance of the gamma distribution are respectively given by $\mu_x = \eta + \kappa\theta$ and $\sigma_x^2 = \kappa\theta^2$.

The design of flood control structures and other hydraulic works is usually done on the basis of a specified annual flood discharge X_T corresponding to a specified return period T (in years) where T = 1/Pr [X ≥

X_T] is the return period of the design flood value X_T. In other words, if a dam is constructed to withstand the 100-year flood X_T (T = 100), say, then the dam will be expected to be overtopped on the average once every 100 years. The probability that the dam will be equal to .01 ($1 - p = Pr [X \geq X_T] = 1/100 = .01$). This shows how the estimation of quantiles X_T(or X_p) and the construction of confidence intervals for these quantiles comes to play an important role in hydraulic design.

This article is to present an approximate method for constructing confidence intervals for quantiles of the gamma distribution. Our method has been applied successfully to the 3-parameter gamma distribution at least for sample sizes and shape parameter values commonly found in flood frequency analysis. From a hydrologic perspective, the method has been discussed in (Ashkar and Bobée 1987) but in the present study we shall give a brief description of the method and apply it to the 2-parameter gamma distribution. It will be shown that for this distribution the method can give excellent results for many sample sizes (n), shape parameter values (κ) and probability levels (p) encountered in many areas of application.

3. THE PROPOSED METHOD

In equations 1 through 7 we present the method without any mathematical justification. The next two sections wil help give further clarification of the method, but it is only after Table 1 is presented, that the practical utility of the method can be fully appreciated.

Let Y be a continuous random variable with probability density function (p.d.f.) $f_Y(y;\ \Theta_1,\ \ldots,\ \Theta_k)$, and cumulative distribution function (c.d.f.) $F_Y(y;\ \Theta_1,\ \ldots,\ \Theta_k)$, where $\Theta_1, \ldots, \Theta_k$ are unknown parameters. Suppose that a method *exists* for constructing confidence intervals for the quantiles Y_p of the random variable Y. In other words, for any 100 $(1 - 2\alpha)$ % confidence level, we can calculate upper and lower confidence limits $U_\alpha(p)$ and $L_\alpha(p)$ such that:

$$P\ [Y_p \geq U_\alpha\ (p)] = \alpha \quad \text{and} \quad P\ [Y_p \leq L_\alpha\ (p)] = \alpha \qquad (1)$$

which implies that:

$$P\ [L_\alpha\ (p) \leq Y_p \leq U_\alpha\ (p)] = 1 - 2\alpha \qquad (2)$$

Suppose that X is a random variable for which a method for constructing confidence intervals for X_p *does not exist*. Let $f_X(x;\ \Theta'_1,\ \ldots,\ \Theta'_m)$ and $F_X(x;\ \Theta'_1,\ \ldots,\ \Theta'_m)$ be the p.d.f. and c.d.f. of X, respectively, where $\Theta'_1, \ldots, \Theta'_m$ are unknown parameters. What we are searching for, are two real numbers $U'_\alpha\ (p)$ and $L'_\alpha\ (p)$ such that:

$$P\ [X_p \geq U'_\alpha\ (p)] = \alpha \quad \text{and} \quad P\ [X_p \leq L'_\alpha\ (p)] = \alpha \qquad (3)$$

or

$$P\ [L'_\alpha\ (p) \leq X_p \leq U'_\alpha\ (p)] = 1 - 2\alpha$$

Assume for the time being that $\theta_1, \ldots, \theta_k$ are *known.* Let p_1 be the probability of Y not exceeding $L_\alpha(p)$ and p_2 the probability of it not exceeding $U_\alpha(p)$, i.e.:

$$P\ [Y \leq L_\alpha\ (p)] = p_1 \quad \text{and} \quad P\ [Y \leq U_\alpha(p)] = p_2 \qquad (4)$$

This means that $L_\alpha(p)$ is the p_1-quantile of Y and $U_\alpha(p)$ is the p_2-quantile of Y:

$$L_\alpha(p) = Y_{p_1} \quad , \quad U_\alpha(p) = Y_{p_2} \tag{5}$$

If $L_\alpha(p)$ and $U_\alpha(p)$ are known, then p_1 and p_2 can be calculated using the equations:

$$p_1 = F_Y\,[L_\alpha(p)] \quad , \quad p_2 = F_Y\,[U_\alpha(p)] \tag{6}$$

If the c.d.f. F_X is known, then this can be used to calculate the p_1- and p_2- quantiles of X, namely X_{p_1} and X_{p_2}. The method we propose consists in taking the desired lower and upper 100(1-2α) % confidence limits for X_p to be equal to:

$$L'_\alpha(p) = X_{p_1} \qquad \text{and} \qquad U'_\alpha(p) = X_{p_2} \tag{7}$$

Before giving a mathematical justification for this approximate method we shall give an example.

4. EXAMPLE

It is well known that when the parent distribution of an observed sample is normal or log normal, exact confidence intervals for X_p can be constructed as described in (Johnson and Welch, 1940). These are based on the noncentral t-distribution.

In other words, if $\{Y_1, \ldots, Y_n\}$ is a random sample of size n from a normal distribution, with mean μ_y and standard deviation σ_y, then the exact

100 (1 - 2α) % confidence interval $[L_\alpha(p), U_\alpha(p)]$ for the p^{th} quantile Y_p, of Y, is given by:

$$[L_\alpha(p), U_\alpha(p)] = [\bar{Y} + S_Y \xi_\alpha(p), \bar{Y} + S_Y \xi_{1-\alpha}(p)] \tag{8}$$

where $\xi_\alpha(p)$ and $\xi_{1-\alpha}(p)$ are obtained using tables of the non-central t-distribution (Locks, Alexander and Byars, 1973; Resnikoff and Lieberman, 1957; Odeh and Owen, 1980 and others).

The following sample, placed in increasing order, gives the maximum discharge during the month of September, over the period 1940-1966, at a gaging station on the Harricana river, Canada. The sample values are in m^3/sec. and the sample size, n, is 27.

19, 23, 27, 33, 39, 39, 40, 43, 50, 50, 51, 61, 62, 63, 65, 66, 71, 82, 85, 86, 89, 93, 101, 106, 117, 119, 126.

The 2- parameter gamma distribution provides a good fit to the sample, and the maximum likelihood (ML) estimates of the parameters θ and κ of this distribution (X) are respectively: $\tilde{\theta} = 14.57$ and $\tilde{\kappa} = 4.59$. The estimate of the coefficient of skewness of X is therefore $\tilde{\gamma} = 2/\sqrt{4.59} = .934$ and estimates of the mean and standard deviation are $\tilde{\mu}_x = 66.88$ and $\tilde{\sigma}_x = 31.22$. Suppose that we wish to construct a 90% confidence interval (α = 0.05) for the 99th percentile of X (p = .99). We choose a "prior" Y to be for instance a normal random variable with mean μ_y and variance σ_y^2. This choice of Y will be discussed in more detail later. From tables of the noncentral t-distribution, with $n = 27$, $p = 0.99$ and $\alpha = 0.05$, we find $L_\alpha(p) = \xi_\alpha(p) = 1.817$ and $U_\alpha(p) = \xi_{1-\alpha}(p) = 3.117$, which by equation 8 are the lower and upper confidence limits for the standardized normal quantile $(Y_p-\mu_y)/\sigma_y$ under the assumption that $\bar{Y} = \mu_y$ and $S_y = \sigma_y$ (we shall also return to this assumption later). The next step is to find Φ (1.817) and Φ (3.117), where Φ is the c.d.f. of the standard normal distribution. Approximation formula (26.2.17) of Zelen and Severo (1970) gives $\Phi\ (1.817) = p_1 \simeq 0.965391$ and $\Phi\ (3.117) = p_2 \simeq 0.999086$. From Tables of Cohen, Helm and Sugg (1969) or Tables of Harter (1964) with $\tilde{\gamma} = .934$ and non-exceedance probabilities $p_1 = 0.965391$ and $p_2 = 0.999086$ we obtain the values $K_{p_1} \simeq 2.13$ and $K_{p_2} \simeq 4.54$, where K_p is the p^{th} quantile of the standardized gamma variate with coefficient of skewness $\tilde{\gamma}$. The values of K can also be calculated using program MDCHI of the Library IMSL (1982) or more conveniently by using the Wilson-Hilferty approximation (Wilson and Hilferty, 1931):

$$K_p \simeq \frac{2}{\tilde{\gamma}} \{[\frac{\tilde{\gamma}}{6} (z_p - \frac{\tilde{\gamma}}{6}) + 1]^3 -1\}$$

Our proposed method consists in taking the lower and upper confidence limits for X_p as follows:

$$L'_\alpha\ (p) = L'_{0.5}\ (0.99) = \tilde{x}_{p_1} = \tilde{\mu}_x + K_{p_1}\tilde{\sigma}_x \simeq 66.88 + 2.13\ (31.22) = 133.4$$

and

$$U'_\alpha (p) = \tilde{\mu}_x + K_{p_2} \tilde{\sigma}_x \simeq 208.6$$

The above numerical calculations are summarized schematically in Figure 1. What we have done, in this example, is that we have used the *exact* 90% confidence interval [1.817, 3.117] for the 99th percentile of the normal distribution ($Y_{.99}$), under the assumption that $\bar{Y} = \mu_y$ and $S_Y = \sigma_y$, to construct an *approximate* 90 % confidence interval [133.4, 208.6] for the corresponding gamma percentile, $X_{.99}$, by making use of the observed sample from the gamma distribution. Had the random variable Y that we started with been Weibull, for instance, rather than normal, we would have obtained a different confidence interval for $X_{.99}$. Had Y on the other hand been lognormal or any other distribution derived from the normal by a monotonic transformation, we would have obtained exactly the same confidence interval as above. We expect that the distribution Y that behaves "nearest" to the gamma distribution (X) at the probability level p, of interest, to yield the most accurate (or least inaccurate) confidence intervals for the gamma quantile X_p. This "best" distribution Y may vary, however, with the value of the probability level p and also with the value of α (confidence level). With the help of simulation, one can choose the best distribution Y for different values of p and α.

We mention that for the above example the method of Bain, Engelhardt and Shiue gives $L'_\alpha(p) \simeq 143.9$ and $U'_\alpha(p) \simeq 202.1$. Had we been interested in a 90% confidence interval for $X_{.01}$ instead of $X_{.99}$ our method would have given [7.9, 22.3] while that of Bain *et al.* gives [7.4, 20.7]. We shall have a word to say about the two methods later.

5. MATHEMATICAL JUSTIFICATION FOR THE PROPOSED APPROACH

We know that for any two distributions X and Y, a relationship exists which gives the quantiles X_p of the distribution X as a function of the corresponding quantiles Y_p of the distribution Y. This relationship is given by:

$$X_p = h\ (Y_p) = F_X^{-1}\ [F_Y\ (Y_p)] \tag{9a}$$

Of course, the inverse of this relationship gives Y_p as a function of X_p:

$$Y_p = h^{-1}\ (X_p) = F_Y^{-1}\ [F_X\ (X_p)] \tag{9b}$$

and both functions $h(\cdot)$ and $h^{-1}(\cdot)$ are increasing functions.

If F_X in the last two equations is estimated by $\tilde{F}_X$ then h will be estimated by $\tilde{h}$ and equations (9a) and (9b) will be approximate equalities rather than strict equalities.

If by using certain assumptions such as $\bar{Y} = \mu_y$ and $S_y = \sigma_y$ we are able to calculate $L_\alpha(p)$ and $U_\alpha(p)$ such that:

$$P\ [L_\alpha\ (p) \le Y_p \le U_\alpha(p)] = 1 - 2\alpha \tag{10}$$

i.e. such that $[L_\alpha(p), U_\alpha(p)]$ is a 100 (1 - 2α) % confidence interval for Y_p, then:

(1) taking $L_\alpha(p) = Y_{p_1}$ and $U_\alpha(p) = Y_{p_2}$ as was done in (5);

(2) making use of the approximate relationship $Y_p \simeq \tilde{h}^{-1}(X_p)$;

and (3) using the fact that $\tilde{h}^{-1}(\cdot)$ is an increasing function; we obtain:

$$\begin{aligned} 1 - 2\alpha &= P\,[L_\alpha(p) \le Y_p \le U_\alpha(p)] \\ &\simeq P\,[Y_{p_1} \le \tilde{h}^{-1}\,(X_p) \le Y_{p_2}] \\ &\simeq P\,[\tilde{h}^{-1}\,(X_{p_1}) \le \tilde{h}^{-1}\,(X_p) \le \tilde{h}^{-1}\,(X_{p_2})] \\ &\simeq P\,\{\tilde{h}[\tilde{h}^{-1}(X_{p_1})] \le \tilde{h}[\tilde{h}^{-1}(X_p)] \le \tilde{h}[\tilde{h}^{-1}(X_{p_2})]\} \\ &\simeq P\,(X_{p_1} \le X_p \le X_{p_2}) \qquad (11) \end{aligned}$$

This last result means that $[X_{p_1}, X_{p_2}]$ is an approximate 100 (1-2α) % confidence interval for X_p, and this is basically why we suggested taking $L'_\alpha(p) = X_{p_1}$ and $U'_\alpha(p) = X_{p_2}$ in (7).

The foregoing mathematical derivations show the logic behind the approach that we are proposing for transforming confidence intervals from one distribution (Y) to another distribution (X) but they do not tell us which distribution Y should be chosen as the "prior" for any given distribution X. This choice of Y when X is gamma distributed will be resolved by simulation, as will now be shown.

6. TESTING THE PROPOSED METHOD BY SIMULATION

Ashkar and Bobée (1987) considered the case of a 3-parameter gamma distribution (X) which is frequently used in flood frequency analysis and their study consisted of two parts:

(1) In the first part, the coefficient of skewness (shape parameter) of the variable X was assumed to be known, and it was shown using simulation that the best distribution Y for producing confidence intervals for the quantiles of X was the *normal distribution* as compared to other choices of Y such as the Weibull (2-parameter) and exponential (1- and 2-parameter). Note that when Y is taken to be 1-parameter exponential, it is assumed that $Y/\bar{Y}$ follows the unit exponential; likewise when Y is Weibull, it is assumed that $(\ell n\ Y - \bar{Y})/\ S_y$ follows the standard Extreme Value distribution.

(2) In the second part of the study by Ashkar and Bobée (1987), the coefficient of skewness of X was assumed to be unknown, and in this case it was shown that taking Y to be *Weibull* (2-parameter) gave fairly accurate confidence intervals for the quantiles of X (3-parameter gamma) for quite a wide range of sample sizes and coefficients of skewness found in practice. Of course one would presume that taking Y to be 3-parameter rather than 2-parameter Weibull would have most probably given better results, but since no simple method exists for constructing confidence intervals for

quantiles of the 3-parameter Weibull, Ashkar and Bobée did not consider this option in their study.

In what follows we shall consider the case where X is a 2-parameter gamma distribution with both parameters unknown. From exploratory simulation experiments we have observed that among the Weibull (2-parameter), normal, and exponential (1- and 2-parameter) distributions, it was the *normal* distribution that gave the best confidence intervals for quantiles X_p of the gamma distribution. From our experience, it is not necessary to carry out a sophisticated Monte Carlo experiment in order to find out which "prior" Y is best for a given distribution X, so it was sufficient from the limited computer runs that we carried out to clearly see that the normal distribution was the best prior for the problem at hand. Although detailed results for the other distributions are not available, we mention that the normal distribution was uniformly superior to all other distributions that were considered (readers who are interested in comparative values of confidence intervals for the 3-parameter gamma, arrived at by normal, exponential and Weibull assumptions can refer to the paper by Ashkar and Bobée, 1987). The procedure that we propose for obtaining confidence intervals for the quantiles of the gamma distribution is therefore to take Y to be normally distributed exactly as was done in the example that we gave earlier.

We present in Table 1 the results of an experiment in which 10,000 samples for each of three sample sizes n = 10, 25, 50 were generated from the standardized gamma distribution (mean zero, variance one) with coefficient of skewness γ = 0.2, 0.5, 0.7, 1.0, 1.5 and 2.0. For each sample size n and coefficient of skewness γ we present the actual frequency with which 90% and 99% confidence intervals constructed using the method proposed in the present study contained the true value of the gamma quantile X_p, for p = 0.002, .01, .05, .1, .5, .9, .95, .99 and .998. The exact number of samples that were generated in each run (10,000) is rather subjective but we believe that it provides a good degree of accuracy. The calculation of K_{p_1} and K_{p_2} in the experiment was done using two approximation formulas: the Wilson-Hilferty transformation, mentioned earlier, and the Cornish-Fisher transformation (Fisher and Cornish, 1960).

Table 1 shows that the method proposed in the present study gives very good results for a wide range of sample sizes (n), probability levels (p) and coefficients of skewness (γ) of the gamma variable. Most of the frequencies reported in this table are based on the Wilson-Hilferty transformation for calculating gamma quantiles, which generally gave more reliable results than the Cornish-Fisher transformation but a few entries in the upper right hand corner of the table are based on the Cornish-Fisher transformation. A few other entries are missing because neither transformation gave reliable results.

A comparison of Table 1 with Table 2 of Bain, Engelhardt and Shieu (1984) shows that the method proposed in the present study brings some clear improvements over the method of Bain *et al.* for values of p greater than about .20. For $p < .20$ one of the two methods might perform slightly better than the other, but with the computer time at our disposal we did not find it necessary to make a detailed comparison between the two. Our main objective was to extend as much as possible the range of values of p, n and

γ for which to test our method. For values of p lower than .20 it is therefore left to the user to choose the method that he or she finds appropriate.

7. CONCLUSION

We close our discussion with the following remarks:

(1) The method proposed in the present study is general; it remains to be seen if it can be useful for distributions other than the 2- or 3-parameter gamma distribution;

(2) The results reported in Table 1 cover only the case where the parameters θ and κ of the gamma distribution are obtained by maximum likelihood. The method was tested and gave approximately the same degree of accuracy when θ and κ were estimated by the method of moments, but due to lack of space, these results are not shown here;

(3) Another method which was not investigated in the present study but which the authors feel can be very useful for transforming confidence intervals from one distribution Y to another distribution X has been proposed by Stedinger (1983), and discussed in Ashkar and Bobée (1987). The method consists in scaling confidence intervals of the quantiles Y_p by multiplying them by a certain factor to obtain a corresponding approximate confidence interval for X_p, the factor being equal to the ratio of the asymptotic standard error of $\hat{X}_p$ to that of

$\tilde{Y}_p$. This method was not tested because it takes considerably more time to program and run on a computer than the method that we have proposed

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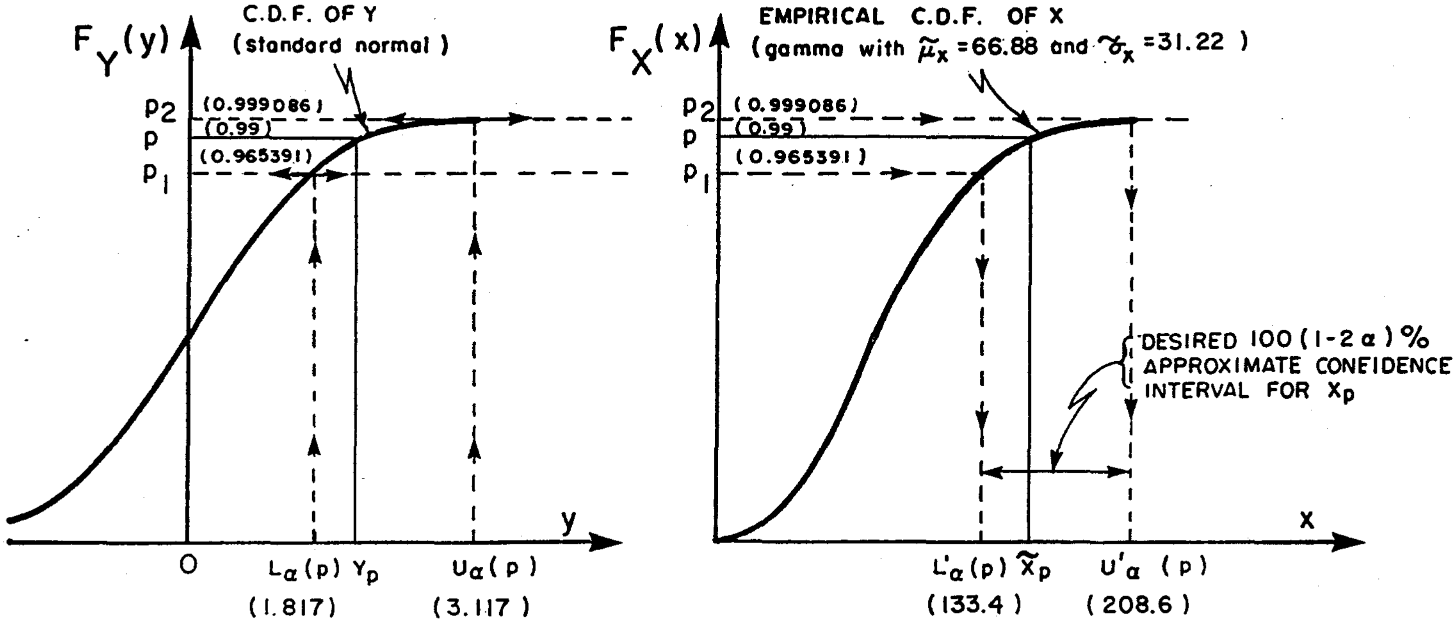

FIGURE 1. Obtaining approximate confidence intervals for X_p using exact confidence intervals for X_p. The information given between parentheses corresponds to the example given in the present study.

Table 1: Frequency (%) with which approximate confidence intervals constructed for the quantiles X_p of a gamma variable X based on a normal variable Y, contained the true value of X_p.

	confidence level	n	SKEW COEFFICIENT OF X: 0.2	0.5	0.7	1.0	1.5	2.0
p = .002	90 %	10	89.83	90.68	90.01	90.06	89.47	**
		25	89.43	90.14	90.26	89.95	88.55‡	**
		50	89.87	90.00	90.48	89.49	86.27‡	**
	99 %	10	99.03	98.98	99.03	99.17	98.94	**
		25	98.88	98.80	98.95	99.12	98.83‡	**
		50	98.95	99.05	98.90	98.96	98.36‡	**
p = .01	90 %	10	89.78.	90.47	89.77	90.02	90.55	91.48‡
		25	89.42	90.11	90.15	90.32	89.49	**
		50	89.79	89.95	90.50	90.38	90.64‡	**
	99 %	10	99.03	98.93	99.02	99.14	99.13	98.40
		25	98.81	98.74	98.98	99.16	98.87	98.95‡
		50	98.93	99.02	98.94	99.07	98.58	**
p = .05	90 %	10	89.45	89.80	89.45	89.80	90.52	90.55
		25	89.38	90.11	90.27	90.13	90.82	89.93
		50	89.74	90.13	90.31	90.35	90.51	88.74
	99 %	10	98.94	98.88	98.91	98.99	99.15	99.18
		25	98.77	98.80	98.97	99.09	99.22	99.03
		50	98.83	99.00	98.99	99.14	99.09	98.71
p = .10	90 %	10	89.40	89.31	89.21	89.65	90.17	90.48
		25	89.30	89.70	89.97	89.81	90.56	90.76
		50	89.82	89.91	90.21	90.18	90.86	90.67
	99 %	10	98.91	98.84	98.81	98.90	96.98	99.18
		25	98.77	98.78	98.91	99.00	99.22	99.25
		50	98.83	96.96	99.01	99.17	99.23	99.25
p = .50	90 %	10	88.43	86.14	88.26	88.43	87.90	86.28
		25	89.53	89.13	89.15	88.87	89.22	89.16
		50	89.72	89.46	89.33	90.04	90.06	89.84
	99 %	10	98.59	98.46	98.67	98.66	98.73	98.45
		25	98.92	98.71	96.96	96.54	98.70	98.75
		50	98.93	98.84	96.83	98.88	99.01	98.86
p = .90	90 %	10	89.09	89.28	89.25	89.58	88.62	88.72
		25	89.76	89.80	89.25	89.35	89.13	89.16
		50	90.05	90.50	90.27	89.73	89.76	89.16
	99 %	10	98.82	98.90	98.82	98.90	98.83	98.93
		25	99.02	99.02	98.99	98.98	98.80	98.77
		50	98.87	98.86	98.82	98.91	98.93	98.82
p = .95	90 %	10	89.32	89.69	89.42	90.06	88.92	89.22
		25	89.95	89.68	89.40	89.68	89.40	89.37
		50	90.11	90.37	90.40	89.90	89.81	89.09
	99 %	10	98.90	99.03	98.92	98.97	98.86	99.00
		25	99.09	99.08	99.00	99.00	98.89	98.97
		50	98.93	98.88	96.84	98.95	98.95	98.87
p = .99	90 %	10	89.61	90.10	89.75	90.26	89.32	89.67
		25	89.87	89.70	89.87	90.01	89.42	89.75
		50	90.03	90.29	90.44	89.99	89.81	89.34
	99 %	10	98.89	89.04	98.89	98.97	98.96	99.09
		25	99.11	99.13	99.02	99.19	98.95	99.11
		50	98.88	98.88	98.95	98.93	98.97	98.90
p = .998	90 %	10	89.85	90.26	89.97	90.32	89.53	89.82
		25	89.95	89.82	89.97	90.25	89.66	89.96
		50	90.08	90.20	90.47	90.06	89.83	89.39
	99 %	10	98.91	99.04	98.93	99.02	99.01	99.06
		25	99.11	99.15	99.00	99.29	98.97	99.09
		50	98.92	98.87	98.92	98.96	98.96	98.91

* The reported frequencies are based on 10 000 simulated samples for each sample size n, skew coefficient γ, and probability level p.

** These frequencies are not reported because neither the Wilson-Hilferty nor the Cornish-Fisher approximation formula for calculating the quantiles of the gamma distribution gave reliable results.

‡ These frequencies are based on the Cornish-Fisher approximation formula. All the orther frequencies are based on the Wilson-Hilferty approximation formula.