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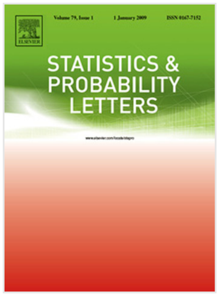

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On copula-based conditional quantile estimators

Bruno Rémillard[a], Bouchra Nasri[b,*], Taoufik Bouezmarni[c]

[a]GERAD, CRM, and Department of Decision Sciences, HEC Montréal, 3000 chemin de la Côte Sainte-Catherine, Montréal(Qc), H3T 2A7. Canada
[b]Department of Water Sciences, Institut national de recherche scientifique, Eau-Terre-Environnement, 490 rue de la couronne, Québec,G1K 9A9. Canada
[c]Department of Statistics, Université de Sherbrooke, 2500 boul. de l'Université, Sherbrooke, J1K 2R1. Canada

Abstract

Recently, two different copula-based approaches have been proposed to estimate the conditional quantile function of a variable Y with respect to a vector of covariates $\boldsymbol{X}$: the first estimator is related to quantile regression weighted by the conditional copula density, while the second estimator is based on the inverse of the conditional distribution function written in terms of margins and the copula. Using empirical processes, we show that even if the two estimators look quite different, their estimation errors have the same limiting distribution. Also, we propose a bootstrap procedure for the limiting process in order to construct uniform confidence bands around the conditional quantile function.

Keywords: Conditional quantile function, copula, quantile regression, bootstrap

1. Introduction

Copulas, or dependence functions, are very popular to model the dependence between variables, because one can remove the effect of marginal distributions, provided the latter are continuous. This is why dependence measures based on the copula are so robust, compared to the traditional Pearson correlation coefficient. Copulas also enter naturally when computing the conditional distribution function of a random variable Y given covariates $\boldsymbol{X} = (X_1, \ldots, X_d)$. See, e.g., Bouyé and Salmon [5] when $d = 1$. This relation between the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$ and the associated

*Corresponding author

copula was used recently to propose conditional quantile estimators, as alternative to the quantile regression methods [11] or the parameter approach [6, 15, 16].

A first copula-based estimator of the conditional quantile was proposed by Noh et al. [19] and is based on a weighted quantile regression method. The asymptotic limiting distribution was proved to be Gaussian. More recently, a more intuitive estimator of the plug-in type was proposed in Kraus and Czado [12], Nasri and Bouezmarni [14], who compared the estimated MISE of various competitors, including the estimator proposed by Noh et al. [19]. From the simulations performed in Kraus and Czado [12], Nasri and Bouezmarni [14], it seems that the plug-in estimator performs better than the other copula-based estimator. However the asymptotic behavior of this estimator was not discussed.

In Section 2, we describe the estimators of Noh et al. [19] and Kraus and Czado [12] and we discuss their implementation. Another closely related parametric estimator proposed in Nasri and Bouezmarni [14] is also discussed. In Section 3, we study the asymptotic limiting distribution of the estimators viewed as stochastic processes over $(0,1)$ and we show that the two semi-parametric estimators have the same limiting distribution. We also propose a bootstrapping method for constructing uniform confidence bands for the conditional quantile functions.

2. Estimation of conditional quantiles

One way to model the dependence between a variable of interest Y and covariates $\mathbf{X} = (X_1, \ldots, X_d)$ is to use dependence functions called copulas; see, e.g., Nelsen [17]. More precisely, suppose that $(Y_1, \mathbf{X}_1), \ldots, (Y_n, \mathbf{X}_n)$ are i.i.d. observations of $(Y, \mathbf{X})$ with (unconditional) continuous margins F_0, $F_1, \ldots, F_d$, and copula C with density c. Set $\mathbf{F}(\mathbf{x}) = (F_1(x_1), \ldots, F_d(x_d))$.

By definition, a copula is a joint distribution function of uniform random variables. According to Sklar's theorem [17], since the margins are assumed to be continuous, there exists a unique copula C so that the joint distribution function of $(Y, \mathbf{X})$ can be written in terms of the copula and the margins viz.

$$P(Y \le y, \mathbf{X} \le \mathbf{x}) = C\{F_0(y), \mathbf{F}(\mathbf{x})\}, \quad y \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d. \tag{1}$$

Note that the copula C is the cdf of $(U, \mathbf{V})$, where $U = F_0(Y)$ and $\mathbf{V} = \mathbf{F}(\mathbf{X})$.

2.1. Copula-based conditional quantiles

Denote by $\mathcal{H}(y,\mathbf{x})$ the conditional distribution function of Y given $\boldsymbol{X}=\mathbf{x}$. The expression of the conditional distribution function $\mathcal{H}$ in terms of the copula function and the marginal distributions appeared explicitly first in a preliminary version of Bouyé and Salmon [5] in the case $d=1$. However, it is easy to extend it to any $d\geq 1$, and one can easily show that

$$\mathcal{H}(y,\mathbf{x}) = \mathrm{P}(Y\leq y|\mathbf{X}=\mathbf{x}) = \mathcal{C}\{F_0(y),\mathbf{F}(\mathbf{x})\}, \quad y\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^d, \tag{2}$$

where $\mathcal{C}(u,\mathbf{v})$ is the conditional distribution function of U given $\mathbf{V}\equiv\mathbf{F}(\mathbf{X})=\mathbf{v}\equiv\mathbf{F}(\mathbf{x})$. In fact, according to Rémillard [20, Proposition 8.6.2], for $u\in[0,1]$ and $\mathbf{v}=(v_1,\ldots,v_d)\in(0,1)^d$,

$$\mathcal{C}(u,\mathbf{v}) = \frac{\partial v_1\cdots\partial v_d C\left(u,v_1,\ldots,v_d\right)}{\partial v_1\cdots\partial v_d C\left(1,v_1,\ldots,v_d\right)},$$

and $\partial_u\mathcal{C}(u,\mathbf{v}) = c(u,\mathbf{v})/\int_0^1 c(z,\mathbf{v})dz$, so $\mathcal{C}(u,\mathbf{v})=\int_0^u c(z,\mathbf{v})dz/\int_0^1 c(z,\mathbf{v})dz$.

Now, the associated conditional quantile function $Q(\alpha,\mathbf{x})$, $\alpha\in(0,1)$, is given by

$$Q(\alpha,\mathbf{x}) = \inf\{y\in\mathbb{R}:\mathcal{H}(y,\mathbf{x})\geq\alpha\}. \tag{3}$$

Using (2), we get that Q depends only on the margins F_0, $\mathbf{F}$ and the copula C viz.

$$Q(\alpha,\mathbf{x}) = F_0^{-1}\left[\mathit{\Gamma}\{\alpha,\mathbf{F}(\mathbf{x})\}\right], \tag{4}$$

where $\mathit{\Gamma}(\alpha,\mathbf{v})$ is the quantile of order α of the distribution function $\mathcal{C}(u,\mathbf{v})$, $u\in[0,1]$, with $\mathbf{v}\in(0,1)^d$ fixed. Note that (4) is the basic equation for defining the plug-in estimator.

Next, using (2), one gets that $Q(\alpha,\mathbf{x})$ is also a solution of

$$\arg\min_a \mathrm{E}\left[\mathfrak{c}\left\{F_0(Y),\mathbf{F}(\mathbf{x})\right\}\rho_\alpha\left(Y-a\right)\right], \tag{5}$$

where $\mathfrak{c}(u,\mathbf{v})=\partial_u\mathcal{C}(u,\mathbf{v})$, $\rho_\alpha(y)=y\left\{\alpha-\mathbb{I}\left(y<0\right)\right\}=(1-\alpha)|y|\mathbb{I}(y<0)+\alpha y\mathbb{I}(y\geq 0)$, $y\in\mathbb{R}$, and $\mathbb{I}$ is the indicator function. The latter equation is used by Noh et al. [19] to construct an estimator of $Q(\alpha,\mathbf{x})$.

2.2. Estimation of the copula and the margins

To estimate the conditional quantile using copulas, one needs to estimate the copula C associated with $(Y, \mathbf{X})$ or $(U, \mathbf{V})$, and the margins $F_0, \mathbf{F}$. First, one can assume that $Y_i = F_0^{-1}(U_i)$ and $X_{ij} = F_j^{-1}(V_{ij})$, where $(U_1, \mathbf{V}_1), \ldots, (U_n, \mathbf{V}_n)$ are i.i.d. observations from copula C.

2.2.1. Estimation of the copula

For sake of simplicity, we assume that the copula belongs to a parametric family $\{C_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \mathcal{O}\}$, so the estimation of the copula is given as $C_{\boldsymbol{\theta}_n}$, where $\boldsymbol{\theta}_n$ is a rank-based consistent estimator [7] of the true parameter $\boldsymbol{\theta}_0$. One can use the pseudo-MLE method proposed by Genest et al. [8]. Consequently, the quantile function $\mathit{\Gamma}(\alpha, \mathbf{v}) \equiv \mathit{\Gamma}_{\boldsymbol{\theta}}(\alpha, \mathbf{v})$ can be estimated by $\mathit{\Gamma}_{\boldsymbol{\theta}_n}(\alpha, \mathbf{v})$, $\alpha \in (0,1)$, $\mathbf{v} \in (0,1)^d$. The parametric family approach is also what Noh et al. [19] and Kraus and Czado [12] considered. In fact, in the case of several covariates, Kraus and Czado [12] used a particular case of a parametric copula family, namely a D-vine model [2, 1], which is a construction of a copula using a given set of parametric bivariate copula families. Note that instead of considering a parametric family of copulas, one could estimate the density of the copula non-parametrically, so that all the conditional quantile estimators discussed here could also be computed. However the convergence is slower and it often suffers from the curse of dimensionality [4, 9], with the possible exception of pair-copula construction [13]. The next step is to estimate the margins.

2.2.2. Estimation of the margins

Motivated by the two-step inference function for margins (IFM) method [10], one could use parametric families to estimate each of the margins. This would make sense in several applications. For copula-based quantile estimators, this approach was suggested in Nasri and Bouezmarni [14], where a parametric copula-based estimator was proposed. Note that as discussed in Noh et al. [18], if the estimation of the margins is incorrect, the estimation of the copula parameter $\boldsymbol{\theta}$ can be biased. One can also consider non-parametric estimators, namely for any $y \in \mathbb{R}$ and any $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$F_{n0}(y) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbb{I}(Y_i \leq y), \quad F_{nj}(x_j) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbb{I}(X_{ij} \leq x_j), \quad j \in \{1, \ldots, d\}, \tag{6}$$

and set $\mathbf{F}_n(\mathbf{x}) = (F_{n1}(x_1), \ldots, F_{nd}(x_d))$. Further note that $F_{n0}(y) = D_n \circ F_0(y)$, where D_n is the empirical distribution function of the U_i's and $\mathbf{F}_n(\mathbf{x}) = \mathbf{B}_n \circ \mathbf{F}(\mathbf{x})$, where $\mathbf{B}_n$ is the vector of empirical marginal distribution functions of $\mathbf{V}_1, \ldots, \mathbf{V}_d$. Noh et al. [19] propose a kernel-based estimator $\hat{F}_{n0}$ for F_0 such that $n^{1/2} \sup_y |\hat{F}_{n0}(y) - F_{n0}(y)| \stackrel{Pr}{\to} 0$ as $n \to \infty$. This was also used in Kraus and Czado [12]. Even if $\hat{F}_{n0}$ is continuous, the precision of the estimation might not be better and there is always the question of the choice of the bandwidth. This is why we will use the estimators given by (6). For the rest of the section, let $\mathbf{x}$ be given and set $\mathbf{v} = \mathbf{F}(\mathbf{x})$. It then follows that $\mathbf{F}_n(\mathbf{x}) = \mathbf{B}_n(\mathbf{v})$. For sake of simplicity, $\mathbf{x}$ or $\mathbf{v}$ might be omitted. We present the copula-based estimators we will study.

2.3. Weighted quantile regression estimator

Surprisingly, the natural plug-in estimator did not appear first in the literature. In fact, Noh et al. [19] proposed a copula-based model mixed with a quantile regression approach using (5) viz.

$$Q_{n,wqr}(\alpha, \mathbf{x}) = \arg\min_a \left[\sum_{i=1}^n \rho_\alpha (Y_i - a)\, \mathfrak{c}_{\boldsymbol{\theta}_n} \{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\} \right], \tag{7}$$

even if the solution is not necessarily unique. In fact they take $c_{\boldsymbol{\theta}_n}(u, \mathbf{v})$ instead of taking $\mathfrak{c}_{\boldsymbol{\theta}_n}(u, \mathbf{v})$ but it leads to the same estimator; see, e.g., (8). However, a unique way to define a solution to (7) is by using the empirical weighted distribution function H_n defined for any $y \in \mathbb{R}$ by

$$H_n(y, \mathbf{x}) = \sum_{i=1}^n \mathbb{I}(Y_i \le y) w_{i,n} = G_n\{F_0(y), \mathbf{v}\}, \text{ with } G_n(u, \mathbf{v}) = \sum_{i=1}^n \mathbb{I}(U_i \le u) w_{i,n},$$

where, for any $i \in \{1, \ldots, n\}$,

$$w_{i,n} = \frac{\mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}}{\sum_{j=1}^n \mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_j), \mathbf{F}_n(\mathbf{x})\}} = \frac{c_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}}{\sum_{j=1}^n c_{\boldsymbol{\theta}_n}\{F_{n0}(Y_j), \mathbf{F}_n(\mathbf{x})\}} = \frac{\mathfrak{c}_{\boldsymbol{\theta}_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}}{\sum_{j=1}^n \mathfrak{c}_{\boldsymbol{\theta}_n}\{D_n(U_j), \mathbf{B}_n(\mathbf{v})\}}. \tag{8}$$

The estimator $Q_{n,wqr}(\alpha, \mathbf{x})$ is then defined as the quantile of level α of H_n, i.e.,

$$Q_{n,wqr}(\alpha, \mathbf{x}) = H_n^{-1}(\alpha, \mathbf{x}) = F_0^{-1} \circ G_n^{-1}(\alpha, \mathbf{v}), \quad \alpha \in (0, 1). \tag{9}$$

If $\hat{a} = \arg\min_a \left[\sum_{i=1}^n \rho_\alpha (Y_i - a)\, \mathfrak{c}_{\boldsymbol{\theta}_n}\{F_{n0}(Y_i), \mathbf{F}_n(\mathbf{x})\}\right]$, then $H_n(\hat{a}, \mathbf{x}) \ge \alpha \ge H_n(\hat{a}-, \mathbf{x})$. Hence $H_n^{-1}(\alpha, \mathbf{x})$ satisfies (7).

It is easy to show that H_n is a consistent estimator of the distribution function $\mathcal{H}(y,\mathbf{x}) = \mathcal{C}\{F_0(y),\mathbf{v}\}$, $y \in \mathbb{R}$. Also G_n is a consistent and asymptotically unbiased estimator of the distribution function $\mathcal{C}(u,\mathbf{v})$, $u \in [0,1]$.

2.4. Plug-in estimators

Expression (4) provides a natural way for estimating the conditional quantile. We now describe both parametric and semi-parametric estimators of $Q(\alpha,\mathbf{x})$.

2.4.1. Parametric estimator

In the parametric approach, we assume that the marginal distributions F_0 and $\mathbf{F}$ belong to parametric families denoted by $F_{0\boldsymbol{\beta}_0}(\cdot)$ and $\mathbf{F}_{\boldsymbol{\beta}}(\cdot)$ respectively. If $\boldsymbol{\beta}_{n0}$ and $\boldsymbol{\beta}_n$ are consistent estimators of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}$, and if $C_{\boldsymbol{\theta}}$, $F_{0\boldsymbol{\beta}_0}(\cdot)$ and $\mathbf{F}_{\boldsymbol{\beta}}(\cdot)$ are continuous functions of the parameters, then for any $y \in \mathbb{R}$, $\check{H}_n(y,\mathbf{x}) = \mathcal{C}_{\boldsymbol{\theta}_n}\left\{F_{0\boldsymbol{\beta}_{n0}}(y),\mathbf{F}_{\boldsymbol{\beta}_n}(\mathbf{x})\right\}$ is clearly a consistent estimator of $\mathcal{H}(y,\mathbf{x})$, yielding

$$Q_{n,p}(\alpha,\mathbf{x}) = \check{H}_n^{-1}(\alpha,\mathbf{x}) = F_{0\boldsymbol{\beta}_{n0}}^{-1}\left[\mathit{\Gamma}_{\boldsymbol{\theta}_n}\{\alpha,\mathbf{F}_{\boldsymbol{\beta}_n}(\mathbf{x})\}\right], \quad \alpha \in (0,1). \tag{10}$$

2.4.2. Semiparametric estimator

Here, the marginal distributions are estimated using (6). Next, $\mathcal{H}(y,\mathbf{x})$ is estimated by

$$\tilde{H}_n(y,\mathbf{x}) = \mathcal{C}_{\boldsymbol{\theta}_n}\{F_{n0}(y),\mathbf{F}_n(\mathbf{x})\} = \tilde{G}_n\{F_0(y),\mathbf{v}\}, \quad y \in \mathbb{R}, \tag{11}$$

where $\tilde{G}_n(u,\mathbf{v}) = \mathcal{C}_{\boldsymbol{\theta}_n}\{D_n(u),\mathbf{B}_n(\mathbf{v})\}$, which is a consistent estimate of $\mathcal{C}_{\boldsymbol{\theta}_0}(u,\mathbf{v})$, $u \in [0,1]$. As a result, the estimation of $Q(\alpha,\mathbf{x})$ is defined for any $\alpha \in (0,1)$ by

$$Q_{n,sp}(\alpha,\mathbf{x}) = \tilde{H}_n^{-1}(\alpha,\mathbf{x}) = F_{n0}^{-1}\left[\mathit{\Gamma}_{\boldsymbol{\theta}_n}\{\alpha,\mathbf{F}_n(\mathbf{x})\}\right] = F_0^{-1} \circ \tilde{G}_n^{-1}(\alpha,\mathbf{v}). \tag{12}$$

3. Asymptotic behavior of the copula-based estimators

In this section we find the asymptotic distribution of the conditional quantile functions for the proposed estimators, extending the results of Noh et al. [19]. As a result, we obtain that the estimation error of the plug-in estimator and the weighted quantile regression estimator converge to the same limiting distribution. We also propose, in Section 3.4, a different bootstrap algorithm that can be used to construct uniform confidence bands about the conditional quantile function.

As before, $\mathbf{x}$ is fixed and $\mathbf{v} = \mathbf{F}(\mathbf{x})$. Throughout this section, we assume that the density $f_0 = F_0'$ exists and is positive everywhere. If the support is not $\mathbb{R}$, just transform Y accordingly. This way $F_0(y) \in (0,1)$ for any $y \in \mathbb{R}$. Also suppose that the density c of the $(d+1)$-dimensional copula C is positive on $(0,1)^{d+1}$. Then $\mathcal{H}(\cdot, \mathbf{x})$ is continuously differentiable with density h satisfying $h(y,\mathbf{x}) = f_0(y)\mathfrak{c}(u,\mathbf{v}) > 0$, for any $y \in \mathbb{R}$. Further set $Q(u,\mathbf{x}) = \mathcal{H}^{-1}(u,\mathbf{x})$ and $\Gamma(u,\mathbf{v}) = \mathcal{C}^{-1}(u,\mathbf{v})$, $u \in (0,1)$.

3.1. Convergence of the parametric estimator

In what follows, $\nabla_{\boldsymbol{\beta}_0} F_{0\boldsymbol{\beta}_0}(y)$ is a p_0-dimensional column vector, $\nabla_{\boldsymbol{\beta}} \mathbf{F}_{\boldsymbol{\beta}}$ is a $p \times d$ matrix, $\nabla_{\mathbf{v}} \mathcal{C}_{\boldsymbol{\theta}}(u,\mathbf{v})$ is a d-dimensional column vector, $\nabla_{\boldsymbol{\theta}} \mathcal{C}_{\boldsymbol{\theta}}(u,\mathbf{v}) = \dot{\mathcal{C}}_{\boldsymbol{\theta}}(u,\mathbf{v})$ is a q-dimensional column vector which represent the partial derivatives with respect to $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}$, $\mathbf{v}$ and $\boldsymbol{\theta}$ of $F_{0\boldsymbol{\beta}_0}$, $\mathbf{F}_{\boldsymbol{\beta}}$, $\mathcal{C}_{\boldsymbol{\theta}}$ and $\mathcal{C}_{\boldsymbol{\theta}}$ respectively. Throughout this section, we assume that these derivatives are continuous, and that $\mathfrak{c}_{\boldsymbol{\theta}}(u,\mathbf{v})$ is continuously differentiable with respect to $u \in (0,1)$.

Set $\mathcal{B}_{n0} = n^{1/2}(\boldsymbol{\beta}_{n0} - \boldsymbol{\beta}_0)$, $\mathcal{B}_n = n^{1/2}(\boldsymbol{\beta}_n - \boldsymbol{\beta})$, and $\boldsymbol{\Theta}_n = n^{1/2}(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0)$. Finally, define $\check{\mathbb{H}}_n(y) = n^{1/2}\left\{\check{H}_n(y,\mathbf{x}) - \mathcal{C}(y,\mathbf{x})\right\}$ for any $y \in \mathbb{R}$, and $\mathbb{Q}_{n,p}(u) = n^{1/2}\left\{Q_{n,p}(u,\mathbf{x}) - Q(u,\mathbf{x})\right\}$, $u \in (0,1)$. The proof of the following theorem, giving the asymptotic behavior of the parametric quantile process, follows readily from the Delta method [21]. To simplify notations, set $\dot{\mathcal{C}}(u,\mathbf{v}) = \nabla_{\boldsymbol{\theta}} \mathcal{C}_{\boldsymbol{\theta}}(u,\mathbf{v})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$ and $\nabla_{\mathbf{v}} \mathcal{C}_{\boldsymbol{\theta}_0}(u,\mathbf{v}) = \nabla_{\mathbf{v}} \mathcal{C}(u,\mathbf{v})$.

Theorem 1. Assume that $(\mathcal{B}_{n0}, \mathcal{B}_n, \boldsymbol{\Theta}_n)$ converges in law to a centered Gaussian vector $(\mathcal{B}_0, \mathcal{B}, \boldsymbol{\Theta})$.[1] Then, as $n \to \infty$, $\check{\mathbb{H}}_n$ converges in $D(\mathbb{R})$[2] to a continuous centered Gaussian process $\check{\mathbb{H}}$, denoted $\check{\mathbb{H}}_n \rightsquigarrow \check{\mathbb{H}} = \check{\mathbb{G}} \circ F_{0\boldsymbol{\beta}_0}$, where

$$\check{\mathbb{G}}(u) = \boldsymbol{\Theta}^\top \dot{\mathcal{C}}(u,\mathbf{v}) + \mathcal{B}^\top \nabla_{\boldsymbol{\beta}} \mathbf{F}_{\boldsymbol{\beta}} \left\{ \boldsymbol{F}_{\boldsymbol{\beta}}^{-1}(\mathbf{v}) \right\} \nabla_{\mathbf{v}} \mathcal{C}(u,\mathbf{v}) + \mathfrak{c}_{\boldsymbol{\theta}_0}(u,\mathbf{v}) \mathcal{B}_0^\top \nabla_{\boldsymbol{\beta}_0} F_{0\boldsymbol{\beta}_0} \left\{ F_{0\boldsymbol{\beta}_0}^{-1}(u) \right\}, \quad u \in [0,1].$$

[1] See, e.g. Joe [10] for sufficient regularity conditions.

[2] Convergence in $D(I)$ means that for any close interval $[a,b] \subset I$, the process converges in law in the Skorokhod topology on $D([a,b])$. In particular, continuous functions of the process converges in law. See, e.g., Billingsley [3].

Furthermore, $\mathbb{Q}_{n,p} \rightsquigarrow \mathbb{Q}_p$ in $D(0,1)$, where $\mathbb{Q}_p(u) = -\frac{\breve{\mathbb{H}}\{Q(u,\mathbf{x})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}$, $u \in (0,1)$. In particular, for any $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u\in[a,b]} |Q_{n,p}(u,\mathbf{x}) - Q(u,\mathbf{x})|$ converges in law to $\sup_{u\in[a,b]} \left|\frac{\breve{\mathbb{H}}\{Q(u,\mathbf{x})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}\right|$.

3.2. Convergence of the semiparametric estimator

We now study the convergence of the process $\mathbb{Q}_{n,sp}(u) = n^{1/2}\{Q_{n,sp}(u,\mathbf{x}) - Q(u,\mathbf{x})\}$, $u \in (0,1)$. Before stating the theorem, define $\mathbb{D}_n(u) = n^{1/2}\{D_n(u) - u\}$, and $\mathbb{B}_n(\mathbf{v}) = n^{1/2}(\mathbf{B}_n(\mathbf{v}) - \mathbf{v})$, $u \in [0,1]$, $\mathbf{v} \in (0,1)^d$. The proof of this theorem follows from the Delta method [21].

Theorem 2. Assume that $(\mathbb{D}_n, \mathbb{B}_n, \boldsymbol{\Theta}_n)$ converges in $D\left([0,1]^{1+d}\right) \times \mathbb{R}^q$ to $(\mathbb{D}, \mathbb{B}, \boldsymbol{\Theta})$, where $\mathbb{B}$ and $\mathbb{B}$ are centered Gaussian processes and $\boldsymbol{\Theta}$ is a centered random vector.[3] Then, as $n \to \infty$, $\tilde{\mathbb{G}}_n$ converges in $D([0,1])$ to $\tilde{\mathbb{G}} = \mathbb{H} + \mathbb{D}\mathfrak{c}_{\boldsymbol{\theta}_0}(\cdot,\mathbf{v})$, where $\mathbb{H}(u) = \boldsymbol{\Theta}^\top \dot{\mathcal{C}}(u,\mathbf{v}) + \mathbb{B}(\mathbf{v})^\top \nabla_{\mathbf{v}} \mathcal{C}(u,\mathbf{v})$, $u \in [0,1]$. Furthermore, $\mathbb{Q}_{n,sp} \rightsquigarrow \mathbb{Q}_{sp}$ in $D(0,1)$, where $\mathbb{Q}_{sp}(u) = -\frac{\tilde{\mathbb{G}}\{\mathit{\Gamma}(u,\mathbf{v})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}$, $u \in (0,1)$. In particular, for any $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u\in[a,b]} |Q_{n,sp}(u,\mathbf{x}) - Q(u,\mathbf{x})|$ converges in law to $\sup_{u\in[a,b]} \left|\frac{\tilde{\mathbb{G}}\{\mathit{\Gamma}(u,\mathbf{v})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}\right|$.

3.3. Convergence of the weighted quantile regression estimator

We now study the convergence of the process $\mathbb{Q}_{n,wqr}(u) = n^{1/2}\{Q_{n,wqr}(u,\mathbf{x}) - Q(u,\mathbf{x})\}$. It extends the results in Noh et al. [19], where only the convergence at a single value was proven. In order to formulate the result, we need to define another sequence of stochastic processes, namely $\mathring{\mathbb{G}}_n(u) = n^{-1/2}\sum_{i=1}^n \{\mathbb{I}(U_i \le u)\mathfrak{c}_{\boldsymbol{\theta}_0}(U_i,\mathbf{v}) - \mathcal{C}_{\boldsymbol{\theta}_0}(u,\mathbf{v})\}$, $u \in [0,1]$. It follows from the theory of stochastic processes [21] that $(\mathbb{D}_n, \mathbb{B}_n, \mathring{\mathbb{G}}_n)$ converges in $D\left([0,1]^{2+d}\right)$ to centered Gaussian processes $(\mathbb{D}, \mathbb{B}, \mathring{\mathbb{G}})$. We can now show that the two estimators have the same limiting distribution.

Theorem 3. Assume that $(\mathbb{D}_n, \mathbb{B}_n, \mathring{\mathbb{G}}_n, \boldsymbol{\Theta}_n)$ converges in $D\left([0,1]^{2+d}\right) \times \mathbb{R}^q$ to centered Gaussian processes $(\mathbb{D}, \mathbb{B}, \mathring{\mathbb{G}}, \boldsymbol{\Theta})$. Then, as $n \to \infty$, $\mathbb{G}_n$ converges in $D([0,1])$ to $\mathbb{G} = \tilde{\mathbb{G}}$. Furthermore, $\mathbb{Q}_{n,wgr} \rightsquigarrow \mathbb{Q}_{n,wqr}$ in $D(0,1)$, where $\mathbb{Q}_{wqr}(u) = -\frac{\mathbb{G}\{\mathit{\Gamma}(u,\mathbf{v})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}$, $u \in (0,1)$. In particular, for any $[a,b] \subset (0,1)$, $n^{1/2} \sup_{u\in[a,b]} |Q_{n,wqr}(u,\mathbf{x}) - Q(u,\mathbf{x})|$ converges in law to $\sup_{u\in[a,b]} \left|\frac{\mathbb{G}\{\mathit{\Gamma}(u,\mathbf{v})\}}{h\{Q(u,\mathbf{x}),\mathbf{x}\}}\right|$.

[3] This assumption is satisfied for most well-behaved rank-based estimator of θ. See, e.g., Genest and Rémillard [7].

Proof. Set $\dot{\mathfrak{c}}(u,\mathbf{v}) = \nabla_{\boldsymbol\theta}\mathfrak{c}_{\boldsymbol\theta}(u,\mathbf{v})|_{\boldsymbol\theta=\boldsymbol\theta_0}$, $\nabla_{\mathbf{v}}\mathfrak{c}(u,\mathbf{v}) = \nabla_{\mathbf{v}}\mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v})$. It suffices to prove the convergence of $\mathbb{G}_n(u) = \sqrt{n}(G_n(u) - \mathcal{C}_{\boldsymbol\theta_0}(u,\mathbf{v}))$. Write $G_n(u) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i \le u)\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}/s_n$, where $s_n = \frac{1}{n}\sum_{i=1}^n \mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\}$.

Set $r_n(u) = \mathfrak{c}_{\boldsymbol\theta_n}\{D_n(u), \mathbb{B}_n(\mathbf{v})\} - \mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v}) - \frac{\{\boldsymbol\Theta_n^\top \dot{\mathfrak{c}}(u,\mathbf{v}) + \partial_u \mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v})\mathbb{D}_n(u) + \nabla_{\mathbf{v}}\mathfrak{c}(u,\mathbf{v})^\top \mathbb{B}_n(\mathbf{v})\}}{n^{1/2}}$, $u \in [0,1]$. By hypothesis, as $n \to \infty$, $n^{1/2} \sup_{u\in[0,1]} |r_n(u)|$ converges in probability to 0. It follows that

$$\begin{aligned}
\mathbb{G}_n(u) &= \frac{n^{-1/2}}{s_n}\sum_{i=1}^n \mathbb{I}(U_i \le u)\{\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\} - \mathfrak{c}_{\boldsymbol\theta_0}(U_i,\mathbf{v})\} + \mathring{\mathbb{G}}_n(u)/s_n - \mathcal{C}_{\boldsymbol\theta_0}(u,\mathbf{v})n^{1/2}(1-1/s_n) \\
&= \{\mathbb{L}_n(u) + \mathring{\mathbb{G}}_n(u) - \mathcal{C}_{\boldsymbol\theta_0}(u,\mathbf{v})\mathbb{L}_n(1) - \mathcal{C}_{\boldsymbol\theta_0}(u,\mathbf{v})\mathring{\mathbb{G}}_n(1)\}/s_n,
\end{aligned}$$

where $\mathbb{L}_n(u) = n^{-1/2}\sum_{i=1}^n \mathbb{I}(U_i \le u)\{\mathfrak{c}_{\boldsymbol\theta_n}\{D_n(U_i), \mathbf{B}_n(\mathbf{v})\} - \mathfrak{c}_{\boldsymbol\theta_0}(U_i,\mathbf{v})\}$. Now,

$$\begin{aligned}
\mathbb{L}_n(u) &= \boldsymbol\Theta_n^\top \left\{\frac{1}{n}\sum_{i=1} \mathbb{I}(U_i \le u)\dot{\mathfrak{c}}(U_i,\mathbf{v})\right\} + \frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i \le u)\mathbb{D}_n(U_i)\partial_u\mathfrak{c}(U_i,\mathbf{v}) \\
&\quad + \mathbb{B}_n(\mathbf{v})^\top \left\{\frac{1}{n}\sum_{i=1}^n \mathbb{I}(U_i \le u)\nabla_{\mathbf{v}}\mathfrak{c}(U_i,\mathbf{v})\right\} + o_P(1) \\
&= \boldsymbol\Theta_n^\top \dot{\mathcal{C}}(u,\mathbf{v}) + \int_0^u \mathbb{D}_n(z)\partial_u\mathfrak{c}(z,\mathbf{v})dz + \mathbb{B}_n(\mathbf{v})^\top \nabla_{\mathbf{v}}\mathcal{C}(u,\mathbf{v}) + o_P(1).
\end{aligned}$$

Next, assuming that $u\mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v}) \to 0$ as $u \to 0$, we have

$$\begin{aligned}
\int_0^u \mathbb{D}_n(z)\partial_z\mathfrak{c}_{\boldsymbol\theta_0}(z,\mathbf{v})dz &= n^{-1/2}\sum_{i=1}^n \int_0^u \partial_z\mathfrak{c}_{\boldsymbol\theta_0}(z,\mathbf{v})\{\mathbb{I}(U_i \le z) - z\}dz \\
&= n^{-1/2}\sum_{i=1}^n \mathbb{I}(U_i \le u)\{\mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v}) - \mathfrak{c}_{\boldsymbol\theta_0}(U_i,\mathbf{v})\} - n^{1/2}\{u\mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v}) - \mathcal{C}_{\boldsymbol\theta_0}(u,\mathbf{v})\} \\
&= \mathfrak{c}_{\boldsymbol\theta_0}(u,\mathbf{v})\mathbb{D}_n(u) - \mathring{\mathbb{G}}_n(u).
\end{aligned}$$

As a result, $\mathbb{L}_n \rightsquigarrow \mathbb{H} + \mathbb{D}\mathfrak{c}(\cdot,\mathbf{v}) - \mathring{\mathbb{G}} = \tilde{\mathbb{G}} - \mathring{\mathbb{G}}$. so, $\mathbb{G}_n \rightsquigarrow \mathbb{G} = \tilde{\mathbb{G}}$ in $D([0,1])$. □

Remark 1. Note that Theorems 2 and 3 are still valid if we choose the kernel distribution for marginals instead of the empirical distribution functions since their asymptotic behavior is the same. This is also true for the bootstrapping procedure defined next.

3.4. Bootstrapping

Algorithm 1 (Bootstrapping $\tilde{\mathbb{G}}$)**.** First, estimate $\boldsymbol{\theta}$ using a regular rank-based estimator $\boldsymbol{\theta}_n$ of the form $\boldsymbol{\theta}_n = \mathcal{T}_n(U_{1,n}, \mathbf{V}_{1,n}, \ldots, U_{n,n}, \mathbf{V}_{n,n})$ in the sense of Genest and Rémillard [7], and set $\mathbf{v}_n = \mathbf{F}_n(x)$. Then, for each $k \in \{1, \ldots, N\}$, repeat the following steps:

- Generate $(U_i^\star, \mathbf{V}_i^\star) \sim C_{\boldsymbol{\theta}_n}$, $i \in \{1, \ldots, n\}$, and compute the empirical margins $D_n^\star$, $\mathbf{F}_n^\star$;
- Calculate the pseudo-observations $U_{i,n}^\star = D_n^\star(U_i^\star)$, $\mathbf{V}_{i,n}^\star = \mathbf{F}_n^\star(\mathbf{V}_i^\star)$, $i \in \{1, \ldots, n\}$;
- Estimate $\boldsymbol{\theta}_n^\star = \mathcal{T}_n\left(U_{1,n}^\star, \mathbf{V}_{1,n}^\star, \ldots, U_{n,n}^\star, \mathbf{V}_{n,n}^\star\right)$;
- Set $\tilde{\mathbb{G}}_n^{(k)}(u) = n^{1/2}\left[\mathcal{C}_{\boldsymbol{\theta}_n^\star}\left\{D_n^\star(u), \mathbf{B}_n^\star(\mathbf{v}_n)\right\} - \mathcal{C}_{\boldsymbol{\theta}_n}(u, \mathbf{v}_n)\right]$, $u \in [0, 1]$.

The next theorem shows the consistency of the proposed bootstrap.

Theorem 4. Under the conditions of Theorem 2, as $n \to \infty$, $\tilde{\mathbb{G}}_n^{(1)}, \ldots, \tilde{\mathbb{G}}_n^{(N)}$ converge to independent copies of $\tilde{\mathbb{G}}$.

Proof. From Genest and Rémillard [7], $(\mathbb{D}_n^\star, \mathbb{B}_n, \mathbb{B}_n^\star, \boldsymbol{\Theta}_n, \boldsymbol{\Theta}_n^\star) \rightsquigarrow \left(\mathbb{D}^\perp, \mathbb{B}, \mathbb{B}^\perp, \boldsymbol{\Theta}, \boldsymbol{\Theta} + \boldsymbol{\Theta}^\perp\right)$, where $\left(\mathbb{D}^\perp, \mathbb{B}^\perp, \boldsymbol{\Theta}^\perp\right)$ is an independent copy of $(\mathbb{D}, \mathbb{B}, \boldsymbol{\Theta})$. Hence, since $n^{1/2}\{\mathbf{B}_n^\star(\mathbf{v}_n) - \mathbf{v}\} = \mathbb{B}_n^\star(\mathbf{v}_n) + \mathbb{B}_n(\mathbf{v})$, it follows from the Delta Method and Theorem 2 that

$$
\begin{aligned}
\tilde{\mathbb{G}}_n^{(k)}(u) &= \dot{\mathcal{C}}(u, v)^\top \boldsymbol{\Theta}_n^\star + \nabla_{\mathbf{v}} \mathcal{C}(u, \mathbf{v})\{\mathbb{B}_n^\star(\mathbf{v}_n) + \mathbb{B}_n(\mathbf{v})\} + \mathfrak{c}_{\boldsymbol{\theta}_0}(u, \mathbf{v})\mathbb{D}_n^\star(u) + o_P(1) \\
&\rightsquigarrow \dot{\mathcal{C}}(u, v)^\top \left(\boldsymbol{\Theta}^\perp + \boldsymbol{\Theta}\right) + \nabla_{\mathbf{v}} \mathcal{C}(u, \mathbf{v})\{\mathbb{B}^\perp(\mathbf{v}) + \mathbb{B}(\mathbf{v})\} + \mathfrak{c}_{\boldsymbol{\theta}_0}(u, \mathbf{v})\mathbb{D}^\perp(u) \\
&= \dot{\mathcal{C}}(u, v)^\top \boldsymbol{\Theta}^\perp + \nabla_{\mathbf{v}} \mathcal{C}(u, \mathbf{v})\mathbb{B}^\perp(\mathbf{v}) + \mathfrak{c}_{\boldsymbol{\theta}_0}(u, \mathbf{v})\mathbb{D}^\perp(u) + \mathbb{H}(u) = \tilde{\mathbb{G}}^\perp(u) + \mathbb{H}(u),
\end{aligned}
$$

where $\tilde{\mathbb{G}}^\perp$ is an independent copy of $\tilde{\mathbb{G}}$, while $n^{1/2}\left\{\mathcal{C}_{\boldsymbol{\theta}_n}(u, \mathbf{v}_n) - \mathcal{C}_{\boldsymbol{\theta}_0}(u, \mathbf{v})\right\} \rightsquigarrow \mathbb{H}$. As a result, $\tilde{\mathbb{G}}_n^{(1)}, \ldots, \tilde{\mathbb{G}}_n^{(N)}$ converge to independent copies of $\tilde{\mathbb{G}}$. □

Remark 2. Note that as shown in Genest and Rémillard [7], most interesting estimators are regular. In particular, estimators of the class $\mathcal{R}_1$: this means that there exists a continuously differentiable function J so that $E[J(U, \mathbf{V})] = 0$ and $\boldsymbol{\Theta}_n = n^{-1/2} \sum_{i=1}^n J\{D_n(U_i), B_n(\mathbf{V}_i)\} + o_P(1)$. For example, pseudo-maximum likelihood estimators, as defined in Genest et al. [8], belong to this class.

3.4.1. Construction of the uniform $100(1-\alpha)\%$ *confidence band for* Q

To construct the uniform confidence band on $[a,b] \subset (0,1)$, we generate N processes $\tilde{\mathbb{G}}^{(k)}$, $k \in \{1,\ldots,N\}$ and they are evaluated at $u \in \mathcal{A} = \{a + j(b-a)/m; j = 0,\ldots,m\}$, where m is fixed but large enough (say $m = 1000$). The density f_0 is estimated with a Gaussian kernel estimator f_{n0}, so $\mathfrak{h}(u) = h\{Q(u,\mathbf{x}),\mathbf{x}\}$ is estimated by $\mathfrak{h}_n(u) = f_{n0} \circ Q_{n,sp}(u,\mathbf{x})\mathfrak{c}_{\boldsymbol{\theta}_n}(u,\mathbf{v}_n)$, when $\mathbf{v}_n = \mathbf{F}_n(\mathbf{x})$. One then computes $b_{k,n} = \max_{u\in\mathcal{A}} \left|\tilde{\mathbb{G}}^{(k)}(u)\right| / \mathfrak{h}_n(u)$, $k \in \{1,\ldots,N\}$, and let $b_n(\alpha)$ be the associated quantile of order $1-\alpha$. The uniform confidence band about $Q(\cdot,\mathbf{x})$ is given by $Q_{n,sp}(u,\mathbf{x}) \pm n^{-1/2} b_n(\alpha)$, $u \in [a,b]$. A 95% confidence interval about a single point $Q(u,\mathbf{x})$ is given by $Q_{n,sp}(u,\mathbf{x}) \pm n^{-1/2} 1.96\hat{\sigma}/\mathfrak{h}_n(u)$ where $\hat{\sigma}^2$ is the sample variance of the values $\tilde{\mathbb{G}}^{(k)}(u)$, $k \in \{1,\ldots,N\}$.

Remark 3. Using our notations, the bootstrap algorithm proposed in Noh et al. [19] yields values $Q^{(k)}_{n,wqr}$, $k \in \{1,\ldots,N\}$, so that $\mathbb{Q}^{(k)}_{n,wqr} = n^{1/2}\left\{Q^{(k)}_{n,wqr} - Q\right\}$ converges to $\mathbb{Q}^{(k)}_{wqr} + \mathbb{Q}_{wqr}$, where $\mathbb{Q}^{(k)}_{wqr}$ is an independent copy of $\mathbb{Q}_{wqr}$. It then follows that their algorithm works for estimating the asymptotic variance σ^2_α, in the sense that what they call $\hat{\sigma}^2_{boot}$ satisfies $\hat{\sigma}^2_{boot} \approx \frac{\sigma^2_\alpha}{n}$ if n and N are large. However, their procedure is slower than the one we propose since we do not need to compute $Y^\star_i = F^{-1}_{n0}(U^\star_i)$ and $\mathbf{X}^\star_i = \mathbf{F}^{-1}_n(\mathbf{V}^\star_i)$, $i \in \{1,\ldots,n\}$. In addition, computing $\tilde{H}_n$ is faster than computing H_n.

4. Conclusion

We have shown that two seemingly different estimators for the conditional quantile function have in fact the same limiting distribution. However, the plug-in estimator is easier and faster to implement, in addition to being more accurate for small samples, as shown by simulations in Kraus and Czado [12] and Nasri and Bouezmarni [14]. Therefore, this is the one we recommend.

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