



Institut national  
de la recherche  
scientifique



UNIVERSITÀ  
DEGLI STUDI  
DI PALERMO

Università degli Studi di Palermo (UNIPA)  
Dipartimento di Ingegneria (ID)  
Université du Québec  
Institut National de la Recherche Scientifique (INRS)

# **Theoretical and experimental studies on quantum correlations and memory effects in composite quantum systems**

**Kobra Mahdavi pour**

Thesis submitted to obtain the double Degree of Philosophy Doctor (Ph.D.)  
in Information and Communication Technologies (ICT)  
and  
in Energy and Material Sciences (EMT)

Directeur de recherche

Prof. Rosario Lo Franco  
University of Palermo, Italy

Codirecteur de recherche

Prof. Roberto Morandotti  
INRS-EMT, Québec, Canada

l' Rights reserved (Kobra Mahdavi pour), year 2024

*To my mother, my hero*

# Acknowledgements

I am deeply grateful to my supervisor, Prof. Rosario Lo Franco, at the University of Palermo. His invaluable guidance, continuous support, and passion have significantly influenced the development of my academic personality. His profound and insightful perspective on problems has taught me invaluable lessons throughout my journey. His belief in me and appreciation for my efforts encouraged me greatly. Beyond his role in academia, he has also been an incredible mentor and a wonderful friend, making this journey both inspiring and enriching.

I am deeply thankful to my supervisor, Roberto Morandotti, at INRS-EMT. His expertise, thoughtful feedback, and consistent guidance have greatly supported me throughout my academic journey. His mentorship has encouraged me to improve and grow in my research efforts.

I thank my colleague, Dr. Farzam Nosrati, for our scientific discussions and his contagious enthusiasm, which served as invaluable guidance during my PhD journey. His simple yet profound approach to problem-solving has taught me a great deal. I am also thankful for his friendship during challenging times.

I want to express my heartfelt gratitude to my colleague and close friend, Mahshid Khazaei Shadfar. Your valuable input and feedback have been incredibly enlightening. The support, love, and companionship you offered me during my gloomiest times, in the indescribable darkness of those days, are beyond measure.

I would also like to thank my colleagues, Dr. Matteo Piccolini and Samira Ebrahimi, not only for their useful discussions but also for the joyful moments we shared.

I would like to express my gratitude to Prof. Ilenia Tinnirello and her research group for providing me with such a friendly environment that I never felt far from home. I am also deeply thankful to Dr. Daniele Croce, not only for his kindness, friendship, and support but also for the joyful and pleasant tea times we shared in the lab.

I sincerely thank my colleagues at INRS-EMT, especially Dr. Stefania Sciara and Dr. Mario Chemnitz, for their constant support and kindness. Their guidance, discussions, and collaboration have greatly improved my research. Working with such talented experts has been an honor, and I have learned so much from our time together. I would like to thank my colleague at INRS-EMT, Agnes George, for our scientific discussions, valuable collaboration, and insightful questions.

I would like to express my gratitude to Prof. Hossein Rangani Jahromi, my respected colleague, for teaching me subtlety, accuracy, and other aspects of research with passion. I deeply appreciate your trust and kindness in sharing your ideas with me.

I would like to express my heartfelt gratitude to my close friend, Dr. Leila Mehrvar. Your unwavering support and kindness during my most devastating and hopeless moments are beyond measure. I will never forget your comforting presence and steadfast companionship when I felt lost.

I would like to thank my close friend, Aida Amini, for her unwavering support and companionship during my gloomiest and darkest days.

I dedicate this thesis to my beloved family: my mother, my hero; my father; my sister; and my brothers. Your unconditional love, support, and understanding have been my greatest sources of strength, especially during challenging times. I am deeply grateful to have such an amazing mother who constantly motivates and encourages me. Your unwavering faith in me has kept me from giving up during tough moments and has inspired me to keep going. In my darkest times, I could always hear your comforting words: "At the end of the darkest night lies the light of day." I am deeply grateful to

my sister for her unwavering support, patience, and belief in me, which means more than words can express. And to my father: the person I am today is the result of the promise I made to you. I have never forgotten it and have always carried your memory in my heart.



# Abstract

This doctoral dissertation presents groundbreaking research on the theoretical and experimental exploration of quantum correlations and memory effects in composite quantum systems. The work is divided into two main parts: Part I investigates the utilization of quantum correlations in identical quantum systems within a quantum information framework, while Part II introduces a novel witness of non-Markovianity and evaluates its validity and efficiency across various systems.

Part I highlights the indistinguishability of identical qubits as a fundamental quantum resource that can be harnessed within the spatially localized operations and classical communication (sLOCC) framework to conditionally generate entanglement. This probabilistic and controllable scheme comprises three steps: initialization, deformation, and post-selection, enabling the generation of different classes of multipartite entangled states starting from a product state of  $N$  spatially distinguishable identical qubits. Using graph-based representations, these schemes are mapped onto colored, complex, and weighted digraphs corresponding to specific experimental setups. Additionally, the analysis explores indistinguishability from an operational perspective, emphasizing its role in quantum metrology for quantum-enhanced phase estimation using a NOON-like state ( $N=2$ ) as a probe. It also demonstrates how quantum walks can achieve optimal phase sensing measurements. Lastly, the study examines experimentally controllable inhomogeneous quantum walk dynamics as a platform for investigating the effects of coherent disorder on quantum correlations between indistinguishable photons, providing insights into dynamic quantum systems.

Part II introduces a new witness of non-Markovianity and examines its validity and efficiency through various examples. Inspired by the observation that non-Markovian effects can accelerate system dynamics and that quantum statistical speed quantifiers can determine the evolution time limit, a novel witness is proposed to characterize the non-Markovian behavior of open quantum systems. This witness is based on the positive change rate of the Hilbert-Schmidt speed (HSS), a specific form of quantum statistical speed. A significant advantage of this witness is that it does not require the diagonalization of the system's evolved density matrix. Its efficiency is tested across low- and high-dimensional systems as well as multiqubit open quantum systems. Furthermore, the study demonstrates the HSS-based witness as a reliable tool for evaluating and detecting global memory effects in both unital and non-unital correlated channels with varying noisy spectral densities. Additionally, it explores the impact of classical correlations between sequences of noisy quantum channels on the non-Markovian memory effect.

The contributions made in this thesis significantly advance the field of quantum information processing by enhancing our understanding of the role of indistinguishability in quantum phenomena and introducing a novel witness for non-Markovianity that provides practical tools for characterizing memory effects in open quantum systems. Together, these findings offer fundamental insights into quantum dynamics and open promising avenues for the development of future quantum technologies.

# Résumé

Cette thèse doctorale présente des recherches novatrices sur l'exploration théorique et expérimentale des corrélations quantiques et des effets de mémoire dans les systèmes quantiques composites. Le travail est divisé en deux parties principales : la première partie étudie l'utilisation des corrélations quantiques dans des systèmes quantiques identiques dans le cadre de l'information quantique, tandis que la deuxième partie propose un nouvel indicateur de non-markovianité et évalue sa validité et son efficacité dans divers systèmes.

La première partie met en avant l'indistinguabilité des qubits identiques en tant que ressource quantique fondamentale pouvant être exploitée dans le cadre des opérations localisées spatialement et communication classique (sOLCC) pour générer de l'intrication de manière conditionnelle. Ce schéma probabiliste et contrôlable se compose de trois étapes : l'initialisation, la déformation et la post-sélection, permettant de générer différentes classes d'états intriqués multipartites à partir d'un état produit de  $N$  qubits identiques spatialement distinguables. En utilisant des représentations basées sur des graphes, ces schémas sont traduits en graphes orientés colorés, complexes et pondérés correspondant à des configurations expérimentales spécifiques. De plus, l'analyse explore l'indistinguabilité d'un point de vue opérationnel, en mettant l'accent sur son rôle en métrologie quantique pour l'estimation de phase améliorée par des moyens quantiques, utilisant un état de type NOON ( $N=2$ ) comme sonde. Elle démontre également comment les marches quantiques peuvent permettre d'obtenir des mesures optimales pour des problèmes de détection de phase. Enfin, l'étude examine les dynamiques de marches quantiques inhomogènes contrôlables expérimentalement, servant de plateforme pour étudier les effets du désordre cohérent sur les corrélations quantiques entre photons indistinguables, offrant ainsi des perspectives sur les systèmes quantiques dynamiques.

La deuxième partie introduit un nouvel indicateur de non-markovianité et examine sa validité et son efficacité à travers divers exemples. Inspiré par l'observation que les effets non-markoviens peuvent accélérer la dynamique des systèmes et que les quantificateurs de vitesse statistique quantique peuvent déterminer la limite du temps d'évolution, un nouvel indicateur est proposé pour caractériser le comportement non-markovien des systèmes quantiques ouverts. Cet indicateur est basé sur le taux de variation positif de la vitesse de Hilbert-Schmidt (HSS), une forme spécifique de vitesse statistique quantique. Un avantage majeur de cet indicateur est qu'il ne nécessite pas la diagonalisation de la matrice densité évoluée du système. Son efficacité est testée dans des systèmes de faible et de haute dimension ainsi que dans des systèmes quantiques ouverts à plusieurs qubits. En outre, l'étude démontre que l'indicateur basé sur la HSS constitue un outil fiable pour évaluer et détecter les effets de mémoire globale dans les canaux corrélés, unitaires et non-unitaires, avec des densités spectrales bruyantes variables. Enfin, elle explore l'impact des corrélations classiques entre des séquences de canaux quantiques bruyants sur l'effet de mémoire non-markovienne.

Les contributions de cette thèse représentent une avancée significative dans le domaine de l'information quantique, en améliorant notre compréhension du rôle de l'indistinguabilité dans les phénomènes quantiques et en proposant un nouvel indicateur de non-markovianité offrant des outils pratiques pour la caractérisation des effets de mémoire dans les systèmes quantiques ouverts. Ces résultats apportent des connaissances fondamentales sur la dynamique quantique et ouvrent des perspectives prometteuses pour le développement des technologies quantiques futures.

# Synopsis

Cette thèse doctorale en cours présente des recherches novatrices axées sur les études théoriques et expérimentales des corrélations quantiques et des effets de mémoire dans les systèmes quantiques composites. La première partie de l'étude explore l'utilisation des corrélations quantiques dans des systèmes quantiques identiques dans le cadre de l'information quantique. Bien que les systèmes quantiques ne soient pas idéalement isolés, leurs interactions avec l'environnement entraînent une perte de cohérence et d'information. Par conséquent, la dynamique de ces systèmes peut être classée en deux types : une dynamique sans mémoire, dite markovienne, et une dynamique avec mémoire, dite non markovienne. Dans la deuxième partie, j'introduis un nouvel outil pour détecter la non-markovianité et j'évalue sa validité et son efficacité dans divers systèmes.

## **Part I: Exploitation des corrélations quantiques des systèmes quantiques identiques dans un scénario d'information quantique**

Dans la première partie de ma thèse de doctorat, j'explore les corrélations quantiques des systèmes quantiques identiques dans des scénarios d'information quantique. Je commence par examiner le rôle de l'indistinguabilité des particules identiques comme ressource pour la génération d'états intriqués multipartites. Ensuite, j'étudie l'indistinguabilité d'un point de vue opérationnel et j'analyse une autre classe importante d'états intriqués multipartites, à savoir l'état NOON, ainsi que son application dans l'estimation de phase améliorée par la quantique. En outre, pour étudier les caractéristiques quantiques basées sur le comportement d'indistinguabilité dans un cadre dynamique, je discute des Marches quantiques en temps discret.

### **L'indistinguabilité des particules identiques comme ressource pour le traitement de l'information quantique**

#### **Indistinguabilité versus identité**

Les particules sont identiques lorsque leurs propriétés intrinsèques, y compris la charge électrique, la masse et le spin total, sont les mêmes. Les sous-systèmes identiques (photons, électrons, atomes, qubits, plasmons, etc.), qu'ils appartiennent à des systèmes bosoniques ou fermioniques, constituent les éléments de base des réseaux quantiques [1, 2, 3]. En général, deux types de grandeurs physiques peuvent être utilisés pour décrire et distinguer les particules identiques : les grandeurs dépendantes de l'état et les grandeurs indépendantes de l'état. Les grandeurs dépendantes de l'état désignent celles qui varient en fonction de l'état des particules (par exemple, la position et la vitesse), tandis que les grandeurs indépendantes de l'état correspondent aux propriétés intrinsèques des particules (par exemple, la masse, la charge électrique, la position, la quantité de mouvement, etc.).

En physique classique, les particules identiques sont distinguables et peuvent être caractérisées individuellement à travers un ensemble de grandeurs physiques uniques et indépendantes de l'état. Bien que deux ou plusieurs particules classiques puissent partager plusieurs propriétés physiques, elles diffèrent néanmoins par leurs positions spatiales. En d'autres termes, il est possible d'attribuer une identité physique à chaque particule classique en lui assignant un nom ou un label unique. Ces labels pour les particules classiques incluent des informations physiques telles que la position et la trajectoire. Le concept physique de l'identité des particules joue un rôle dans la dynamique. Cela est nécessaire pour garantir que les particules sont individuellement accessibles via des observables. En mécanique classique, cette situation est satisfaite grâce à des techniques de détection sensibles à la position spatiale, permettant d'identifier et de suivre les trajectoires des particules identiques sans affecter leur dynamique.

En mécanique quantique, les particules élémentaires sont décrites à l'aide de grandeurs physiques dépendantes de l'état et indépendantes de l'état. Contrairement à la mécanique classique, les trajectoires des particules identiques ne peuvent pas être observées sans altérer leur dynamique et leur comportement ; de plus, elles ne peuvent pas être attribuées

de manière unique à des particules spécifiques. La description ondulatoire et probabiliste permet aux fonctions d'onde de différentes particules de se chevaucher spatialement, entraînant une probabilité non nulle d'occuper simultanément la même région de l'espace. Dans cette situation, utiliser des mesures reposant sur des propriétés dépendantes de l'état, telles que celles basées uniquement sur les positions des particules dans les régions de chevauchement, ne permet pas à l'observateur de déterminer quelle particule spécifique correspond au résultat mesuré. Prenons l'exemple de sources synchronisées de photons A et B, chacune émettant des photons uniques qui entrent en collision simultanément dans une région spatiale restreinte de détection avec une certaine probabilité. Si un détecteur de photons unique s'active dans cette région, il est impossible de déterminer la source du photon détecté, et, par conséquent, les particules sont considérées comme indistinguables. En conclusion, alors que l'identité est une propriété intrinsèque des particules, l'indistinguabilité est liée au processus de mesure et à la perspective du détecteur.

### Approche sans étiquette

L'approche standard utilise des labels non physiques (non observables) pour distinguer les particules identiques, modifiant ainsi la description de leur dynamique. Cela nécessite également l'application du postulat de symétrisation par rapport à la permutation des labels non physiques. Selon ce postulat, l'état global décrivant un ensemble de particules identiques doit rester inchangé lorsque les rôles de deux particules quelconques sont échangés. Les particules bosoniques présentent une symétrie sous échange de particules, tandis que les états fermioniques sont antisymétriques lors d'échanges similaires. Considérons un exemple simple : un système composé de deux particules identiques décrites dans  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Selon l'Éq. 1.1, un seul des états suivants est autorisé. Les états sous la forme de produits  $|\Psi\rangle = |\phi\rangle_1 \otimes |\psi\rangle_2$  sont interdits en mécanique quantique pour les particules identiques. Le principe d'exclusion de Pauli est une conséquence immédiate du postulat de symétrisation, stipulant que deux ou plusieurs fermions ne peuvent pas occuper simultanément le même état quantique. Un tel postulat a des conséquences significatives en théorie quantique des champs, dans les condensats de Bose-Einstein et en physique des particules. Cependant, une autre conséquence du postulat de symétrisation est que deux particules identiques, qu'il s'agisse de fermions ou de bosons, sont toujours intriquées. Il s'agit d'une conclusion mathématique évidente. L'approche standard engendre des difficultés méthodologiques et pratiques dans la caractérisation de l'intrication des particules identiques, car elle utilise des labels non physiques et impose le postulat de symétrisation de l'état. Pour relever ces défis, une approche alternative, connue sous le nom d'"approche sans étiquette" pour les particules identiques, a été proposée [4, 5]. Au lieu d'utiliser des propriétés non observables, cette approche repose sur des observables pour déterminer l'état.

Pour résumer brièvement l'approche sans étiquette, supposons deux particules identiques situées dans des régions spatiales distinctes, avec  $|\xi\rangle$  et  $|\chi\rangle$  représentant respectivement leurs états. Les mesures sur une seule particule sont effectuées à l'aide de dispositifs localisés dans les régions  $L$  et  $R$ , où le postulat de symétrisation n'a aucun impact sur les prédictions physiques [6, 7, 8, 9]. Dans ce scénario, la matrice de densité peut être obtenue en supposant que  $L$  et  $R$  sont des labels réels et distinguables, sans appliquer la symétrisation à ces labels. De plus, le résultat d'une mesure effectuée dans la région  $L(R)$  ne peut pas provenir d'une particule située dans  $R(L)$  [9]. L'état global peut ainsi être factorisé sous la forme  $|\Psi\rangle = |\xi\rangle L \otimes |\chi\rangle R$ , que ce soit pour les bosons ou les fermions. Par conséquent, les particules identiques se comportent comme des particules non identiques. Néanmoins, lorsque les particules sont parfaitement distinguables, l'état global ne peut plus être écrit sous forme d'état produit.

Un état à une particule peut être caractérisé par un ensemble complet d'observables dans l'approche sans étiquette. L'état de deux particules identiques peut être représenté, sans labels non observables, comme une liste d'états à une particule  $|\phi\rangle_1$  et  $|\phi\rangle_2$ , sous la forme de  $|\Psi\rangle = \frac{|\phi_1, \phi_2\rangle}{\mathcal{N}}$ , où  $\mathcal{N}$  est le facteur de normalisation.

L'étape suivante consiste à définir l'action des opérateurs dans l'approche sans étiquette. Considérons l'opérateur à une particule  $O^{(1)}$  agissant sur l'état global de deux particules, une à la fois, ce qui donne  $O^{(1)}|\Psi\rangle = |O^{(1)}\phi_1, \phi_2\rangle + |\phi_1, O^{(1)}\phi_2\rangle$ , de manière similaire à l'approche basée sur les étiquettes [8]. Dans un espace d'état à une particule avec une base  $\beta^{(1)} = |\psi_k\rangle, k = 1, 2, \dots$ , l'opération de l'opérateur à une particule est définie sous une forme générale comme  $\hat{O}^{(1)} = \sum_{j,k} |\psi_j\rangle \langle \psi_k|$ . Ainsi, un produit scalaire symétrique entre des espaces d'états de différentes dimensions peut être établi comme suit

$$\langle \psi_k | \phi_1, \phi_2 \rangle := \langle \psi_k | \phi_1 \rangle \langle \phi_2 \rangle + \eta \langle \psi_k | \phi_2 \rangle \langle \phi_1 \rangle, \quad (1)$$

où  $\eta = 1(-1)$  correspond aux statistiques bosoniques ou fermioniques. L'équation illustre l'état pur réduit non normalisé d'une seule particule résultant de la projection d'un état à deux particules sur  $|\psi_k\rangle$  (mesure projective d'une particule). De manière similaire, nous définissons l'amplitude de probabilité pour deux particules comme un produit scalaire symétrique entre les espaces d'états d'un vecteur d'état à deux particules, exprimée comme une combinaison linéaire de produit s d'amplitudes à une particule, définie comme suit,

$$\langle \Phi'_1, \Phi'_2 | \Phi_1, \Phi_2 \rangle := \langle \Phi'_1 | \Phi_1 \rangle \langle \Phi'_2 | \Phi_2 \rangle + \eta \langle \Phi'_1 | \Phi_2 \rangle \langle \Phi'_2 | \Phi_1 \rangle. \quad (2)$$

L'équation découle de l'effet d'indistinguabilité, où l'amplitude de probabilité de trouver une particule dans l'état  $\Phi'_1(\Phi'_2)$  résulte de la présence d'une particule dans  $\Phi_1(\Phi_2)$ .

Nous pouvons définir la trace partielle à une particule  $Tr^{(1)}$  d'un système, qui est interprétée physiquement comme l'ensemble statistique de tous les états réduits normalisés obtenus après une mesure projective sur les états de base. Cette

opération correspond à la mesure d'une particule d'un sous-système sans enregistrer le résultat, comme

$$Tr^{(1)}[|\Psi\rangle\langle\Psi|] := \sum_k \langle\psi_k|\phi_1, \phi_2\rangle\langle\phi_1, \phi_2|\psi_k\rangle \quad (3)$$

en conséquence, la matrice de densité réduite à une particule peut être déterminée comme,

$$\rho^{(1)} = \frac{Tr^{(1)}[|\Psi\rangle\langle\Psi|]}{\mathcal{M}}, \quad (4)$$

où  $\mathcal{M}$  est une constante de normalisation telle que  $Tr^{(1)}\rho^{(1)} = 1$ . Selon la définition ci-dessus, la trace partielle représente une opération physique sur l'état du système basée sur des mesures projectives effectives, contrairement à la trace partielle effectuée avec des labels non physiques lorsque les paramètres de mesure sont ambigus.

En conséquence des résultats préliminaires fondamentaux issus de l'approche sans étiquettes, nous pouvons définir l'intrication  $E_M(|\Psi\rangle)$  de l'état pur global par rapport aux mesures effectuées dans la région localisée  $M$  [6, 10]. Celle-ci est déterminée par l'entropie de von Neumann de la matrice de densité réduite à une particule donnée dans l'Éq. 1.5. Ainsi, nous avons :

$$E_M(|\Psi\rangle) := S(\rho_M^{(1)}) = - \sum_j \lambda_j \log_2 \lambda_j, \quad (5)$$

où  $S(\rho_M^{(1)})$  est l'entropie de von Neumann de  $\rho_M^{(1)}$  et  $\lambda_j$  en sont les valeurs propres. En raison de l'indistinguabilité spatiale, la quantité de  $E_M$  est supposée dépendre de  $M$ . Ainsi, lorsque les particules sont soit spatialement séparées, soit dans le même mode, le mode localisé où la mesure est effectuée n'affecte pas le résultat, et nous supposons que l'intrication obtenue est intrinsèque, une intrication absolue du système.

En conclusion, l'approche sans étiquettes permet de déterminer l'intrication des particules identiques en utilisant les mêmes notions communément adoptées pour les particules distinguables, notamment l'entropie de von Neumann de l'état réduit après avoir effectué une trace partielle, surmontant ainsi les inconvénients liés à l'utilisation d'étiquettes non physiques.

### Opérations localisées spatialement et communication classique (sOLCC)

Un cadre opérationnel bien établi est utilisé pour exploiter les propriétés quantiques telles que l'intrication et la cohérence dans des systèmes composites avec des particules non identiques. Ce cadre, dans le contexte des opérations locales et de la communication classique (LOCC) [11], repose sur l'adressage et l'exécution d'opérations locales sur des modes orthogonaux. Cependant, étant donné que les particules identiques ne peuvent pas être adressées individuellement, en particulier lorsqu'elles se chevauchent spatialement, l'idée de "localité des particules" inhérente à LOCC devient dénuée de sens. Ainsi, un cadre opérationnel adapté basé sur les opérations localisées spatialement et la communication classique (sOLCC) [12] peut à son tour quantifier les propriétés quantiques des systèmes composites de particules identiques qui se chevauchent spatialement. Résumé brièvement le cadre sOLCC en commençant par deux particules identiques préparées indépendamment dans l'état global suivant comme,  $|\Psi\rangle = |\psi_1 \uparrow, \psi_2 \downarrow\rangle$ , où  $\psi_1$  et  $\psi_2$  sont des fonctions d'onde spatiales illustrant les degrés de liberté externes, et  $\uparrow$  et  $\downarrow$  indiquent le pseudospin représentant les états internes. En conséquence, pour activer et accéder à l'intrication au sein de ce système, des mesures locales sur des particules uniques, insensibles aux états de pseudospin, sont effectuées dans les régions spatiales  $L$  et  $R$ , comme illustré dans la Fig. 1.1. Le projecteur utilisé pour le processus de comptage local des particules est défini comme,

$$\hat{\Pi}^{(2)} = \sum_{\sigma, \tau = \uparrow, \downarrow} |L\sigma, R\tau\rangle\langle L\sigma, R\tau|, \quad (6)$$

qui a projeté l'état  $|\Psi\rangle$  sur une base à deux particules  $\beta_{LR} = \{|L\uparrow, R\uparrow\rangle, |L\uparrow, R\downarrow\rangle, |L\downarrow, R\uparrow\rangle, |L\downarrow, R\downarrow\rangle\}$ . L'état pur normalisé projeté  $|\Psi_{LR}\rangle$  est alors :

$$|\Psi_{LR}\rangle = \frac{\hat{\Pi}^{(2)}|\Psi\rangle}{\sqrt{\langle\Psi|\hat{\Pi}^{(2)}|\Psi\rangle}} = \frac{lr'|L\uparrow, R\downarrow\rangle + \eta l'r|L\downarrow, R\uparrow\rangle}{\sqrt{|lr'|^2 + |l'r|^2}}, \quad (7)$$

$l = \langle L|\psi_1\rangle$ ,  $r' = \langle R|\psi_1\rangle$ ,  $l' = \langle L|\psi_2\rangle$  and  $r = \langle R|\psi_2\rangle$  sont les amplitudes de probabilité de trouver une particule dans les sites L et R. En raison de la post-sélection, la probabilité de succès de sOLCC, représentée par  $P_{LR}$ , est obtenue comme

$$P_{LR} = P_L P'_R + P'_L P_R, \quad (8)$$

où  $P_L = |l|^2$ ,  $P_R = |r|^2$ ,  $P'_L = |l'|^2$  and  $P'_R = |r'|^2$ , sont les probabilités de trouver une particule dans les régions L et R, respectivement, au sein des fonctions d'onde  $\psi_1$  and  $\psi_2$ . Après avoir effectué la trace partielle sur une région, telle que L,

dans la base  $\{|L \downarrow\rangle, |L \uparrow\rangle\}$ , suivie d'une projection sur la région R décrite par la base  $\{|R \downarrow\rangle, |R \uparrow\rangle\}$ , le résultat est une matrice de densité réduite donnée par :

$$\rho_R^{(1)} = \frac{1}{P_L P'_R + P'_L P_R} (P'_L P_R |R \uparrow\rangle \langle R \uparrow| + P'_L P_R |R \downarrow\rangle \langle R \downarrow|). \quad (9)$$

Par la suite, l'entropie de von Neumann  $S(\rho^{(1)}) = E_{LR}(|\Psi\rangle) = -\text{Tr}(\rho_{LR}^{(1)} \log_2 \rho_{LR}^{(1)})$  fournit l'intrication opérationnelle comme

$$E_{LR}(|\Psi\rangle) = -\frac{P_L P'_R}{P_L P'_R + P'_L P_R} \log_2 \frac{P_L P'_R}{P_L P'_R + P'_L P_R} - \frac{P'_L P_R}{P_L P'_R + P'_L P_R} \log_2 \frac{P'_L P_R}{P_L P'_R + P'_L P_R}, \quad (10)$$

qui révèle les corrélations quantiques entre les pseudospins des particules, observées à travers des mesures locales. Le degré de cette intrication dépend des probabilités de trouver les particules dans les régions localisées L et R.

Discutons de l'étendue de l'intrication en termes de chevauchement spatial des fonctions d'onde  $\psi_1$  et  $\psi_2$ . Le premier scénario est celui où il n'y a aucun chevauchement spatial, dans lequel les fonctions d'onde  $\psi_1$  et  $\psi_2$ , sont spatialement séparées et localisées autour des sites L et R, respectivement; ainsi,  $P'_L = P_R = 0$ . Dans ce scénario, l'état  $|\Psi_{LR}\rangle = |L \uparrow, R \downarrow\rangle$  est séparable, ce qui entraîne une intrication nulle  $E(|\Psi_{LR}\rangle) = 0$ . Cela est dû à la séparation spatiale et à l'adressage local des particules. Un autre exemple est lorsque les fonctions d'onde  $\psi_1$  et  $\psi_2$  se chevauchent partiellement, différentes situations se présentent. Si une mesure locale sur L ou R est effectuée en dehors de la région de chevauchement (par exemple,  $P'_L = 0$  ou  $P_R = 0$ ), l'intrication est nulle. Dans le cas où des mesures locales sont effectuées dans la région de chevauchement, l'intrication entre les pseudospins des particules est obtenue conditionnellement avec une probabilité  $P_{LR}$  donnée par l'Éq. 1.10. Enfin, les deux fonctions d'onde se chevauchent complètement, et c'est lorsque les deux particules peuvent être trouvées avec la même probabilité non nulle dans la même région spatiale. Dans un tel scénario, par exemple, lorsque  $P_L = P_R = P'_L = P'_R = 1/2$ , l'intrication opérationnelle maximale  $E(|\Psi_{LR}\rangle) = 1$  est obtenue avec une probabilité de succès  $P_{LR} = 1/2$ . Cette analyse révèle que la quantification de l'intrication opérationnelle de l'Éq. 1.12 n'est pas seulement une propriété intrinsèque, mais dépend également de la structure de l'état et de la méthode de mesure.

Le cadre sOLCC dans l'approche sans étiquettes permet l'identification, la manipulation et l'exploitation de l'indistinguabilité des particules identiques. Cela fait de l'indistinguabilité des systèmes élémentaires identiques (particules) une caractéristique quantique intrinsèque qui joue un rôle important dans le traitement de l'information quantique, notamment dans la génération d'intrication et l'estimation de phase. Dans les sections suivantes, je discuterai brièvement de ces processus.

## Génération d'intrication

L'indistinguabilité des particules identiques qu'elle soit obtenue par chevauchement spatial ou par identité des chemins photoniques constitue une véritable ressource quantique pouvant être exploitée pour générer des états intriqués bipartites et multipartites. Bien que l'identité des chemins photoniques repose sur une interaction directe entre des particules indistinguables, la porte d'intrication basée sur l'indistinguabilité dépend uniquement du chevauchement spatial de deux particules indépendantes avec des pseudospins opposés, qui ne se rencontrent qu'au niveau de la détection [12]. Ici, nous discutons brièvement de l'indistinguabilité de cette porte d'intrication pour la génération d'états intriqués bipartites, démontrée à la fois expérimentalement [13, 14] et théoriquement [12]. Considérons deux particules identiques, provenant de sources distinctes, qui sont initialement dans une forme non corrélée donnée par  $|\psi_1 \uparrow, \psi_2 \downarrow\rangle$ , qui est donnée dans l'approche sans étiquettes. La fonction d'onde de chaque particule est distribuée de manière cohérente et contrôlée dans deux régions opérationnelles distantes, L et R, à l'aide d'un séparateur de faisceau,  $|\psi_1\rangle \rightarrow |\psi_{1D}\rangle$ ,  $|\psi_2\rangle \rightarrow |\psi_{2D}\rangle$ , ainsi, l'état global prend la forme suivante  $|\Psi\rangle = |\psi_{1D} \uparrow, \psi_{2D} \downarrow\rangle$ , où  $|\psi_{1D}\rangle = l|\psi_L\rangle + r|\psi_R\rangle$  ( $|l|^2 + |r|^2 = 1$ ) et  $|\psi_{2D}\rangle = l'|\psi'_L\rangle + r'|\psi'_R\rangle$  ( $|l'|^2 + |r'|^2 = 1$ ), avec  $|\psi_x\rangle$  and  $|\psi'_x\rangle$ , désignant les deux fonctions d'onde situées dans la région spatiale  $x$ . La mesure sOLCC, illustrée dans la Fig. 1.2, peut être réalisée par comptage de photons uniques. Étant donné que les chemins des particules ne partagent aucune information passée et ne se rencontrent qu'au niveau des détecteurs, ces derniers sont insensibles aux degrés de liberté intégrés. Par conséquent, les particules apparaissent indistinguables aux yeux des détecteurs. Ainsi, le projecteur donné dans l'Éq. 2.1  $\hat{\Pi}^{(2)} = \sum_{\sigma, \tau=\uparrow, \downarrow} |L\sigma, R\tau\rangle \langle L\sigma, R\tau|$  pour activer l'état souhaité est appliqué sur le  $|\Psi\rangle$  et conduit à l'état quantique donné  $|\Psi_{LR}\rangle = \frac{l r' |L\uparrow, R\downarrow\rangle + \eta l' r |L\downarrow, R\uparrow\rangle}{\sqrt{|l r'|^2 + |\eta l' r|^2}}$  dans l'Éq. 1.9 avec une probabilité de  $P_{LR} = |l r'|^2 + |r l'|^2$ , où  $\eta$  fait référence aux statistiques fermioniques ou bosoniques.

Étant donné l'importance des états intriqués multipartites tels que les états W, Dicke, GHZ et les états de cluster dans divers domaines des technologies quantiques, notamment l'informatique quantique [15, 16], la communication quantique [17, 18, 19], la métrologie quantique [20] et la téléportation quantique [21], la question essentielle est de savoir si nous pouvons générer ces états intriqués pour les systèmes bosoniques et fermioniques en exploitant l'indistinguabilité spatiale des particules identiques. Selon les résultats précédents décrits dans [12], le formalisme sOLCC peut être étendu

à des scénarios généraux impliquant  $N$  particules réparties dans  $N$  régions. Cette extension facilite l'étude de l'intrication multipartite entre plusieurs particules identiques. Dans le chapitre 2, je discuterai de ce sujet en détail, car il constitue le principal résultat de ma thèse de doctorat : le schéma général contrôlable pour générer des états intriqués multipartites. J'expliquerai également comment représenter ce schéma mathématique de manière plus intuitive et visuelle. En utilisant une représentation basée sur les graphes, nous traduisons et cartographions notre schéma en graphes dirigés colorés, complexes et pondérés, chacun correspondant à une configuration expérimentale [22].

D'un point de vue opérationnel, les particules sont également considérées comme indistinguables si elles partagent le même mode dans le champ électromagnétique tel que la polarisation, le temps, la fréquence, la position et la quantité de mouvement lorsqu'elles sont caractérisées par une interférence à deux particules [23]. En conséquence, on peut noter qu'une autre classe importante d'états intriqués multipartites est l'état  $N00N$  [24, 25]. Dans cet exemple,  $N$  particules indistinguables sont dans une superposition entre deux chemins ou modes, toutes se trouvant soit dans un chemin, soit dans l'autre mode. Comme les états  $N00N$  permettent d'améliorer la résolution spatiale et de phase avec un facteur d'échelle  $N$ , soit  $\frac{1}{N}$ , ils sont au cur de la détection quantique [26] et de la métrologie quantique [27, 28], en particulier pour l'estimation de phase quantique [29]. Dans la section suivante, je discuterai brièvement de l'estimation de phase.

## Théorie de l'estimation de phase

La mesure précise de grandeurs physiques, telles que l'intensité d'un champ, d'une force ou du temps, constitue l'un des piliers fondamentaux de la physique fondamentale et de ses applications, notamment en métrologie, en navigation par satellite et en géopositionnement.

Les mesures précises sont souvent réalisées en convertissant les grandeurs physiques en décalages de phase, estimés à l'aide de techniques interférométriques [29, 30]. Dans le scénario d'estimation de phase, le paramètre inconnu est un décalage de phase  $\phi$  entre deux chemins ou modes différents (ou, plus généralement, tout degré de liberté) des photons. La limite ultime atteignable pour estimer  $\phi$  est donnée par la borne de Cramér-Rao [31].

$$\langle \Delta\phi \rangle^2 \geq \frac{1}{\mu F_c(\phi)}, \quad (11)$$

où  $\mu$  représente un nombre asymétriquement grand de répétitions de l'expérience. En effet, la variance constitue une erreur intrinsèque dans les processus d'estimation. Ici,  $F_c(\phi)$  correspond à l'information de Fisher (classique), définie comme suit

$$F_c(\phi) = \sum_i \frac{[\partial_\phi p_i(\phi)]^2}{p_i(\phi)}, \quad (12)$$

où la règle de Born  $p_i(\phi) = \text{Tr}[\rho_\phi E_i]$  représente la distribution de probabilité associée à un ensemble de mesures à valeurs d'opérateurs positifs (POVMs)  $\{E_i\}$  qui satisfont  $\sum_i E_i = \mathbf{I}$ . Dans cette information de Fisher, l'erreur sur l'estimation d'un paramètre  $\phi$  Cela dépend à la fois de l'état initial et de l'ensemble des mesures. Par conséquent, on peut optimiser  $F_c(\phi)$  sur toutes les POVM possibles afin de maximiser l'information de Fisher classique et d'évaluer les limites ultimes avec précision. Cette limite ultime est connue sous le nom de borne de Cramér-Rao quantique (QCRB)

$$\langle \Delta\phi \rangle^2 \geq \frac{1}{\mu F_C(\phi)} \geq \frac{1}{\mu F_Q(\phi)}, \quad (13)$$

où l'information de Fisher quantique (QFI) est donnée par  $F_Q(\phi) = \text{Tr}[\hat{L}_\phi^2 \hat{\rho}_\phi]$  [24], en introduisant une borne supérieure sur l'information de Fisher classique (précision) pour toute mesure quantique. Ici,  $\hat{L}_\phi$  est l'opérateur autoadjoint de la dérivée logarithmique symétrique (SLD) satisfaisant l'équation  $\frac{\partial \hat{\rho}_\phi}{\partial \phi} = \frac{1}{2}(\hat{L}_\phi \hat{\rho}_\phi + \hat{\rho}_\phi \hat{L}_\phi)$ . On peut réécrire l'état  $\hat{\rho}_\phi$  dans sa base propre comme  $\rho_\phi = \sum_i \lambda_i |e_i\rangle \langle e_i|$ , et ensuite obtenir l'information de Fisher quantique sous la forme suivante

$$F_Q(\phi) = 2 \sum_{i \neq j} \frac{1}{\lambda_i + \lambda_j} \left| \left\langle e_i \left| \frac{\partial \hat{\rho}_\phi}{\partial \phi} \right| e_j \right\rangle \right|^2, \quad (14)$$

où, dans le cas d'un état quantique pur, l'expression ci-dessus se simplifie en [20]

$$F_Q(\phi) = 4 \left[ \langle \partial_\phi \psi_\phi | \partial_\phi \psi_\phi \rangle - |\langle \partial_\phi \psi_\phi | \psi_\phi \rangle|^2 \right]. \quad (15)$$

Jusqu'ici, j'ai parlé de la manière dont l'indistinguabilité joue un rôle important dans la génération de l'entrelacement. Il est crucial de comprendre comment les caractéristiques quantiques basées sur l'indistinguabilité se comportent dans un cadre dynamique, en particulier lorsqu'on considère des bosons se propagent à travers un système non homogène. Une plateforme théorique appropriée pour mener une telle étude est fournie par la marche quantique (QW), qui offre un modèle de propagation cohérente très général. Ainsi, dans la section suivante, je discuterai du modèle de la marche quantique.

## Marche quantique pour le traitement de l'information quantique

Les marches quantiques sont des outils avancés pour la construction d'algorithmes quantiques [32], le calcul quantique [33], et fournissent un aperçu des phénomènes complexes dans la nature, tels que le processus de photosynthèse [34, 35], et expliquent la faible perte lors du transport dans les systèmes de collecte de lumière [36]. Dans ce qui suit, je donne une brève introduction au contexte des marches quantiques.

**Marche aléatoire classique** - Une marche aléatoire est le mouvement aléatoire d'un marcheur sur une ligne, où, en lançant une pièce, le marcheur décide s'il doit se déplacer vers la droite ou vers la gauche. Les notions de + et - sont utilisées pour représenter le résultat de face ou pile et, en conséquence, se déplacer vers la droite ou vers la gauche. De même,  $P_+$  and  $P_-$  sont utilisés pour représenter les probabilités de chaque issue, et pour une pièce ordinaire non biaisée, on peut s'attendre à ce que  $P_+ = P_- = \frac{1}{2}$ . Avant de lancer la pièce, il y a une chance  $P_+$  que le marcheur prenne le chemin de droite ou, avec une probabilité  $P_-$  qu'il prenne le chemin de gauche. Cependant, après le lancer de la pièce, il n'y a plus d'ambiguïté, et le marcheur effectue un mouvement vers la droite ou vers la gauche avec une certitude absolue. Si le marcheur est initialement à la position  $x = 0$ , se déplacer vers la droite et vers la gauche correspond à avancer vers la position  $x = +1$  et  $x = -1$ , respectivement. Le marcheur peut continuer à se déplacer le long d'un arbre de décision en utilisant des lancers de pièce. Cette procédure est connue sous le nom de marche aléatoire classique.

Les marches quantiques (MQ) l'analogue quantique des marches aléatoires classiques permettent une variété d'applications, y compris des modèles universels de calcul quantique [37], des dispositifs universels de mesure quantique [38], et des simulateurs quantiques [39].

### Marche quantique en temps discret

En général, le modèle de la marche quantique en temps discret (marche quantique en temps discret) est défini sur l'espace de Hilbert conjoint  $H = H_p \otimes H_c$  de la position ( $H_p$ ) et de la pièce ( $H_c$ ) du marcheur. L'état du système est donc défini par  $|\psi\rangle_0 = \sum_k c_k |k\rangle$ , où  $c_k$  st l'amplitude de probabilité de trouver un photon dans le mode  $k$ . Chaque mode consiste en deux degrés de liberté  $|x, c\rangle$ , où la position des particules ( $|x\rangle = \dots, |-1\rangle, |0\rangle, |1\rangle, \dots$ ) et l'état de la pièce  $|c\rangle = |R\rangle, |L\rangle$ , où  $|R\rangle$  et  $|L\rangle$  représentent respectivement les particules se déplaçant vers la droite et vers la gauche. L'opérateur d'évolution unitaire à un instant donné est  $\hat{U}(t) = \hat{S} \cdot \hat{C}(x, t)$ . Cet opérateur inclut l'opérateur de décalage conditionnel [40]

$$\hat{S} = \sum_k |x+1, R\rangle \langle x, R| + |x-1, L\rangle \langle x, L|, \quad (16)$$

et l'opérateur de la pièce pour le coupleur dynamique à chaque position synthétique et à chaque étape de temps [40]

$$\hat{C}(x, t) = \sum_x \begin{pmatrix} \cos[\theta(x, t)] & -i \sin[\theta(x, t)] \\ -i \sin[\theta(x, t)] & \cos[\theta(x, t)] \end{pmatrix}, \quad (17)$$

où  $\theta(x, t)$  détermine les coefficients de transmission et de réflexion. Enfin, l'action répétée de l'opérateur unitaire sur l'état initial à l'étape zéro  $t = 0$  déplacera la particule vers l'état au temps  $t$ , donné par

$$|\psi(t)\rangle = \hat{U}(t)\hat{U}(t-1) \dots \hat{U} |\psi(0)\rangle. \quad (18)$$

### Le rôle du désordre

Le désordre dans les Marches Quantiques en Temps Discret (MQTD) fait référence à l'introduction d'aléa ou d'irrégularité dans l'évolution du marcheur. Dans une marche quantique standard, l'évolution suit un ensemble fixe d'opérateurs unitaires, généralement l'opérateur de pièce et l'opérateur de déplacement. Dans une marche quantique ordonnée, le comportement de dispersion est balistique, ce qui signifie que la variance de la position du marcheur croît quadratiquement avec le temps,  $\sigma(x)^2 \propto t^2$ , contrairement aux marches aléatoires classiques, où la variance évolue linéairement. Cette mise à l'échelle quadratique indique que le marcheur quantique se propage beaucoup plus rapidement dans l'espace des positions. Cependant, lorsque du désordre est introduit, l'évolution s'écarte de ces règles standard, entraînant divers effets, notamment la localisation, la décohérence et des modifications du comportement de transport.

**Localisation** L'une des conséquences les plus notables du désordre est la localisation d'Anderson, où le marcheur quantique se retrouve confiné dans une région spécifique du réseau au lieu de se propager librement. Ce phénomène résulte d'un désordre statique (indépendant du temps), qui perturbe les schémas d'interférence, conduisant à une distribution de probabilité restant localisée autour de certaines positions. Cet effet a été étudié expérimentalement sur des plateformes de marches quantiques photoniques [41, 42]. Dans le Chapitre.4, j'explore plus en détail linfluence du désordre statique aléatoire sur la localisation induite par le désordre dans les marches quantiques.

#### Décohérence

Le désordre peut être introduit par des fluctuations aléatoires, souvent issues de l'interaction du marcheur avec son environnement. Cela perturbe les interférences constructives, entraînant une perte progressive de cohérence. Toutefois, dans certains cas, la décohérence joue un rôle bénéfique, notamment dans le développement d'algorithmes quantiques [43]. Par exemple, la décohérence induite par la mesure ou par des interactions avec l'environnement constitue un cadre naturel



pour étudier la transition des marches quantiques aux marches aléatoires classiques [44, 45]. Il est à noter que le désordre peut également être imposé de manière cohérente, un concept exploré dans le Chapitre.4.

#### **Diffusion anormale et normale**

Dans une marche quantique ordonnée, le déplacement quadratique moyen du marcheur suit un schéma de croissance balistique. Cependant, en présence de désordre, le comportement de dispersion peut changer de manière radicale. La diffusion anormale fait référence à des écarts par rapport à la mise à l'échelle linéaire standard observée dans le transport classique. Les marches quantiques peuvent présenter une superdiffusion, où le marcheur se propage plus rapidement que dans une marche aléatoire classique, ou une subdiffusion, où le désordre entrave l'exploration efficace du réseau [44, 45, 46, 47].

#### **Amélioration non classique**

Bien que le désordre soit souvent considéré comme une source de bruit, il peut également être exploité pour des applications quantiques spécifiques. Dans le traitement de l'information quantique, le désordre peut renforcer l'intrication entre le marcheur et les états de la pièce ou introduire des corrélations entre différentes parties du système [48]. Dans le Chapitre IV de ma thèse de doctorat, nous montrons comment un désordre contrôlé peut optimiser les corrélations quantiques, en particulier dans le cas de deux photons indistinguables se propageant dans des modes distincts d'un réseau de marche quantique. Cela met en évidence le potentiel du désordre en tant que ressource pour les technologies quantiques, plutôt que comme un simple obstacle.

### **La marche quantique en temps discret comme un dispositif de mesure généralisé**

Dans le modèle de la marche quantique en temps discret, l'évolution des photons sur le réseau dépend de l'état quantique de la pièce (c'est-à-dire du mode quantique). Contrôler le mode quantique de la pièce permet de contrôler les marcheurs (les photons) et, par conséquent, l'évolution du système, ce qui permet la conception de mesures optimales à valeurs d'opérateurs positifs (POVM). De plus, il a été démontré que les MQD peuvent être utilisées pour généraliser les POVM. Dans ce scénario, la mesure d'un marcheur quantique à une position donnée correspond à un élément de la mesure POVM sur un certain état de la pièce. Alternativement, une POVM générale peut être construite en concevant des opérateurs de pièce  $\hat{C}(x, t)$  et en mesurant ensuite les positions des marcheurs après certaines étapes. Mathématiquement, les éléments de la POVM  $E_i$  sont donnés par  $E_i = \text{Tr}_{anc}[(1 \otimes \sigma)\pi_i]$ , où 1 est l'opérateur identité,  $\sigma$  est l'état de l'ancilla (système supplémentaire) et correspond à la position du marcheur, et  $\pi_i$  est le projecteur de von Neumann sur l'espace de Hilbert conjoint position-pièce. Dans la théorie des marches quantiques, une étape unique d'un marcheur quantique partant d'une position définie peut être considérée comme une mesure projective de von Neumann de l'état de la pièce. Spécifiquement, le projecteur  $\pi_i$  à la position  $x = i$ , i.e.,  $\sigma_i = |x = i\rangle \langle x = i|$ , c'est-à-dire,  $\pi_i = \hat{U}(t)^\dagger [|x = i\rangle \langle x = i| \otimes 1] \hat{U}(t)$  [49]. Enfin, la probabilité de mesurer l'élément  $i$ -ème de la POVM pour un état initial  $\rho$  est  $p_i = \text{Tr}(E_i \rho)$  et correspond à la mesure projective  $M_i = |x, c\rangle \langle x, c|$  sur l'état final du marcheur quantique après l'évolution à  $t$ -ième étape  $\rho_t = \hat{U}(t)^\dagger \rho \hat{U}(t)$ , c'est-à-dire  $p_i = \text{Tr}(M_i \rho_t)$ , où la  $t$ -ième étape est donnée par le retour du marcheur.

### **Implémentation de la marche quantique en temps discret en optique quantique**

Une large gamme de systèmes quantiques physiques a été proposée pour implémenter des marches quantiques, y compris les atomes piégés [50], les ions piégés [51, 52], les qubits supraconducteurs [53, 54, 55], la résonance magnétique nucléaire [56, 57], les faisceaux laser et l'optique volumique [58, 59, 60, 61, 62] et les systèmes photoniques intégrés [63, 64, 65, 66, 67, 68]. Par la suite, comme dans ma thèse, l'accent sera mis sur l'implémentation des marche quantique en temps discret dans des systèmes optiques, et je discuterai de certains exemples de ceux-ci.

#### **Marche quantique en temps discret dans les systèmes optiques**

Dans les systèmes optiques, diverses méthodes de codage des degrés de liberté, tels que les modes spatiaux (ou trajectoires), la polarisation, les intervalles temporels, le moment angulaire orbital (OAM), et la fréquence, ont été utilisées pour implémenter des marche quantique en temps discret. Les plateformes photoniques comprennent des systèmes optiques volumétriques, des systèmes à fibres et des systèmes photoniques intégrés, représentant des progrès prometteurs dans l'implémentation photonique des marche quantique en temps discret. Dans l'implémentation optique volumétrique, plusieurs méthodes différentes ont été utilisées, où les impulsions laser sont simulées comme des marcheurs. Dans un exemple, la polarisation des photons et le mode spatial sont utilisés respectivement comme état de la pièce et de position, avec des plaques optiques et des déplaceurs de faisceau (BD) utilisés pour l'opération de la pièce et l'opérateur de décalage conditionnel [69, 62]. Dans un autre exemple, la pièce et la position sont encodées respectivement dans la polarisation et les intervalles temporels des photons, pour implémenter des marche quantique en temps discret unidimensionnelles et bidimensionnelles avec des opérations de pièce contrôlables [58, 59].

Une autre possibilité est d'utiliser d'autres degrés de liberté (DOFs) des photons, où le moment angulaire de spin (SAM) et le moment angulaire orbital sont utilisés pour coder les états de la pièce et de la position afin d'implémenter une marche quantique en temps discret unidimensionnelle (marche quantique en temps discret) [70]. Alternativement, des modes d'OAM comme sites de réseau et la polarisation comme état de la pièce sont utilisés pour implémenter une marche quantique en temps discret avec un faisceau laser dans un interféromètre à anneau [71].

Cependant, les systèmes optiques volumétriques ont des limitations et des défis en termes de scalabilité et de stabilité, surtout sur un grand nombre d'étapes. Les systèmes à fibres sont des candidats prometteurs pour simuler les marches quantiques, car ils peuvent facilement utiliser des boucles de fibres pour implémenter des états quantiques codés dans le temps. Dans l'implémentation de la marche quantique en temps discret sur la ligne dans une configuration de boucle de fibres avec des opérations de pièce réglables, les états de la pièce et de la position sont encodés dans la polarisation du photon et le domaine temporel [72]. Un autre exemple d'utilisation des fibres implique un système de boucle de fibres couplées comme un réseau synthétique composé de boucles de fibres couplées et d'une porte dynamique, où la dimension synthétique est donnée par les modes temporels [73]. De tels dispositifs ont été utilisés pour des tâches telles que la préparation et la manipulation d'états de biphotons, ainsi que pour la réalisation d'interférences à deux et quatre niveaux dans des intervalles temporels entre deux photons [74]. Dans le chapitre 3, je discuterai de la façon dont nous utilisons une telle plateforme pour simuler la marche quantique en temps discret pour l'estimation de phase améliorée par la quantique. De plus, une autre approche alternative implique l'utilisation d'un interféromètre à double Sagnac multipassage en optique volumétrique [75]. Dans le chapitre 4 de ma thèse, je discuterai du rôle du désordre cohérent dans les réseaux photoniques à travers la dynamique contrôlable des marches quantiques inhomogènes.

## Génération d'états d'intrication multipartite authentiques grâce à l'indiscernabilité de particules identiques

Dans le Chapitre 2, nous explorons la génération d'états multipartites authentiquement intriqués en exploitant l'indiscernabilité des particules identiques. L'indiscernabilité des particules identiques, obtenue grâce à un chevauchement spatial, constitue une véritable ressource quantique pour générer des états intriqués bipartites et multipartites. La génération d'intrication bipartite basée sur l'indiscernabilité spatiale implique des qubits bosoniques ou fermioniques indépendants qui se rencontrent (se chevauchent spatialement) uniquement au niveau des régions de détection [12, 13, 76]. Dans ce travail, nous allons au-delà des cas bipartites et tripartites pour proposer un cadre global et contrôlable permettant de générer des états multipartites intriqués tels que les états W, Dicke, GHZ et cluster avec des statistiques bosoniques et fermioniques, qui sont essentiels pour les tâches de traitement de l'information quantique.

Notre cadre théorique repose sur trois étapes clés : l'initialisation de qubits identiques indépendants, la déformation pour répartir spatialement ces qubits dans des régions distinctes, et la post-sélection pour activer l'état intriqué souhaité. La déformation, étape centrale, peut être réalisée selon deux approches : les conceptions sur site et distante (Fig. 2.1). Dans la conception sur site (Fig. 2.1 (a)), tous les qubits subissent un opérateur de déformation global, redistribuant spatialement les qubits tout en préservant leurs degrés de liberté internes. La conception distante (Fig. 2.1 (b)) applique des opérateurs de déformation individuels à chaque qubit, les répartissant dans différentes régions spatiales. Les deux approches exploitent les chevauchements spatiaux pour générer l'intrication, la post-sélection garantissant que l'état souhaité émerge de manière probabiliste.

En utilisant des représentations basées sur des graphes dans le cadre des opérations localisées spatialement et de la communication classique (sLOCC), nous démontrons mathématiquement une traduction directe des schémas de génération d'états intriqués spécifiques en graphes orientés colorés, complexes et pondérés, chacun correspondant à une configuration expérimentale donnée. Dans cette représentation graphique, les sommets représentent des régions spatiales ou des états de particules, et les arêtes codent les chevauchements spatiaux et les transformations de pseudospin. Par exemple, le graphe orienté complet (Fig. 2.3 (a)) et le graphe orienté en étoile (Fig. 2.3 (b)) sont utilisés pour générer des états W, les variations de connectivité et de poids des arêtes influençant la probabilité de succès. De même, la Fig. 2.6 (a) illustre le schéma de génération de l'état GHZ, où les arêtes orientées et les boucles représentent les transformations des qubits. Les figures Fig. 2.4 (a) et Fig. 2.4 (b) présentent les probabilités de succès et les fidélités des différents schémas pour l'état W, mettant en évidence les avantages de l'optimisation des configurations de graphes orientés pour des états intriqués spécifiques. Ces représentations permettent une exploration efficace des schémas de génération alternatifs et l'optimisation des ressources d'intrication.

Enfin, l'article aborde la faisabilité expérimentale des schémas proposés, en suggérant des implémentations avec des systèmes photoniques et d'autres plateformes quantiques telles que les qubits supraconducteurs et les condensats de Bose-Einstein. Les considérations pratiques, telles que les sources de photons uniques, le contrôle des fonctions d'onde et la détection basée sur le cadre sLOCC, sont également examinées, ainsi que les avantages de cette méthode par rapport aux approches traditionnelles de génération d'intrication. Les auteurs mettent en avant le potentiel d'avancées supplémentaires grâce à l'optimisation automatisée des structures graphiques pour améliorer à la fois la fidélité et les taux de succès des états intriqués sur des systèmes quantiques variés.

# Réseaux photoniques synthétiques pour l'estimation de phase à paramètre unique

Dans le chapitre 3, j'examine théoriquement les états intriqués dans le temps pour l'estimation quantique de phase et la conception de mesures optimales qui saturent la borne de Cramér-Rao (CRB) dans des réseaux photoniques synthétiques (SPLs) en implémentant des marches quantiques en temps discret contrôlées (DTQWs). Un SPL est réalisé expérimentalement à l'aide de boucles de fibre couplées et d'une porte dynamique, où la dimension synthétique est définie par les modes temporels. Au sein de ce SPL, des DTQWs contrôlées sont utilisées pour effectuer des mesures optimales de type opérateurs à valeurs positives (POVMs) pour l'estimation de phase à un seul paramètre.

Très récemment, un réseau photonique synthétique (SPL) basé sur un système de boucles de fibres couplées dynamiquement a été utilisé comme plateforme pour des tâches de génération et de traitement, telles que la préparation d'états biphotoniques, leur manipulation, et la réalisation d'interférences à deux et quatre niveaux dans le domaine des boîtes temporelles entre deux photons [74]. Dans ce projet, je démontre théoriquement une estimation de phase améliorée par la mécanique quantique en utilisant un état intriqué en boîte temporelle, spécifiquement un état de type NOON avec un nombre de photons égal à deux ( $N = 2$ ), donné sous la forme de l'Eq. 3.1, comme sonde. Cela est réalisé en effectuant des mesures optimales à travers un schéma de marche quantique contrôlée sur des réseaux photoniques synthétiques (voir Fig.3.3). L'utilisation d'un système de boucles de fibres couplées avec des portes dynamiques permet un contrôle précis de l'évolution de la marche quantique afin d'effectuer correctement les POVMs optimaux pour atteindre la meilleure précision dans l'estimation d'une phase inconnue. Je montre également que la mise en uvre du schéma optimal présenté dans la Fig.3.3, pour la lumière classique et l'état intriqué en boîte temporelle, atteint l'information de Fisher maximale, saturant ainsi la limite quantique de Cramér-Rao. Ce résultat confirme les performances du SPL implémenté.

Ce résultat prouve de manière concluante que notre SPL est une plateforme prometteuse pour divers protocoles d'information quantique, tels que l'estimation de phase multi-paramètres [77], la tomographie d'état quantique [78], et d'autres phénomènes liés aux marches quantiques (QWs) en quasi-symétrie PT, notamment l'interférence quantique [79, 80], la discrimination d'état quantique [81], et la transformation quantique ultrarapide [82, 83].

## Amélioration des corrélations bosoniques non classiques dans un réseau de marche quantique grâce au contrôle expérimental du désordre

Dans le Chapitre. 4, nous examinons théoriquement et expérimentalement le rôle du désordre et des inhomogénéités dans les réseaux quantiques. Comprendre le comportement des caractéristiques quantiques basé sur l'indistinguabilité dans un cadre dynamique est essentiel. La présence de désordre et d'inhomogénéités dans les réseaux quantiques s'est souvent révélée être avantageuse de manière inattendue, tant pour les ressources quantiques que classiques [84, 36, 85, 86, 87, 88]. Ces effets proviennent généralement de l'interaction avec un environnement externe.

Malgré diverses études [89], les stratégies visant à améliorer les corrélations à deux particules dans des modes distincts d'un réseau de marches quantiques via le contrôle du désordre restent inexplorées. Nous comblons cette lacune en injectant deux photons indiscernables (un biphoton) dans une marche quantique unidimensionnelle inhomogène en temps discret (1D DTQW) et en contrôlant expérimentalement des configurations spécifiques de désordre tout en maintenant le système isolé. Comme démontré expérimentalement dans la Fig.4.4, il est possible d'ajuster l'amélioration à deux modes et totale de la non-classicalité en position et en intensité. Cela représente un réseau adaptatif dont l'évaluation des paramètres détermine la focalisation des ressources non classiques dans des modes spécifiques. De plus, dans la Fig.4.9, nous illustrons numériquement que la moyenne sur des configurations aléatoires de désordre entraîne une diminution des corrélations quantiques initiales du biphoton. Nos résultats démontrent clairement que la présence de désordre facilite l'amélioration dynamique des corrélations quantiques du biphoton de manière contrôlable, ouvrant la voie à son application potentielle dans des scénarios d'information quantique.

## Part II: Caractériser l'effet mémoire dans le système quantique ouvert

Dans la Partie II de ma thèse de doctorat, je me concentre sur la caractérisation des effets mémoire dans les systèmes quantiques ouverts. Cette partie est composée de chapitres basés sur des articles précédemment publiés dans des revues scientifiques réputées. Cependant, avant d'aborder les résultats, je présente une brève introduction au contexte.

L'interaction inévitable des systèmes quantiques avec leurs environnements induit la décohérence et la dissipation. La dynamique du système peut être sans mémoire, ce que l'on appelle le régime markovien, ou elle peut être associée au flux d'information de l'environnement vers le système, appelé le régime non markovien. Les effets non markoviens sont présents dans divers systèmes réalistes tels que les systèmes optiques quantiques [90], les systèmes solides [91, 92, 93], les qubits supraconducteurs à flux et le contrôle quantique, la biologie quantique [94, 95] et la chimie physique [96, 97], l'optique quantique : matériaux à bande interdite photonique. De plus, il est montré que la non-markovianité peut être une ressource pour les tâches d'information quantique [98, 99, 100, 101, 102].

## Système quantique ouvert et cartes dynamiques

Un système quantique ouvert  $S$  [103, 104, 105] peut être considéré comme un sous-système d'un système plus grand comprenant  $S$  et un autre sous-système  $E$ , son environnement, comme illustré dans la Fig. 5.1. L'espace de Hilbert du système combiné  $S + E$  est représenté par le produit tensoriel de leurs espaces de Hilbert individuels comme

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E, \quad (19)$$

où  $\mathcal{H}_S$  et  $\mathcal{H}_E$  désignent respectivement l'espace de Hilbert du système et de l'environnement. L'état physique du système total est représenté par  $\rho_{SE}$  un état dans l'espace de Hilbert  $\mathcal{H}_{SE}$  qui satisfait  $\rho_{SE} \geq 0$  et  $\text{Tr} \rho_{SE} = 1$ . En conséquence, l'état correspondant de  $S$  et  $E$  peut être obtenu par une trace partielle sur les espaces de Hilbert  $\mathcal{H}_E$  et  $\mathcal{H}_S$ , c'est-à-dire respectivement comme  $\rho_S = \text{Tr}_E \rho_{SE}$  et  $\rho_E = \text{Tr}_S \rho_{SE}$ . Nous supposons que le système total  $S + E$  est un système fermé régi par le Hamiltonien suivant

$$H = H_S \otimes I_E + I_S \otimes H_E + H_I, \quad (20)$$

où  $H_S$  et  $H_E$  désignent respectivement le Hamiltonien libre du système et de l'environnement,  $H_I$  est le Hamiltonien d'interaction. Ainsi, l'évolution unitaire correspondante est donnée par

$$U(t) = \exp(-iHt), \hbar = 1. \quad (21)$$

La dynamique du système total peut être déterminée à partir de l'équation de von Neumann sous la forme suivante

$$\frac{d\rho_{SE}(t)}{dt} = -i[H, \rho_{SE}(t)], \quad (22)$$

ce qui donne la solution générale

$$\rho_{SE}(t) \equiv U(t)\rho_{SE}(0)U^\dagger(t). \quad (23)$$

Nous supposons que l'état initial du système total est non corrélé avec l'état produit tensoriel, c'est-à-dire  $\rho_{SE}(0) = \rho_S \otimes \rho_E(0)$ . En effectuant une trace partielle sur les degrés de liberté de l'environnement, on obtient la matrice de densité réduite du système à tout instant  $t \geq 0$

$$\rho_S = \text{Tr}_E(U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t)). \quad (24)$$

L'équation ci-dessus définit une carte linéaire en considérant un état environnemental initial fixe et un temps fixe sous la forme de

$$\Phi_t : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S). \quad (25)$$

Cette carte prend n'importe quel état initial du système ouvert  $\rho(0)$  et le transforme en l'état correspondant du système ouvert  $\rho_S(t)$  à l'instant  $t$ , comme suit

$$\rho_S(0) \rightarrow \rho_S(t) = \Phi_t \rho_S(0), \quad (26)$$

$\Phi_t$  est appelée une carte dynamique quantique. Il s'agit d'une opération physique qui transforme l'état physique en d'autres états physiques. Elle est à la fois positive et préservant la trace.

On peut envisager le scénario général dans lequel le système  $S$  est initialement corrélé avec un système annexe  $A$ , qui n'est pas affecté par l'évolution gouvernée par l'interaction entre  $S$  et  $E$ . Par conséquent, une transformation physique doit également préserver la physicalité des états  $S - A$ . Pour ce faire, pour tout système annexe avec un espace de Hilbert  $\mathcal{H}_A$ , nous devons

$$\rho_{SA}(0) \in S(\mathcal{H}_{SA}) \rightarrow \Phi_t \otimes I_A(\rho_{SA}(0)) = \rho_{SA}(t) \in S(\mathcal{H}_{SA}) \quad \text{for any } t \geq 0, \quad (27)$$

étendre la condition de l'Équation 5.8, qui exige que  $\Phi_t$  soit complètement positive (CP) et préservant la trace (CPTP). En théorie de l'information quantique, les cartes possédant cette propriété sont également connues sous le nom d'opérations quantiques ou de canaux quantiques [106].

La propriété CP peut être exprimée sous la forme suivante

$$\Phi(t) \text{ is CP} \iff \Phi(t) \otimes I(X_{SA}) \geq 0 \quad \text{for any } X_{SA} \geq 0, \quad (28)$$

où  $X_{SA} \in S(\mathcal{H}_{SA})$  et le système annexe  $A$  a la même dimension que  $S$ . Ainsi, cette carte dynamique décrit l'évolution d'un système quantique ouvert entre l'état initial et l'état final. Par conséquent, la propriété la plus forte d'une carte complètement positive est qu'elle transforme à la fois l'état physique  $S$  en un autre état physique  $S$  et tous les états physiques  $S - A$  en des états physiques de  $S - A$ .

Dans le cas où le paramètre temporel  $t$  varie sur un intervalle de  $0$  à  $T$ , qui peut être fini ou infini, une famille à un paramètre de cartes dynamiques peut être décrite sous la forme suivante

$$\Phi = \{\Phi_t | 0 \leq t \leq T, \Phi_0 = I\}, \quad (29)$$

et  $I$  désigne la carte unité ; de plus, l'état initial de l'environnement  $\rho_E(0)$  reste fixé pour former  $\Phi_t$ . Cette famille contient toutes les informations sur la dynamique de tous les systèmes ouverts sur l'intervalle  $[0, T]$ .

Une forme plus détaillée de la dynamique peut être obtenue en examinant la décomposition diagonale de l'état initial de l'environnement  $\rho_E(t_0) = \sum_i p_i |E_i\rangle \langle E_i|$ , où  $|E_i\rangle$  représente la base orthonormée dans l'espace de Hilbert de  $\mathcal{H}_E$  et  $\sum_i p_i = 1$ . On peut supposer que l'Hamiltonien du système-environnement  $\mathcal{H}_{SE}$  est indépendant du temps ; par conséquent, en utilisant l'Équation 5.6, la matrice de densité réduite du système peut être décrite par

$$\rho_S(t) = \sum_{ij} p_i \langle E_j | U(t) | E_i \rangle \rho_S(0) \langle E_i | U^\dagger(t) | E_j \rangle. \quad (30)$$

En introduisant les opérateurs de Kraus  $K_{ij} := \sqrt{p_i} \langle E_j | U(t) | E_i \rangle$  [107], L'équation ci-dessus peut être représentée sous la forme suivante

$$\rho_S(t) = \sum_{ij} K_{ij} \rho_S(0) K_{ij}^\dagger. \quad (31)$$

Comme l'évolution globale est unitaire, les opérateurs de Kraus satisfont à la contrainte de complétude, c'est-à-dire  $\sum_{ij} K_{ij} K_{ij}^\dagger = I_S$ .

## Évolution markovienne et non-markovienne

Les évolutions quantiques markoviennes, similaires aux processus markoviens classiques, ne sont pas affectées par l'histoire passée du système, et ainsi, l'information perdue n'est pas récupérable. Par conséquent, les évolutions quantiques markoviennes présentent des propriétés phénoménologiques sans mémoire, ce qui signifie qu'aucun retour d'information de l'environnement  $E$  vers le système quantique ouvert ne peut se produire. Avant de poursuivre, je discuterai de certaines propriétés de l'évolution.

Plongeons dans le concept de divisibilité [108]. Une évolution, donnée par la famille de cartes CPTP  $\Phi = \{\Phi_t\}_{t \geq 0}$ , décrit l'évolution du système à tout instant final  $t$  et ne fournit aucun opérateur pour faire évoluer le système dans le domaine temporel  $0 < s < t$ . Par conséquent, l'évolution  $\Phi = \{\Phi_t\}_{t \geq 0}$  est définie comme divisible si et seulement si, pour tout  $0 \leq s \leq t$  il existe une carte linéaire  $V_{t,s} : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S)$  telle que

$$\Phi_t(\cdot) = (V_{t,s} \circ \Phi_s)(\cdot) \equiv V_{t,s}(\Phi_s(\cdot)), \quad (32)$$

où  $V_{t,s}$  est appelé une carte intermédiaire de  $\Phi$  entre  $s$  et  $t$ . Il convient également de noter que toute évolution inversible est divisible, ce qui signifie qu'une carte intermédiaire existe toujours et est donnée par

$$V_{t,s} = \Phi_t \circ \Phi_s^{-1}. \quad (33)$$

La famille des cartes dynamiques est appelée divisible en P si  $V_{t,s}$  est positive, et divisible en CP si  $V_{t,s}$  est complètement positive pour tous  $t \geq s \geq 0$ .

De plus, sur la base de la définition de la propriété de semi-groupe, qui indique que la carte peut être divisée en une infinité d'étapes temporelles, chacune identique et indépendante des étapes passées et futures [109], la carte dynamique représente une dynamique sans mémoire. En conséquence, l'évolution  $\Phi = \{\Phi_t\}_{t \geq 0}$  est considérée comme un semi-groupe dynamique si et seulement si, pour tous  $t_1, t_2 \geq 0$  la relation suivante est vérifiée

$$\Phi_{t_1+t_2}(\cdot) = \Phi_{t_1} \Phi_{t_2}(\cdot). \quad (34)$$

Il convient également de noter que toute évolution qui satisfait la propriété de semi-groupe est divisible en CP ; cependant, l'inverse n'est pas vrai.

## Les évolutions markoviennes

Pour décrire la markovianité quantique, il est d'abord nécessaire de comprendre le concept de processus markovien dans le domaine classique. Basé sur la théorie des probabilités classiques [110, 111], un processus stochastique  $X(t), x \geq 0$ , qui prend des valeurs dans l'ensemble discret  $\{x_i\}_{i \in N}$ , peut être décrit par une hiérarchie de distributions de probabilités conjointes  $P_n = P_n(x_n, t_n; x_{n-1}, t_{n-1}; \dots, x_1, t_1)$  pour tous  $n \in N$  et les instants  $t_n \geq t_{n-1} \geq \dots \geq t_1 \geq 0$ , connue sous le nom de hiérarchie de Kolmogorov. La distribution  $P_n$  donne la probabilité que le processus prenne la valeur  $x_1$  au temps  $t_1$ , la valeur  $x_2$  au temps  $t_2$  et ainsi de suite, la valeur  $x_n$  au temps  $t - n$ . Une hiérarchie peut représenter un processus stochastique lorsque les conditions de cohérence de Kolmogorov, ainsi que d'autres conditions telles que la positivité et la normalisation, sont satisfaites, comme

$$\sum_{x_m} P_n(x_n, t_n; x_m, t_m; \dots, x_1, t_1) = P_{n-1}(x_n, t_n; x_{n-1}, t_{n-1}; \dots, x_1, t_1). \quad (35)$$

Cela relie la distribution de probabilité à  $n$  points  $P_n$  à la distribution de probabilité à  $n - 1$  points  $P_{n-1}$ . Un processus stochastique  $X(t)$  est dit markovien lorsque la probabilité conditionnelle est décrite par

$$P_{1|n}(x_{n+1}, t_{n+1} | x_n, t_n; \dots x_1, t_1) = \frac{P_{n+1}(x_{n+1}, t_{n+1} \dots x_1, t_1)}{P_n(x_n, t_n; \dots x_1, t_1)}, \quad (36)$$

satisfaisant

$$P_{1|n}(x_{n+1}, t_{n+1} | x_n, t_n; \dots x_1, t_1) = P_{1|1}(x_{n+1}, t_{n+1} | x_n, t_n). \quad (37)$$

La condition pour un processus markovien souligne que la probabilité que le processus stochastique prenne la valeur  $x_{n+1}$  au temps  $t_{n+1}$ , étant donné qu'il a pris les valeurs  $x_i$  aux temps précédents  $t_i$ , ne dépend que de la valeur la plus récente  $x_n$  au temps  $t_n$ . Dans ce contexte, le processus markovien est considéré comme sans mémoire, car l'historique passé avant  $t_n$  est sans importance pour prédire l'avenir une fois que la valeur  $x_n$  au temps  $t_n$ .

Supposons un système classique à dimension finie, où la variable stochastique  $t$  et la variable stochastique  $X$  peuvent prendre une valeur parmi  $\{i\}_{i=1}^d$ . Ainsi, le vecteur de probabilité du temps initial est  $\mathbf{p}(0) = (p_1(0), p_2(0), \dots, p_d(0))$ , où  $p_i(0)$  représente la probabilité initiale que soit dans l'état  $i$ . L'évolution suivante au fil du temps  $t$  est donc donnée par la matrice stochastique  $T(t, 0)$ , comme  $\mathbf{p}(t) = T(t, 0)\mathbf{p}(0)$ . L'élément  $ij$ -ième de  $T(t, 0)$  représente la probabilité de transition de l'état  $i$  au temps initial vers l'état  $j$  au temps  $t$ . Les processus markoviens sont caractérisés par des matrices stochastiques P-divisibles, qui peuvent être décomposées en matrices stochastiques intermédiaires  $T(s, t)$  [112]

$$T(t, 0) = T(s, t)T(s, 0) \quad \text{for any } s \leq t. \quad (38)$$

Ces processus sont considérés comme stochastiquement P-divisibles car les matrices  $T(s, t)$  sont nécessaires pour maintenir le sens physique de l'évolution de la distribution de probabilité. Le terme "stochastiquement" est utilisé pour différencier cette caractéristique de la P-divisibilité que l'on trouve dans les canaux quantiques. Pour la généralisation de l'Équation 5.20 au domaine quantique, il est nécessaire que  $V(s, t)$  dans l'Équation 5.14 soit considéré comme CPTP, ce qui conduit à la définition de l'évolution markovienne quantique basée sur la CP-divisibilité.

Pour illustrer cette définition, considérons l'évolution CP-divisible  $\Phi_t(\cdot)$  et son intermédiaire  $V(t, s)$ , qui sont des cartes CPTP. La matrice de densité du système peut être définie comme l'application consécutive de deux cartes sous la forme suivante

$$\rho_s(t) = V_{t,s}(\Phi_s(\rho_s(0))), \quad (39)$$

Lorsque la CP-divisibilité est satisfaite, aucun retour d'information ne peut se produire entre deux instants  $s$  et  $t$ . De plus, les évolutions qui satisfont la propriété de semi-groupe ont des cartes intermédiaires  $V_{t,s}$  qui dépendent uniquement de la durée de l'intervalle de temps  $t - s$  plutôt que des temps spécifiques  $s$  et  $t$ . Dans ce cas, en raison du taux constant de perte d'information, ces évolutions sont considérées comme étant sans mémoire.

Ainsi, d'après la définition du processus markovien, l'évolution quantique est définie comme non-markovienne si et seulement si la carte dynamique  $\Phi_t(\cdot)$  n'est pas divisible et que la condition de CP-divisibilité est violée. Cela signifie qu'un retour d'information peut se produire.

## Équation maîtresse

Une autre approche pour décrire l'évolution du système quantique ouvert consiste à utiliser l'équation différentielle pour  $\frac{d}{dt}\rho_S(t)$  connue sous le nom d'équation maîtresse. La solution de cette équation donne la même évolution temporelle de la matrice de densité réduite que celle obtenue avec les cartes dynamiques. L'équation maîtresse a la forme générale suivante

$$\frac{d}{dt}\rho_S(t) = \kappa_t(\rho_S(t)), \quad (40)$$

où  $\kappa_t$  est l'opérateur générateur [113]. L'opérateur générateur pour toute évolution qui satisfait la propriété de semi-groupe et est indépendante du temps, c'est-à-dire  $\kappa_t = \kappa$ , est donné par [114, 113]

$$\kappa(\rho_S(t)) = -i[H_S, \rho_S(t)] + \sum_i \gamma_i \left[ A_i \rho_S(t) A_i^\dagger - \frac{1}{2} \{A_i^\dagger A_i, \rho_S(t)\} \right], \quad (41)$$

où  $\gamma_i$  sont les taux non négatifs et  $A_i$  sont appelés opérateurs de Lindblad ou opérateurs de saut. Cette équation a été étendue aux évolutions différentiables, c'est-à-dire markoviennes, CP-divisibles. Dans ce scénario, l'équation maîtresse, connue sous le nom d'équation de Lindblad, et son générateur prennent la forme suivante:

$$\kappa_t(\rho_S(t)) = -i[H_S(t), \rho_S(t)] + \sum_i \gamma_i(t) \left[ A_i(t) \rho_S(t) A_i^\dagger(t) - \frac{1}{2} \{A_i^\dagger(t) A_i(t), \rho_S(t)\} \right], \quad (42)$$

où tous les taux et opérateurs sont généralement dépendants du temps. Le  $\kappa_t$  est sous la forme de Lindblad lorsque  $\gamma_i(t) \geq 0$  pour tous les  $i$ , et est sous la forme généralisée de Lindblad lorsque l'un des taux  $\gamma_i(t)$  est négatif dans un ou plusieurs intervalles de temps.

Où tous les taux et opérateurs sont généralement dépendants du temps. Le  $\kappa_t$  est sous la forme de Lindblad lorsque  $\gamma_i(t) \geq 0$  pour tout  $i$ , et sous la forme généralisée de Lindblad lorsque l'un des taux  $\gamma_i(t)$  est négatif dans un ou plusieurs intervalles de temps

$$\begin{aligned}\Phi_t &= T \left[ \int_0^t \kappa_\tau d\tau \right], \\ V_{t,s} &= T \left[ \int_s^t \kappa_\tau d\tau \right].\end{aligned}\tag{43}$$

Donc, l'évolution  $\Phi_t$  est supposée être Markovienne si et seulement si le générateur correspondant  $\kappa_t$  est sous la forme de Lindblad. Cependant, cette affirmation n'est vraie que pour les évolutions inversibles [108]. Des dynamiques CP-divisibles (Markoviennes) non inversibles avec des générateurs sous la forme de Lindblad généralisée existent. De plus, toute dynamique sous la forme d'une équation maître de Lindblad généralisée où un ou plusieurs taux  $\gamma_i(t)$  sont négatifs pour certains instants de temps est non-Markovienne; toutes les dynamiques non-Markoviennes ne peuvent pas être décrites par la solution de l'équation maître généralisée.

En résumé, l'équation maître de Lindblad est largement considérée comme le modèle standard pour les dynamiques Markoviennes, sans mémoire. Cela est dû à sa propriété de semi-groupe, qui garantit que l'évolution future de l'état est indépendante de ses états passés. En revanche, lorsque les taux dépendants du temps dans l'équation maître de Lindblad généralisée prennent temporairement des valeurs négatives, la divisibilité de la carte dynamique correspondante est rompue. Par conséquent, la carte intermédiaire  $V_{t,s}$  dans l'Éq. 5.14 n'est plus positive, et la carte dynamique est qualifiée de non-Markovienne.

## Témoin de la non-Markovianité quantique

Jusqu'à ce point, j'ai abordé à la fois la dynamique de la Markovianité quantique et de la non-Markovianité. Dans cette section, j'explorerai deux approches pour quantifier la non-Markovianité. La première se concentre sur l'indivisibilité et la violation de la carte dynamique. La deuxième approche est basée sur l'effet mémoire dans la dynamique des systèmes quantiques ouverts, qui est lié à l'échange d'informations entre les systèmes quantiques et leur environnement.

### Non-Markovianité quantique et indivisibilité

Il existe plusieurs approches pour quantifier et détecter la non-Markovianité, en fonction de la manière dont elles proposent différentes propriétés liées à la divisibilité de la carte dynamique. Ces approches incluent les mesures géométriques [115], l'optimisation de la norme de la matrice de Helstrom [116], et la mesure RHP [117]. Bien que les deux premières méthodes soient compliquées à calculer en pratique, la mesure RHP est computationnellement simple. Dans ce qui suit, j'aborderai cette mesure plus en détail.

La mesure de RHP, introduite par Rivas, Huelga et Plenio, est basée sur la notion de divisibilité : une carte complètement positive préservant la trace est divisible si elle peut être définie comme

$$\Phi_{t+\tau,0} = \Phi_{t+\tau,t} \Phi_{t,0},\tag{44}$$

et  $\Phi_{t+\tau,t}$  est complètement positive pour tout temps  $t, \tau \geq 0$  et satisfait la loi de composition

$$\Phi_{s,t} = \Phi_{s,u} \Phi_{u,t},\tag{45}$$

pour tout  $s \geq u \geq t$ . Après multiplication à droite par l'inverse  $\Phi_{t,0}^{-1}$  des deux côtés de l'équation ci-dessus, nous avons

$$\Phi_{t+\tau,t} = \Phi_{t+\tau,0} \Phi_{t,0}^{-1}.\tag{46}$$

Si ces cartes restent complètement positives à tout moment  $t$ , l'évolution temporelle est considérée comme Markovienne. Cependant, pour l'évolution non-Markovienne, il doit exister un moment  $t$  où  $\Phi_{t+\tau,t}$  n'est pas complètement positive. Étant donné que l'idée de base derrière la RHP est de quantifier à quel point les dynamiques intermédiaires  $\Phi_{t+\tau,t}$  ne sont pas complètement positives pour chaque instant  $t$ , mesurer l'écart de la carte intermédiaire par rapport aux cartes complètement positives caractérise l'étendue de la non-Markovianité de l'évolution temporelle.

Pour ce faire, afin d'évaluer le degré de non-positivité complète des cartes intermédiaires  $\Phi_{t+\tau,t}$ , on peut utiliser l'isomorphisme de Choi-Jamiokowski [118, 119]. Supposons l'état maximally intriqué sous la forme de  $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{n=d-1} |n\rangle \langle n|$  (où  $d$  est la dimension du système) ; nous relierons  $\Phi(t+\tau, t)$  à une matrice (Choi-Jamiokowski) construite selon le rôle

$$[\Phi(t+\tau, t) \otimes I](|\psi\rangle \langle \psi|).\tag{47}$$

Pour ce faire, afin d'évaluer le degré de non-positivité complète des cartes intermédiaires  $\Phi_{t+\tau,t}$ , on peut utiliser l'isomorphisme de Choi-Jamiokowski [118, 119]. Supposons l'état maximally intriqué sous la forme de  $|\psi\rangle =$

$\frac{1}{\sqrt{d}} \sum_{k=0}^{n=d-1} |n\rangle \langle n|$  (où  $d$  est la dimension du système) ; nous relierons  $\Phi(t + \tau, t)$  à une matrice (ChoiJamiokowski) construite selon le rôle

$$\|[\Phi(t + \tau, t) \otimes \mathbb{I}] (|\psi\rangle \langle \psi|)\|_1 = \begin{cases} = 1 & \text{iff } \Phi(t + \tau, t) \text{ is } CP, \\ > 1 & \text{otherwise} \end{cases} \quad (48)$$

Selon [117], on peut définir la fonction  $g(t)$  par la dérivée à droite de la norme de trace comme suit.

$$g(t) := \lim_{\epsilon \rightarrow 0^+} \frac{\|[\Phi(t + \tau, t) \otimes \mathbb{I}]\|_1 - 1}{\epsilon}, \quad (49)$$

Par conséquent,  $g(t) > 0$  pour un certain temps  $t$  si et seulement si l'évolution est non-Markovienne. Ainsi, la quantité totale de non-Markovianité sur l'intervalle de temps  $t \in I$  peut être exprimée comme suit

$$\mathcal{N}_{RHP}^I = \int_I g(t) dt. \quad (50)$$

### Non-Markovianité quantique et flux d'information

Un autre critère pour discuter de la non-Markovianité quantique est basé sur la présence d'effets mémoire dans la dynamique d'un système quantique ouvert, qui sont liés à l'échange d'information entre le système ouvert et son environnement. Dans une évolution Markovienne, le système ouvert perd continuellement de l'information vers l'environnement. Cependant, dans une évolution non-Markovienne, le processus est caractérisé par le retour d'information de l'environnement vers le système ouvert [120, 121]. Dans cette approche, la non-Markovianité quantique repose sur la notion de mémoire quantique. Plus précisément, elle fait référence à l'information transférée vers l'environnement sous forme de corrélations système-environnement ou de changements dans les états de l'environnement, qui sont ensuite récupérés par le système.

À cet égard, l'approche BLP, proposée par Breuer, Laine et Piilo dans [121, 120], identifie la dynamique non-Markovienne avec des caractéristiques physiques spécifiques de l'interaction système-environnement. Dans ce cadre, la dynamique non-Markovienne est décrite comme une évolution temporelle pour le système ouvert déterminée par le retour temporaire d'information de l'environnement vers le système. Dans ce scénario, le retour d'information peut se manifester sous la forme d'une augmentation de la distinguabilité des paires d'états quantiques évolutifs. Cette indistinguabilité peut être quantifiée par la distance de trace entre deux états quantiques  $\rho_1$  et  $\rho_2$  pour un temps  $t > 0$ , sous la forme de

$$D(\rho_1, \rho_2) = \frac{1}{2} \|\rho_1 - \rho_2\|_1, \quad (51)$$

où  $0 \leq D(\rho_1, \rho_2) \leq 1$ ,  $D(\rho_1, \rho_2) = 0$  si et seulement si  $\rho_1 = \rho_2$  et  $D(\rho_1, \rho_2) = 1$  si et seulement si  $\rho_1$  et  $\rho_2$  sont orthogonaux. Pour le processus dynamique divisible, la distinguabilité des états diminue de manière monotone, ce qui peut être interprété comme une perte continue d'information du système vers l'environnement. Cependant, selon BLP, la carte dynamique  $\Phi_t$  est considérée comme non-Markovienne s'il existe une paire d'états quantiques  $\rho_i(t) = \Phi_{t,0} \rho_i$  ( $i = 1, 2$ ), telle que la distinguabilité entre eux augmente, ce qui est exprimé en termes de flux d'information sous la forme de

$$\sigma(\rho_1(t), \rho_2(t)) = \frac{d[D(\rho_1(t), \rho_2(t))]}{dt}. \quad (52)$$

Où un flux d'information négatif indique la perte d'information du système vers l'environnement et un flux positif désigne un flux inversé d'information. Par conséquent, selon la définition de BLP, l'intégration sur tous les flux positifs regroupe toute l'information qui revient au système pour une paire initiale donnée ; ainsi, le degré de l'effet mémoire peut être défini comme.

$$\mathcal{N}(\Phi) = \max_{\rho_1, \rho_2} \int_{\sigma > 0} \sigma(t) dt. \quad (53)$$

Étant donné la simplicité et l'interprétation physique intuitive de la distance de trace, elle a été utilisée pour détecter les propriétés non-Markoviennes dans la dynamique des systèmes de qubit [122, 123, 124, 125] et de qutrit [126] couplés à des environnements bosoniques. De plus, la caractéristique cruciale de la mesure BLP permet d'évaluer directement expérimentalement la non-Markovianité à travers des mesures tomographiques de différents états initiaux à divers moments pendant l'évolution [127, 128, 90, 129].

Les quantificateurs de distance statistique quantique et de vitesse jouent un rôle essentiel dans la théorie de l'information quantique [106, 130]. Ils sont utilisés pour évaluer la distinguabilité des états quantiques [106, 130, 131, 132], quantifier les corrélations initiales, le flux d'information et les effets non-Markoviens dans les évolutions quantiques [132, 133, 134], et dériver des limites sur le temps d'évolution [135, 136, 137, 138]. Parmi ceux-ci, le témoin mentionné ci-dessus, la distance de trace, est une distance statistique quantique bien définie qui fournit une interprétation opérationnelle de l'indistinguabilité de deux états quantiques et est utilisée pour observer la non-Markovianité.

En s'appuyant sur cette compréhension et inspirés par le fait que les effets non-Markoviens peuvent accélérer la dynamique des systèmes [134, 139, 140, 141, 142, 143], tandis que les quantificateurs de vitesse statistique quantique



peuvent dériver la limite du temps d'évolution, dans le chapitre 6, nous introduisons un nouveau témoin comme caractérisateur du comportement non-Markovien des systèmes quantiques ouverts. Ce témoin repose sur le taux de variation positif de la vitesse de Hilbert-Schmidt (HSS) comme cas particulier de la vitesse statistique quantique. L'avantage de ce témoin est qu'il ne nécessite pas la diagonalisation de la matrice de densité évoluée du système. En s'appuyant sur cette compréhension et inspirés par le fait que les effets non-Markoviens peuvent accélérer la dynamique des systèmes [134, 139, 140, 141, 142, 143], tandis que les quantificateurs de vitesse statistique quantique peuvent dériver la limite du temps d'évolution, dans le chapitre 6, nous introduisons un nouveau témoin comme caractérisateur du comportement non-Markovien des systèmes quantiques ouverts. Ce témoin repose sur le taux de variation positif de la vitesse de Hilbert-Schmidt (HSS) comme cas particulier de la vitesse statistique quantique. L'avantage de ce témoin est qu'il ne nécessite pas la diagonalisation de la matrice de densité évoluée du système.

Dans le chapitre 6, nous avons examiné l'efficacité du témoin dans plusieurs exemples pratiques de systèmes quantiques ouverts : un qubit soumis à du bruit covariant en phase et des canaux de Pauli, deux qubits indépendants interagissant localement avec des cavités perméables, et des atomes à trois niveaux de type V et  $\Lambda$  (qutrits) dans une cavité dissipative. Nous avons constaté que le témoin de non-Markovianité basé sur la HSS identifie les effets mémoire en totale concordance avec le témoin BLP basé sur la distance de trace [120], détectant ainsi les retours d'information système-environnement.

De plus, dans le chapitre 7, nous avons évalué la validité des témoins de non-Markovianité basés sur la HSS dans des systèmes de haute dimension. À cet effet, nous avons calculé la HSS pour un qutrit (système à trois niveaux) et un qudit (système à  $d$  niveaux, tel qu'un état cohérent de spin) interagissant avec des réservoirs thermiques et sous-vide comprimé. Les résultats démontrent que notre mesure proposée de la non-Markovianité, basée sur la HSS, est efficace pour les systèmes de dimension  $d > 2$  ainsi que pour les systèmes de qubits ( $d = 2$ ). Par conséquent, ces résultats nous permettent d'étudier notre mesure pour des systèmes multipartites, tels que les systèmes qubit-qudit ou qudit-qudit, interagissant avec divers environnements.

Dans le Chapitre 8, nous approfondissons notre analyse pour démontrer que la vitesse de Hilbert-Schmidt (HSS), une mesure efficace de la non-Markovianité, constitue un outil fiable pour évaluer les corrélations dans les canaux unitaux et non unitaux avec différentes densités spectrales bruitées. Pour les canaux unitaux, nous considérons des cas tels que les canaux de Pauli et de dépolarisation, tandis que pour les canaux non unitaux, nous nous concentrons sur les canaux d'amortissement d'amplitude. En outre, nous examinons l'influence de la mémoire classique résultant d'applications corrélées de canaux quantiques sur l'évolution non markovienne du système. De plus, nous montrons que dans les canaux unitaux bruités avec un grand nombre de qubits, la sensibilité de la HSS à l'intensité des corrélations classiques entre des utilisations consécutives du canal est significativement réduite. Cette observation suggère que les systèmes quantiques de grande taille pourraient présenter une sensibilité moindre aux effets accélérateurs des corrélations classiques sur l'évolution des états.

## Observer les effets non-markoviens des processus quantiques grâce à la vitesse de Hilbert-Schmidt

Dans le chapitre 6, inspirés par le fait que les effets non-markoviens peuvent accélérer la dynamique des systèmes quantiques et que les limites du temps d'évolution peuvent être déduites à l'aide de quantificateurs de vitesse statistique quantique, nous introduisons un nouvel indicateur pour caractériser la non-markovianité dans les systèmes quantiques en utilisant la vitesse de Hilbert-Schmidt (HSS), qui est un type particulier de vitesse statistique quantique. L'avantage de cet indicateur réside dans le fait qu'il ne nécessite pas la diagonalisation de la matrice de densité évoluée du système. Dans la définition de notre indicateur, le signe positif de la dérivée temporelle de la HSS, notée  $\chi(t)$ , c'est-à-dire  $\chi(t) > 0$ , indique une augmentation non monotone de la vitesse statistique quantique, correspondant à un retour d'information de l'environnement vers le système. Cette non-monotonie constitue la signature du comportement non-markovien.

Notre étude démontre que l'indicateur basé sur la HSS détecte systématiquement les effets non-markoviens, tels que les retours d'information entre le système et l'environnement, en accord étroit avec la mesure bien établie de Breuer-Laine-Piilo (BLP) [120]. L'efficacité de cet indicateur est vérifiée à travers plusieurs exemples utiles de systèmes quantiques ouverts : un qubit soumis à un bruit phase-covariant et un canal de Pauli, deux qubits indépendants interagissant localement avec des cavités fuyantes, ainsi que des atomes à trois niveaux de type V et  $\Lambda$  (qutrits) dans une cavité dissipative. Par exemple, dans le cas du bruit phase-covariant, la non-markovianité apparaît lorsque les taux de décroissance (tels que  $\gamma_1(t), \gamma_2(t), \gamma_3(t)$ ) deviennent négatifs au cours de l'évolution, entraînant ainsi un  $\chi(t)$ , positif. Cela indique une inversion temporaire du flux d'énergie ou d'information de l'environnement vers le système.

De plus, dans les Fig.6.1 et Fig.6.2, nous présentons la dynamique de la HSS en la comparant à la distance de trace et aux fonctions de cohérence. Dans les régimes non-markoviens, toutes les mesures montrent un comportement oscillatoire avec des régions de dérivée positive, indiquant des effets de mémoire. Nous constatons que l'indicateur de non-markovianité basé sur la HSS identifie les effets de mémoire en parfait accord avec la mesure BLP basée sur la distance de trace [120], détectant ainsi les retours d'information entre le système et l'environnement. Nos résultats concluent que la mesure HSS n'est pas seulement un indicateur valide de la non-markovianité, mais offre également des avantages pratiques pour l'analyse des systèmes multipartites et de haute dimension [144].

## Effets de mémoire dans les systèmes de haute dimension fidèlement identifiés par un témoin basé sur la vitesse de Hilbert-Schmidt

Dans le Chapitre 7, nous explorons l'efficacité du témoin basé sur la HSS pour détecter les effets non-markoviens dans les systèmes quantiques ouverts de haute dimension et multipartites. Nous validons la mesure HSS à travers divers systèmes, tels que des qudits simples et des systèmes hybrides qubit-qutrit, interagissant avec des environnements thermiques, de vide comprimé et de bruit télégraphique aléatoire. Nos résultats démontrent que la mesure HSS est en accord avec les indicateurs de non-markovianité établis, tels que la négativité et la perturbation induite par la mesure (MID), capturant efficacement les effets de mémoire à travers des comportements non monotones.

Notre investigation étend l'utilité de la mesure basée sur la HSS aux systèmes de haute dimension, où elle identifie avec précision les intervalles de dynamiques non-markoviennes, tels que le phénomène de re-cohérence dans les systèmes à qudit unique et le gel ainsi que la reprise des corrélations quantiques dans les systèmes hybrides qubit-qutrit. Par exemple, dans le cas des systèmes hybrides qubit-qutrit sous un environnement de vide comprimé, comme illustré dans la Fig.7.2, la HSS capture de manière fiable les comportements oscillatoires correspondant aux effets de mémoire, en accord avec les tendances dynamiques de la négativité et de la perturbation induite par la mesure (MID). Nous analysons également des scénarios impliquant des états initiaux mixtes et démontrons la robustesse de la mesure HSS, même lorsque l'intrication subit une mort soudaine, que les mesures conventionnelles comme la négativité ne parviennent pas à capturer efficacement (voir Fig.7.4).

En conclusion, le témoin basé sur la HSS s'impose comme un outil puissant pour identifier la non-markovianité, offrant à la fois des avantages théoriques et pratiques. Son efficacité computationnelle et sa cohérence avec les indicateurs établis le rendent particulièrement adapté aux systèmes quantiques complexes. Nous proposons des pistes futures, notamment l'extension de cette méthode à des scénarios comportant des corrélations initiales entre le système et son environnement, mettant en lumière son potentiel pour faire progresser les sciences de l'information quantique et explorer les effets de mémoire quantique dans des contextes variés.

## Observation des effets de mémoire globaux des canaux bruités corrélés multi-qubits à l'aide de la vitesse de Hilbert-Schmidt

Dans le chapitre 8, nous explorons la validité du témoin basé sur la vitesse de Hilbert-Schmidt (HSS) pour détecter l'effet de mémoire global dans les canaux bruités corrélés. En théorie de l'information quantique, les canaux quantiques avec mémoire (classique) se caractérisent par des corrélations entre des applications successives du canal dans une séquence de systèmes quantiques. Ce concept est distinct des effets de mémoire non markoviens. Dans les canaux non markoviens corrélés, l'effet de mémoire global provient à la fois des corrélations classiques entre des utilisations consécutives du canal quantique et des effets de mémoire intrinsèques régis par la densité spectrale du canal.

L'interaction entre les facteurs de corrélation et les effets de mémoire non markoviens dans les canaux purs colorés dans un scénario de décohérence a été discutée pour la première fois dans [145]. Reconnaisant l'importance des canaux non markoviens corrélés pour améliorer la limite de vitesse quantique [146], augmenter la capacité des canaux et atténuer le bruit dans la correction d'erreurs quantiques [147], nous approfondissons notre analyse dans le chapitre 8 pour démontrer que la vitesse de Hilbert-Schmidt (HSS), un témoin robuste de la non-Markovianité, capture efficacement l'effet de mémoire global dans les canaux bruités unitaux et non unitaux. Pour les cas unitaux, nous examinons les canaux de Pauli et de dépolarisation, tandis que pour les cas non unitaux, nous nous concentrons sur les canaux d'amortissement d'amplitude. La HSS reflète de manière cohérente les corrélations classiques, indépendamment de la base utilisée. Les fluctuations de sa dynamique sont liées à l'évolution non markovienne quantique du système, tandis que l'amplification de ces dynamiques signale la présence de corrélations classiques entre des utilisations successives du canal quantique unital (voir Fig.8.2a, Fig.8.3a, et Fig. 8.4a).

De plus, nous avons étudié l'influence des corrélations classiques sur l'effet de mémoire non markovien des canaux bruités corrélés. Pour les canaux unitaux, les corrélations classiques n'affectent pas la durée des revivals temporaires (voir Fig. 8.2a, Fig.8.3a, Fig.8.4a). En revanche, dans les canaux non unitaux, l'augmentation des corrélations classiques modifie les intervalles de temps auxquels apparaissent les effets non markoviens (voir Fig.8.5). Cela fait de la HSS un outil efficace pour distinguer les canaux unitaux des canaux non unitaux.

Étant donné que la HSS est facilement calculable et ne nécessite pas de diagonalisation, nous avons étendu notre analyse à plusieurs qubits, allant jusqu'à huit qubits pour les canaux bruités unitaux. Le comportement observé dans les canaux bruités à deux qubits est reproduit ici : les corrélations classiques amplifient la dynamique de la HSS, quel que soit le nombre de qubits. Cependant, comme le montrent les Fig.8.6a), Fig.8.7a) et Fig. 8.8a), l'impact des corrélations classiques entre plusieurs applications de canaux bruités dans les systèmes à grand nombre de qubits devient moins significatif. Au-delà de dix qubits, la HSS devient moins sensible aux corrélations classiques et reflète principalement l'effet de mémoire non markovien.

## Remarques conclusives

Cette thèse doctorale présente des recherches centrées sur l'étude théorique et expérimentale des corrélations quantiques et des effets de mémoire dans les systèmes quantiques composites. La première partie explore l'utilisation de l'indistinguabilité des systèmes quantiques identiques en tant que véritable ressource pour le traitement de l'information quantique, ainsi que l'étude des propriétés quantiques basées sur l'indistinguabilité dans un cadre dynamique. La deuxième partie introduit un nouveau témoin de non-Markovianité et évalue sa validité et son efficacité dans divers systèmes.

La première partie comprend trois chapitres qui explorent divers aspects du traitement de l'information quantique en exploitant l'indistinguabilité des particules identiques et en examinant le comportement des propriétés quantiques dans un cadre dynamique. **Chapitre 2** présente un schéma contrôlable et complet capable de générer, à travers trois étapes (préparation, déformation et post-sélection) une large classe d'états intriqués multipartites, notamment les états W, Dicke, GHZ et cluster, pour les statistiques bosoniques et fermioniques. Dans **Chapitre 3**, je discute de l'estimation de phase améliorée par la quantique à l'aide d'états intriqués dans le domaine temporel et de l'utilisation des marches quantiques en temps discret comme dispositif de mesure généralisé pour réaliser des POVM optimales dans des systèmes de boucles de fibres couplées. **Chapitre 4** propose des dynamiques de marches quantiques inhomogènes expérimentalement contrôlables comme plateforme pour explorer l'influence du désordre cohérent sur les corrélations quantiques entre des photons indiscernables, offrant des perspectives précieuses sur les systèmes quantiques dynamiques.

L'indistinguabilité des particules identiques constitue une ressource précieuse pour la génération d'états intriqués bipartites [12, 13, 14] et multipartites [148]. Grâce aux mesures sLOCC, cette ressource devient exploitable pour diverses tâches de traitement de l'information quantique [12, 13, 149, 150, 151, 152], car les particules identiques deviennent distinguables et adressables individuellement via leurs régions spatiales. En utilisant les fonctions d'onde spatiales, il est possible de concevoir des opérateurs de test d'intrication expérimentalement réalisables pour analyser les états générés [153]. Notre approche algorithmique exploite l'indistinguabilité des particules identiques pour générer une large classe d'états intriqués multipartites, incluant les états W, Dicke, GHZ et cluster. En adoptant une représentation basée sur des graphes, nous traduisons et cartographions le schéma en graphes orientés colorés, complexes et pondérés, chacun correspondant à une configuration expérimentale. Cette méthode facilite l'analyse de l'état final, l'évaluation de son intrication véritable et l'estimation des ressources en termes de probabilité de post-sélection. En outre, elle permet d'optimiser les ressources tout en prenant en compte les contraintes et les statistiques des particules, afin d'explorer des formes arbitraires de génération d'états multipartites. Par conséquent, ce cadre constitue une interface efficace pour optimiser la génération d'états intriqués multipartites avec des probabilités de succès spécifiques et une fidélité maximale.

Le flux de lumière dans le réseau photonique synthétique (SPL) mis en œuvre via un système de boucles de fibres couplées permet de simuler des marches quantiques en temps discret contrôlées (DTQWs). Récemment, cette plateforme a été utilisée pour des tâches telles que la préparation d'états bi-photoniques, leur manipulation et la réalisation d'interférences à deux et quatre niveaux dans le domaine temporel entre deux photons [74]. En nous appuyant sur ces études, nous avons développé des modèles théoriques et expérimentaux pour la mise en œuvre de l'estimation de phase quantique en utilisant un SPL avec des états intriqués dans le domaine temporel comme états sondes, tout en réalisant des POVMs optimales à l'aide de DTQWs. Nos résultats établissent le SPL comme une plateforme prometteuse pour divers protocoles d'information quantique, notamment l'estimation de phase multi-paramètres [77], la tomographie d'états quantiques [78], et des phénomènes liés aux marches quantiques dans une quasi-symétrie PT, tels que l'interférence quantique [79, 80], la discrimination d'états quantiques [81], et les transformations quantiques ultrarapides [82, 83].

Des dynamiques de marches quantiques inhomogènes et contrôlables peuvent être conçues pour étudier l'effet du désordre cohérent sur les corrélations quantiques entre deux photons indiscernables. Nous avons démontré qu'avec une configuration appropriée du désordre, les corrélations quantiques des biphotons sont renforcées entre deux modes du réseau par rapport à une marche quantique ordonnée. En perspective, cela ouvre la voie à de nombreuses pistes d'exploration supplémentaires, notamment l'étude de différentes configurations de désordre permettant divers types d'améliorations, favorisant ainsi une manipulation spatiale et temporelle des corrélations quantiques. Par ailleurs, la question de savoir si le désordre peut améliorer les corrélations quantiques entre plus de deux particules reste ouverte, avec le potentiel de fournir un cadre de référence dans le contexte des théories des ressources quantiques.

Dans la Partie II de ma thèse, trois articles de recherche sont présentés. Le **Chapitre 6** introduit un témoin novateur de non-Markovianité, inspiré par l'observation que les effets non markoviens peuvent accélérer la dynamique d'un système et que les quantificateurs de vitesse statistique quantique fixent des limites au temps d'évolution. Ce témoin repose sur le taux de variation positif de la vitesse de Hilbert-Schmidt (HSS), un cas particulier de vitesse statistique quantique. Un avantage clé de ce témoin est qu'il élimine la nécessité de diagonaliser la matrice de densité évoluée du système. L'efficacité de ce témoin est démontrée à travers plusieurs exemples de systèmes quantiques ouverts, incluant un qubit soumis à un bruit à covariance de phase et des canaux de Pauli, deux qubits indépendants interagissant localement avec des cavités dissipatives, ainsi que des atomes à trois niveaux de type V et  $\Lambda$  (qutrits) dans des cavités dissipatives. Le témoin basé sur la HSS identifie systématiquement les effets de mémoire, concordant pleinement avec le témoin BLP basé sur la distance de trace [120] et détectant efficacement les flux d'information système-environnement [144].

Dans le **Chapitre 7**, nous explorons l'efficacité de ce témoin pour détecter la non-Markovianité dans divers systèmes quantiques ouverts de haute dimension et multipartites. Notre analyse démontre que les oscillations temporelles du témoin basé sur la HSS s'alignent remarquablement bien avec celles de la négativité quantique et d'autres mesures de corrélation

quantique. Par conséquent, ce témoin constitue un outil fiable et précis pour caractériser les effets de mémoire dans l'évolution quantique des systèmes de haute dimension.

Enfin, étant donné l'importance des canaux non markoviens corrélés pour améliorer la limite de vitesse quantique [146], augmenter la capacité des canaux et atténuer le bruit dans la correction d'erreurs quantiques [147], nous étendons notre analyse dans le **Chapitre 8** pour démontrer que la vitesse de Hilbert-Schmidt (HSS), en tant que mesure efficace de la non-Markovianité, est un outil fiable pour évaluer les corrélations dans les canaux unitaires et non unitaires sous différentes densités spectrales bruitées. Cette étude approfondit le comportement des systèmes multiqubits sous divers canaux bruyants, en particulier les canaux unitaires. À mesure que le nombre de qubits augmente, l'influence des corrélations classiques diminue. Ce phénomène peut être comparé à des fluctuations induites par le bruit parmi les qubits individuels qui s'annulent en moyenne, réduisant ainsi l'impact global sur la dynamique HSS. Physiquement, ce comportement est similaire à celui des grands systèmes macroscopiques où les perturbations locales sont atténuées par la taille du système. De plus, cette interaction entre la cohérence quantique et le bruit classique souligne les dynamiques complexes au sein des systèmes quantiques. La diminution de la sensibilité de la HSS aux corrélations classiques dans les systèmes à qubits élevés suggère un potentiel de traitement de l'information quantique plus stable et robuste. À mesure que le nombre de qubits augmente, le système présente une forme de  $\bar{n}$  moyenne de bruit  $\bar{z}$ , conduisant à un état quantique global plus cohérent. Cette interaction complexe entre cohérence quantique et bruit classique offre de nouvelles perspectives pour optimiser le traitement de l'information quantique dans les systèmes quantiques à grande échelle. Elle met en lumière le potentiel de résilience au bruit et de stabilité dans les applications avancées de l'informatique quantique.

En conclusion, cette thèse apporte des contributions significatives au domaine du traitement de l'information quantique en approfondissant notre compréhension du rôle de l'indistinguabilité dans les tâches d'information quantique. De plus, elle présente un témoin de non-Markovianité novateur, facilement calculable et ne nécessitant pas la diagonalisation de la matrice de densité, offrant ainsi des outils pratiques pour caractériser les effets de mémoire dans divers systèmes réalistes, notamment les systèmes optiques quantiques, le contrôle quantique et l'optique quantique. Collectivement, ces résultats fournissent des perspectives fondamentales sur la dynamique quantique et ouvrent la voie au développement des technologies quantiques futures.

# Contents

|                                                                                                              |               |
|--------------------------------------------------------------------------------------------------------------|---------------|
| <b>List of figures</b>                                                                                       | <b>IV</b>     |
| <b>Abstract</b>                                                                                              | <b>V</b>      |
| <b>Résumé</b>                                                                                                | <b>VI</b>     |
| <b>Synopsis</b>                                                                                              | <b>XXIV</b>   |
| <b>List of Publications</b>                                                                                  | <b>XXVIII</b> |
| <b>General introductory remarks</b>                                                                          | <b>1</b>      |
| <b>I Exploitation of quantum correlations of identical quantum systems in quantum information scenario</b>   | <b>3</b>      |
| <b>1 Introductory remarks I</b>                                                                              | <b>4</b>      |
| 1.1 Indistinguishability of identical particles as a resource for quantum information processing . . . . .   | 4             |
| 1.1.1 Indistinguishability vs identity . . . . .                                                             | 4             |
| 1.1.2 No-label approach . . . . .                                                                            | 4             |
| 1.1.3 Spatially localized operations and classical communication . . . . .                                   | 6             |
| 1.2 Entanglement generation . . . . .                                                                        | 7             |
| 1.3 Phase estimation theory . . . . .                                                                        | 8             |
| 1.4 Quantum walk for quantum information processing . . . . .                                                | 9             |
| 1.4.1 Discrete time quantum walk (DTQW) . . . . .                                                            | 9             |
| 1.4.2 The role of disorder . . . . .                                                                         | 9             |
| 1.4.3 Discrete time quantum walk as a generalized device measurement . . . . .                               | 10            |
| 1.4.4 Implementation of discrete-time quantum walk in quantum optics . . . . .                               | 10            |
| <b>2 Generation of genuine multipartite entangled states via indistinguishability of identical particles</b> | <b>12</b>     |
| 2.1 Introduction . . . . .                                                                                   | 12            |
| 2.2 Indistinguishability-based entangling gate . . . . .                                                     | 14            |
| 2.3 Graph picture of the multipartite entangling gate . . . . .                                              | 15            |
| 2.3.1 Graph picture of bipartite entanglement generation . . . . .                                           | 16            |
| 2.4 Multipartite entanglement generation schemes . . . . .                                                   | 17            |
| 2.4.1 $W$ state . . . . .                                                                                    | 17            |
| 2.4.2 Symmetric Dicke state . . . . .                                                                        | 19            |
| 2.4.3 GHZ state . . . . .                                                                                    | 20            |
| 2.4.4 Cluster state . . . . .                                                                                | 20            |
| 2.5 Discussion on Experimental Feasibility . . . . .                                                         | 22            |
| Appendix . . . . .                                                                                           | 23            |
| A Bell State . . . . .                                                                                       | 24            |

|           |                                                                                                                       |           |
|-----------|-----------------------------------------------------------------------------------------------------------------------|-----------|
| B         | $W$ state with three and four qubits . . . . .                                                                        | 24        |
| C         | Symmetric Dicke state of four particles . . . . .                                                                     | 27        |
| <b>3</b>  | <b>Synthetic photonic lattices for single parameter phase estimation</b>                                              | <b>29</b> |
| 3.1       | Introduction . . . . .                                                                                                | 29        |
| 3.2       | Temporal synthetic photonic lattice using a coupled-fiber loop system . . . . .                                       | 30        |
| 3.3       | Realization of discrete-time quantum walk in synthetic photonic lattice . . . . .                                     | 31        |
| 3.4       | Phase estimation in synthetic photonic lattice . . . . .                                                              | 31        |
| 3.4.1     | Light pulse . . . . .                                                                                                 | 31        |
| 3.4.2     | Time-bin entangled state . . . . .                                                                                    | 32        |
| 3.5       | Conclusion . . . . .                                                                                                  | 33        |
| <b>4</b>  | <b>Enhancing nonclassical bosonic correlations in a quantum walk network through experimental control of disorder</b> | <b>34</b> |
| 4.1       | Introduction . . . . .                                                                                                | 34        |
| 4.2       | Theoretical model . . . . .                                                                                           | 35        |
| 4.3       | Numerical Results . . . . .                                                                                           | 36        |
| 4.4       | Experimental Setup . . . . .                                                                                          | 36        |
| 4.5       | Experimental Results . . . . .                                                                                        | 37        |
| 4.6       | Conclusion . . . . .                                                                                                  | 38        |
| Appendix  | . . . . .                                                                                                             | 38        |
| A         | Quantitative analysis of violation . . . . .                                                                          | 38        |
| B         | Numerical results . . . . .                                                                                           | 39        |
| C         | Numerical results for Single realization . . . . .                                                                    | 41        |
| D         | Experimental Results . . . . .                                                                                        | 41        |
| <b>II</b> | <b>Characterizing memory effects in open quantum systems</b>                                                          | <b>43</b> |
| <b>5</b>  | <b>Introductory remarks II</b>                                                                                        | <b>44</b> |
| 5.1       | Open quantum system and dynamical maps . . . . .                                                                      | 44        |
| 5.2       | Markovian and non-Markovian evolution . . . . .                                                                       | 45        |
| 5.3       | Master equation . . . . .                                                                                             | 47        |
| 5.4       | Quantum non-Markovianity witness . . . . .                                                                            | 47        |
| <b>6</b>  | <b>Witnessing non-Markovian effects of quantum processes through Hilbert-Schmidt speed</b>                            | <b>50</b> |
| 6.1       | Introduction . . . . .                                                                                                | 50        |
| 6.2       | Hilbert-Schmidt speed (HSS) . . . . .                                                                                 | 51        |
| 6.3       | HSS-based non-Markovianity measure . . . . .                                                                          | 52        |
| 6.4       | Qualitative analysis of non-Markovianity . . . . .                                                                    | 53        |
| 6.4.1     | One-qubit systems . . . . .                                                                                           | 53        |
| 6.4.2     | Two-qubit system . . . . .                                                                                            | 55        |
| 6.4.3     | One-qutrit systems . . . . .                                                                                          | 56        |
| 6.5       | Conclusions . . . . .                                                                                                 | 59        |
| Appendix  | . . . . .                                                                                                             | 59        |
| A         | Two-qubit evolved density matrix . . . . .                                                                            | 60        |
| B         | Solutions for $\Lambda$ -type three-level system . . . . .                                                            | 60        |
| <b>7</b>  | <b>Memory effects in high-dimensional systems faithfully identified by Hilbert-Schmidt speed-based witness</b>        | <b>61</b> |
| 7.1       | Introduction . . . . .                                                                                                | 61        |
| 7.2       | Preliminaries . . . . .                                                                                               | 62        |
| 7.2.1     | Non-Markovianity definition . . . . .                                                                                 | 62        |
| 7.2.2     | HSS-based witness of non-Markovianity . . . . .                                                                       | 63        |
| 7.2.3     | Quantum entanglement measure . . . . .                                                                                | 63        |
| 7.2.4     | Quantum correlation quantifier: Measurement-induced disturbance . . . . .                                             | 64        |
| 7.3       | Analyzing the efficiency of the HSS witness in high-dimensional systems with finite Hilbert spaces . . . . .          | 64        |
| 7.3.1     | Single-qudit interacting with a quantum environment . . . . .                                                         | 64        |
| 7.3.2     | Hybrid qubit-qutrit system interacting with various quantum and classical environments . . . . .                      | 65        |
| 7.4       | Conclusions . . . . .                                                                                                 | 72        |
| Appendix  | . . . . .                                                                                                             | 73        |
| A         | Pure hybrid qubit-qutrit evolved density matrix . . . . .                                                             | 73        |
| B         | Mixed hybrid qubit-qutrit evolved density matrix . . . . .                                                            | 75        |

|          |                                                                                                          |           |
|----------|----------------------------------------------------------------------------------------------------------|-----------|
| <b>8</b> | <b>Witnessing global memory effects of multiqubit correlated noisy channels by Hilbert-Schmidt speed</b> | <b>77</b> |
| 8.1      | Introduction . . . . .                                                                                   | 77        |
| 8.2      | Correlated quantum channels . . . . .                                                                    | 78        |
| 8.3      | Characterizing Non-Markovianity with Hilbert-Schmidt speed . . . . .                                     | 79        |
| 8.4      | Two-qubit Correlated unital noisy quantum channels . . . . .                                             | 80        |
| 8.4.1    | Model I: Two-qubit Pauli channels under two different noisy environments . . . . .                       | 80        |
| 8.4.2    | Model II: Two-Qubit Depolarizing Channels under Colored Noise Reservoir . . . . .                        | 82        |
| 8.5      | Two-Qubit Correlated Non-Unital Noisy Quantum Channel . . . . .                                          | 83        |
| 8.6      | Multiqubit correlated unital noisy quantum channel . . . . .                                             | 84        |
| 8.7      | Conclusion . . . . .                                                                                     | 86        |
| Appendix | . . . . .                                                                                                | 87        |
| A        | All Bases . . . . .                                                                                      | 87        |
| B        | Two-qubit correlated unital noisy quantum channels . . . . .                                             | 88        |
| C        | Two-qubit correlated non-unital noisy quantum channels . . . . .                                         | 89        |
| <br>     |                                                                                                          |           |
|          | <b>Conclusive remarks</b>                                                                                | <b>89</b> |
| <br>     |                                                                                                          |           |
|          | <b>References</b>                                                                                        | <b>92</b> |

# List of publications

Here, I present my full publication list, which includes five published papers and three currently under submission. During my Ph.D. research, I collaborated with numerous colleagues. Therefore, I have explicitly detailed my contributions to each paper in the list.

1. **Kobra Mahdavi pour**, Farzam Nosrati, Stefania Sciara, Roberto Morandotti, Rosario Lo Franco, Generation of genuine multipartite entangled states via indistinguishability of identical particles, PRX Quantum, 5(4), 040350, 2024.  
*The author is the main contributor to this project. In particular, she performed the main calculations and numerical simulations, contributed to proposing ideas, and prepared the paper.*
2. Agnes George, Nicola Montaut, **Kobra Mhadavipour**, Farzam Nosrati, Stefania Sciara, Monika Monika, Mario Chemnitz, Muniz, Ulf Peschel, Rosario Lo Franco, Roberto Morandotti, QuantumEnhanced Phase Estimation via Controlled-Discrete-Time Quantum Walk in Synthetic Photonic Lattices. To be submitted.  
*The author is the main contributor to this project. In particular, she performed the main calculations and numerical simulation, contributed to proposing ideas about the performed experiment, and prepared the paper.*
3. Alessandro Laneve, Farzam Nosrati, Andrea Gerdali, **Kobra Mahdavi pour**, Federico Pegoraro, Mahshid Khazaei Shadfar, Rosario Lo Franco, Paolo Mataloni, Enhancing nonclassical bosonic correlations in a quantum walk network through experimental control of disorder, Phys. Rev. Res. 3 p. 033235, 2021.  
*The author provided feedback on the performed calculations and ideas, and contributed to the preparation of the paper.*
4. Hossein Rangani Jahromi, **Kobra Mahdavi pour**, Mahshid Khazaei Shadfar, Rosario Lo Franco, Witnessing non-Markovian effects of quantum processes through Hilbert-Schmidt speed, Phys. Rev. A 102 p. 022221, 2020.  
*The author provided feedback on the performed calculations and ideas, and contributed to the preparation of the paper.*
5. **Kobra Mahdavi pour**, Mahshid Khazaei Shadfar, Hossein Rangani Jahromi, Roberto Morandotti, Rosario Lo Franco, Memory Effects in High-Dimensional Systems Faithfully Identified by Hilbert-Schmidt Speed-Based Witness, Entropy 24. 2022.  
*The author is the main contributor to this project. In particular, she performed the main calculations and numerical simulations, contributed to proposing ideas, and prepared the paper.*
6. **Kobra Mahdavi pour**, Samira Nazifkar, Hossein Rangani Jahromi, Rosario Lo Franco, Witnessing global memory effects of multiqubit correlated noisy channels by Hilbert-Schmidt speed. arXiv:2501.05918 .  
*The author is the main contributor to this project. In particular, she performed the main calculations and numerical simulations, contributed to proposing ideas, and prepared the paper.*

**The following publications are not included in this thesis**

7. Farzam Nosrati, Alessandro Laneve, Mahshid Khazaei Shadfar, Andrea Gerdali, **Kobra Mahdavi pour**, Federico Pegoraro, Paolo Mataloni, Rosario Lo Franco, Readout of quantum information spreading using a disordered quantum walk J. Opt. Soc. Am. B 38 pp. 25702578, 2021.  
*The author provided feedback on the performed calculations, ideas, and contributed to the preparation of the paper.*
8. **Kobra Mahdavi pour**, Farzam Nosrati, Agnes George, Stefania Sciara, Nicola Montaut, Roberto Morandotti, Rosario Lo Franco, Discrete-time quantum walk in the presence of photon loss. To be submitted.  
*The author is the main contributor to this project. In particular, she performed the main calculations and numerical simulations, contributed to proposing ideas, and prepared the paper.*



# List of Figures

- 1.1 Operational entanglement. Two identical particles with opposite pseudospins (internal states) have spatial wave functions  $\psi_1$  and  $\psi_2$  with a given degree of overlap. The entanglement between pseudospins is operationally defined by local measurements in two separated localized spatial regions L and R. . . . . 6
- 1.2 Theoretical schematic. a. The simplest setup is designed to produce entanglement via sLOCC due to spatial indistinguishability. b. The general spatial distribution of particles in the state  $|\Psi\rangle = |\psi_{1D}\uparrow, \psi_{2D}\downarrow\rangle$  before sLOCC, with  $|\psi_{1D}\rangle = l|\psi_L\rangle + r|\psi_R\rangle$ , and  $|\psi_{2D}\rangle = l'|\psi'_L\rangle + r'|\psi'_R\rangle$ . [13] . . . . . 8
- 2.1 **Illustration of the deformation in both on-site and remote schemes.** (a) Illustration of an on-site designed  $N$ -mode deformation, denoted as  $\hat{D}_N$ , where  $N$  identical qubits, each possessing an internal degree of freedom  $\sigma_i = \{\uparrow, \downarrow\}$ , pass through  $N$  modes of this deformation. At the end, qubits arrive in  $N$  detection regions  $R_1, \dots, R_N$ , for sLOCC projection measurement to generate the output state. The inset illustrates the potential for implementing any  $N$ -mode deformation by combining  $D_2$  operators. (b) Illustration of remote deformation design using  $N$  deformations  $D_N^{(i)}$  ( $i = 1, \dots, N$ ) that send  $N$ -qubits into  $N$  individual regions for postselection, generating the output state. . . . . 13
- 2.2 Graph representations of entangling protocol in the case of two particles. (a) bigraph with  $G_b = (V \cup U, E)$ , where  $V = \{|\psi\uparrow\rangle, |\psi\downarrow\rangle\}$  and  $U = \{|\chi_1\rangle, |\chi_2\rangle\}$ , edge  $e_{ij} \in E$  connects a node  $u_i \in U$  and a node  $v_j \in V$  with complex weight  $r_{ij}$ . Panels (b) and (c) display perfect matching sub-graphs that correspond to states  $|R_1\uparrow, R_2\downarrow\rangle$  and  $|R_1\downarrow, R_2\uparrow\rangle$ . (d) Colored digraphs (with self-loops) with nodes  $U = \{u_1, u_2\}$ , where each node is a binary relation between the single-qubit state and the measurement region  $u_i = (|\varphi_i\rangle, |\chi_i\rangle)$ . Blue and red correspond to down and up pseudospins, respectively. . . . . 16
- 2.3 **Digraph representations of four-particle  $W$  state generation.** The digraph illustrates four quantum nodes  $u_j$  ( $j = 1, \dots, 4$ ), representing a binary state and measurement region. Red (blue) edges indicate the detection of a qubit with an up (down) pseudospin. The presence of  $\eta$  on red edges denotes a  $\pi$  phase shift between connected nodes for fermions. (a) The complete digraph illustrates mutually oriented edges between every pair of nodes, each with equal weighting. (b) The star graph displays mutually oriented edges only between the center node and the other nodes. (c) The QFT digraph exhibits mutually oriented edges between each pair with a specific weighting as  $\omega_N^n$  ( $\omega_N = e^{2i\pi/N}$ ). . . . . 18
- 2.4 **Success probability and fidelity of generating  $W$  states with different schemes and particle statistics.** (a) Success probability  $\mathcal{P}(|W^{(N)}\rangle)$  as a function of the number of particles  $N$  for different  $W$  state generation schemes, including the complete digraph for bosons (green star), the star digraph for bosons (red asterisk), the QFT digraph for bosons (blue circle), the star digraph for fermions (black circle), and the QFT digraph for fermions (red square). (b) Fidelity as a function of the maximum probability of success  $\mathcal{P}_{\max}(|W^{(N)}\rangle)$  for three bosons (blue circles) and four bosons (red stars) in the form of genuine  $W$  states in the complete digraph. The dashed lines correspond to the bound that guarantees genuine multipartite  $W$  entanglement. . . . . 19
- 2.5 **Symmetric Dicke state generation.** (a) Colored complete digraph  $U = \{u_1, \dots, u_4\}$  for a four-qubit symmetric Dicke state. (b) Fidelity as a function of maximum probability of success  $\mathcal{P}_{\max}(|D^{(N)}\rangle)$  for four (blue circles) and six (red stars) bosons in the form of a genuine symmetric Dicke state with two pseudospins down in the complete digraph. The dashed lines correspond to the bound that guarantees genuine multipartite symmetric Dicke entangled states. . . . . 20
- 2.6 (a) **Digraph representations of  $N$ -qubit GHZ state.** Colored digraph (with self-loops) with nodes  $U = \{u_1, u_2 \dots u_N\}$ . Also, the self-loops and oriented edges have different colors. The presence of  $\eta$  shows a  $\pi$  phase shift between connected nodes for fermions. Indeed, this phase shift is not necessary when working with bosons. (b) **Trade-off between fidelity and sLOCC probability.** Fidelity as a function of maximum probability of success  $\mathcal{P}_{\max}(|GHZ^{(N)}\rangle)$  for three (blue), four bosons (red), five (green) and six (black) bosons in the form of genuine GHZ state . . . . . 21

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |    |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| 2.7  | <b>Graph representations of <math>N</math>-qubit cluster states for (a) bosons and (b) fermions.</b> The colored digraph (with self-loops) has nodes $U = \{u_1, u_2, \dots, u_N\}$ . (a) In the bosonic design, the dashed-oriented edge represents a $\pi$ phase shift between connected nodes. (b) For fermions, the dashed-oriented edge represents a $\pm(-1)^{N/2}$ phase shift between connected nodes, depending on the number of particles. <b>(c) Trade-off between fidelity and sLOCC probability.</b> Fidelity as a function of maximum probability of success $\mathcal{P}_{max}( C^{(N)}\rangle)$ for four (blue square), six bosons (red stars), and eight (black circle) qubits in the form of a genuine cluster state. The solid lines correspond to the fitted data. . . . . | 22 |
| 2.8  | Perfect matching sub-bigraphs for generation of the three-particle $W$ state [154, 155]. <b>a)</b> Represents the state $ R_1 \uparrow, R_2 \downarrow, R_3 \downarrow\rangle$ . <b>b)</b> Represents the state $ R_1 \downarrow, R_2 \uparrow, R_3 \downarrow\rangle$ . <b>c)</b> Represents the state $ R_1 \downarrow, R_2 \downarrow, R_3 \uparrow\rangle$ . . . . .                                                                                                                                                                                                                                                                                                                                                                                                                       | 25 |
| 3.1  | A coupler connects two polarization-maintaining fiber loops of lengths 120 m (600 ns) for the long loop and 100 m (500 ns) for the short loop. An arbitrary function generator (AFG) controls the couplers transmittance dynamically. The setup also includes a pulsed laser, periodically poled lithium niobate (PPLN) waveguides to generate photons via cascaded second harmonic generation (SHG) and spontaneous parametric down-conversion (SPDC) processes, a phase modulator, a wavelength division multiplexer (WDM), an oscilloscope, a photodetector (PD), and two superconducting nanowire single-photon detectors (SNSPDs). The oscilloscope and PDs are used in the classical part of the experiment [74]. . . . .                                                                | 30 |
| 3.2  | The coupled fiber-loop system consists of long and short loops (shown by green and red colors, respectively) connected by an ultra-fast optical dynamical coupler and $\tau$ is the time delay among adjacent pulses. Two dynamical gates, Gate 1 and Gate 2, control the in-out flow of light in the long loop and short loop. (b) Typical representation of fully controllable QW network with dynamical coupler $\hat{C}(x, t)$ where is a function of the discrete synthetic position $x$ (arrival time of photons) in the lattice and the number of round trips $t$ . . . . .                                                                                                                                                                                                             | 31 |
| 3.3  | The schematic of controlled DTQW, which corresponds to performing optimal POVM measurements for single parameter phase estimation. In the first round trip, the coin operator is set to act as transmission $\hat{T}$ and reflection, $\hat{R}$ respectively, in the synthetic positions 1 and $-1 \hat{T}\hat{R}$ . In the second-round trip, the coin operator is set to act as Fourier coin $\hat{F}$ in the synthetic 0. In the detection level (third-round trip), the coin operator is set as reflection $\hat{R}$ in the synthetic position $x = 1$ , and the synthetic position $x = -1$ is set as the identity $\hat{I}$ . . . . .                                                                                                                                                    | 32 |
| 4.1  | <b>Network representation of a disordered QW according to <math>p</math>-diluted model.</b> The green bars stand for the presence of $\pi$ phases on the path which can be added to or removed from the network. . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | 35 |
| 4.2  | <b>Numerical simulation compression between order and optimal disordered QW.</b> <b>a)</b> Maximum Achievable Violation (MAV) and <b>b)</b> maximum achievable Total Violation versus the number of steps (discrete time) $t$ for the ordered (blue circles) and optimal disordered (red squares) QW. . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | 35 |
| 4.3  | <b>2D sketch of the experimental setup.</b> BS: beam splitter, BD: beam displacer, RP: rotating glass plates, MM: moving mirror, S1: Sagnac Interferometer for odd steps, S2: Sagnac Interferometer for even steps. Blue and red beams circulate in opposite directions and impinge on the BS in the same horizontal point but at different heights along the $z$ direction, due to the effect of BDs. . . . .                                                                                                                                                                                                                                                                                                                                                                                 | 36 |
| 4.4  | <b>Comparison between the theoretical and experimental violation matrices at the 6th step for an enhancing disorder configuration.</b> Numerical simulations are performed taking to account experimental parameters. The expected peak in the value of $V$ is experimentally found, while the measured output coincidence distributions reach globally a similarity value of $97.5(\pm 0.2)\%$ . . . . .                                                                                                                                                                                                                                                                                                                                                                                      | 37 |
| 4.5  | <b>Experimental results compression between the order and the enhanced disordered configuration QW.</b> Experimental results for (dark dot) <b>a)</b> Maximum Achievable Violation (MAV) and <b>b)</b> maximum achievable Total Violation versus the number of steps. The trends are compared with the simulation for optimal disorder (red squares) and ordered evolution (blue circles). The expected results are obtained by numerical simulations performed accounting for experimental parameters, so that the theoretical trends show some discrepancies with respect to the ones in Fig. 4.2. . . . .                                                                                                                                                                                   | 37 |
| 4.7  | <b>Numerical simulation of Violation matrix.</b> Violation matrix at step 15 for indistinguishable photons in the <b>a)</b> ordered and <b>b)</b> completely disordered case $p = 1$ . The disordered matrix has been computed by averaging over 10000 disorder configurations. . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | 39 |
| 4.8  | <b>Numerical simulation of coincidences matrix.</b> Step 15 output distributions of indistinguishable photons in the <b>a)</b> ordered and completely <b>b)</b> disordered case. The disordered matrix has been computed by averaging over 10000 disorder configurations. . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | 40 |
| 4.9  | <b>Numerical simulation of coincidences matrix.</b> Step 15, step 10 and step 6 plot of the average Total Violation, computed over 500 disorder configurations, as a function of the disorder level $p$ . . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | 40 |
| 4.10 | <b>Numerical simulation of violation landscape.</b> Plot of the maximum violation achievable for each output modes pair at step 9, obtained by comparing $10^6$ different disorder realizations. . . . .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | 41 |

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| 4.11 | <b>violation matrices at the 5th.</b> Theoretical and experimental violation matrices at the 5th step for <b>a) and b)</b> ordered evolution and <b>c) and d)</b> disordered evolution. The disorder configuration has been chosen randomly. . . . .                                                                                                                                                                                                                                                           | 42 |
| 4.12 | <b>Numerical simulation of coincidences matrix.</b> 5th step coincidence of <b>a)</b> theoretical <b>b)</b> experimental measurement outputs for the ordered quantum walk. 6th step output coincidence matrix <b>c)</b> theoretical <b>d)</b> experimental measurement outputs for the optimal disordered. The similarities between theoretical and experimental coincidences distributions are $98.2(\pm 0.2)\%$ for case (a) and $97(\pm 5)\%$ for case (b). . . . .                                         | 42 |
| 5.1  | An open quantum system is described by the Hilbert space $\mathcal{H}_S$ with Hamiltonian $H_S$ being coupled to an environment with the Hilbert space $\mathcal{H}_E$ with Hamiltonian $H_E$ through an interaction Hamiltonian $H_I$ . . . . .                                                                                                                                                                                                                                                               | 45 |
| 6.1  | Dynamics of Hilbert-Schmidt speed $HSS(\rho_\varphi(t))$ (blue solid line), trace distance $D(\rho_1(t), \rho_2(t))$ (red dot-dashed line) and coherence characteristic function $P(t)$ (amplified by 20 times for comparison, green dashed line) as a function of the dimensionless time $\gamma_0 t$ for the two-qubit system in the strong coupling regime, with $\lambda = 1.25\gamma_0$ . . . . .                                                                                                         | 56 |
| 6.2  | Dynamics of Hilbert-Schmidt speed $HSS(\rho_\varphi(t))$ (blue solid line) and trace distance $D(\rho_1(t), \rho_2(t))$ (red dashed line) as a function of the dimensionless time $\gamma t$ for the V-type three-level atom, with $\lambda = 5 \times 10^{-3}\gamma$ and $\theta = 0.6$ . . . . .                                                                                                                                                                                                             | 58 |
| 7.1  | Illustration of the composite qubit(A)-qutrit(B) system; Blue dashed lines represent entanglement between the subsystems. The bipartite system can interact either with independent local environments $E_A$ , $E_B$ or with a common environment $E_C$ . . . . .                                                                                                                                                                                                                                              | 66 |
| 7.2  | Evolution of the negativity, MID and HSS as a function of dimensionless time $\tau = \omega_0 t$ when each subsystem of the hybrid qubit-qutrit system, starting from the initial pure state, is independently subject to a squeezed vacuum reservoir. The values of the other parameters are $\alpha = 0.1$ , $\omega_c = 20\omega_0$ , $r = 0.3$ , $\phi = \pi$ and $s = 3$ . . . . .                                                                                                                        | 67 |
| 7.3  | Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when each subsystem is independently coupled to a squeezed vacuum reservoir, with HSS (obtained from the initial pure state) for different values of the entanglement parameter $p$ . In all plots the remaining parameters are $\alpha = 0.1$ , $s = 3$ , $\omega_c = 20\omega_0$ , $r = 0.3$ . . . . .                                                                                 | 68 |
| 7.4  | Evolution of negativity, MID and HSS as a function of dimensionless time $\tau = \nu t$ when each subsystem of the hybrid qubit-qutrit system, starting from the initial pure state, is independently subject to a random telegraph noise in non-Markovian regime $q = 0.1$ . . . . .                                                                                                                                                                                                                          | 69 |
| 7.5  | Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when each subsystem is independently coupled to a random telegraph noise, with HSS (obtained from the initial pure state) for different values of the entanglement parameter $p$ in the non-Markovian regime: $q = 0.1$ . . . . .                                                                                                                                                        | 70 |
| 7.6  | Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when its subsystems are subject to a common RTN source, with HSS (obtained from the initial pure state) for different values of the entanglement parameter $p$ in the non-Markovian regime: $q = 0.1$ . . . . .                                                                                                                                                                          | 71 |
| 7.7  | Evolution of negativity, MID and HSS as a function of dimensionless time $\tau$ when the subsystems of the hybrid qubit-qutrit system, starting from the initial pure state, are independently subject to composite classical-quantum environments. The values of the other parameters are given by $\alpha = 0.1$ , $\omega_c = 20\omega_0$ , $r = 0.3$ , and $\nu = 100$ . . . . .                                                                                                                           | 72 |
| 7.8  | Comparing the evolution of the negativity and MID, computed for the initial mixed state of the hybrid qubit-qutrit system, when the subsystems are independently subject to composite classical-quantum environments, with the HSS obtained from the initial pure state for different values of the entanglement parameter $p$ in the non-Markovian regime: $q = 0.1$ . The values of the other parameters are given by $\alpha = 0.1$ , $s = 3$ , $\omega_c = 20\omega_0$ , $p = 0$ and $\nu = 100$ . . . . . | 73 |
| 8.1  | Illustration of the action of correlated noisy channels. The parameter $\mu$ is the classical correlation factor between consecutive uses of a quantum channel on the sequence of quantum systems. . . . .                                                                                                                                                                                                                                                                                                     | 79 |
| 8.2  | a) Dynamics of HSS as a function of dimensionless time $\tau = \frac{t}{2\nu}$ for two-qubit Pauli channels with memory under the colored pure dephasing evolution in the standard basis with $\nu = 1$ . Three different values of the correlation factor $\mu$ are considered. b) HSS-based measure $\mathcal{N}_{\text{HSS}}$ as a function of the correlation factor $\mu$ . . . . .                                                                                                                       | 81 |
| 8.3  | a) The dynamic of HSS as a function of dimensionless time $\tau = \omega t$ for a two-qubit Pauli channel with memory under squeezed vacuum reservoir with $\alpha = 0.5$ , $s = 4$ , $r = 0.5$ and $\theta = \frac{3\pi}{2}$ . b) HSS-based measure $\mathcal{N}_{\text{HSS}}$ as a function of the correlation factor $\mu$ . . . . .                                                                                                                                                                        | 81 |

|      |                                                                                                                                                                                                                                                                                                                                                                                                 |    |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| 8.4  | a) Dynamics of HSS as a function of dimensionless time $\tau = \frac{t}{2\nu}$ for a two-qubit depolarizing channel with memory under the evolution of colored noise, for $\theta = 0.5$ and $\phi = \pi$ . b) HSS-based measure $\mathcal{N}_{\text{HSS}}$ as a function of the correlation factor $\mu$ . . . . .                                                                             | 82 |
| 8.5  | a) Dynamics of HSS as a function of dimensionless time $\tau = \lambda t$ for two-qubit amplitude damping channels with memory in the standard basis under the evolution of a Lorentzian reservoir for $a = 4$ and $\phi = \pi$ . b) HSS-based measure $\mathcal{N}_{\text{HSS}}$ as a function of the correlation factor $\mu$ . . . . .                                                       | 84 |
| 8.6  | a) Dynamics of HSS for multiqubit correlated Pauli channel under colored pure dephasing reservoirs with constant reservoir parameters $\nu = 1$ at dimensionless time $\tau = 1.62$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. b) Range of variation $\delta$ versus the number of qubits. . . . .                  | 84 |
| 8.7  | a) Dynamics of HSS for multiqubit correlated Pauli channel under squeezed vacuum reservoirs: $\alpha = 0.5$ , $s = 4$ , $r = 0.5$ , and $\theta = \frac{3\pi}{2}$ at dimensionless time $\tau = 0.2$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. b) Range of variation $\delta$ versus the number of qubits. . . . . | 85 |
| 8.8  | a) Dynamics of HSS for multiqubit correlated depolarizing channel under colored noise reservoirs: $\eta = 0.5$ , $\Phi = \frac{\pi}{2}$ at dimensionless time $\tau = 1.6$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. b) Range of variation $\delta$ versus the number of qubits. . . . .                           | 85 |
| 8.9  | Dynamics of HSS as a function of dimensionless time $\tau = \frac{t}{2\nu}$ for correlated multi-qubit Pauli channels under colored pure dephasing evolution in the standard basis with $\nu = 1$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. . . . .                                                                | 86 |
| 8.10 | Dynamics of HSS as a function of dimensionless time $\tau = \omega t$ for correlated multi-qubit Pauli channels under squeezed vacuum evolution in the standard basis with $\alpha = 0.5$ , $s = 4$ , $r = 0.5$ , and $\theta = \frac{3\pi}{2}$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. . . . .                  | 87 |
| 8.11 | Dynamics of HSS as a function of dimensionless time $\tau = \frac{t}{2\nu}$ for correlated multi-qubit depolarizing channels under colored noise evolution in the standard basis with $\theta = 0.5$ and $\phi = \frac{\pi}{2}$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to $N = 2, 3, 4, 6, 7$ , and 8 qubits. . . . .                                  | 88 |

# List of Tables

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |    |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----|
| 2.1 | <b>Digraph configurations for generating <math>W</math> states with three and four qubits.</b> This table outlines diverse digraph configurations utilized to generate four-qubit $W$ states involving identical bosonic and fermionic qubits. It includes their corresponding fidelities ( $F$ ) and sLOCC probabilities ( $\mathcal{P}$ ). The presence of $\eta$ , highlighted in red, shows a $\pi$ phase shift between interconnected nodes for fermions. Moreover, edge weights labeled with $\omega_N$ denote $e^{2i\pi/N}$ . . . . . | 26 |
| 2.2 | <b>Digraph configurations for generating four-qubit Dicke states.</b> The table summarizes digraph configurations utilized to generate four-qubit Dicke states, involving identical bosonic and fermionic qubits, along with their corresponding fidelities ( $F$ ) and sLOCC probabilities ( $\mathcal{P}$ ). The presence of $\eta$ , highlighted in red, indicates a $\pi$ phase shift between interconnected nodes for fermions. Moreover, edge weights labeled with $\omega_N$ denote $e^{2i\pi/N}$ . . . . .                           | 27 |

# General introductory remarks

This ongoing doctoral dissertation presents pioneering research focusing on the theoretical and experimental studies of quantum correlations and memory effects in composite quantum systems. Part I of the study investigates the utilization of quantum correlations in identical quantum systems within a quantum information framework. Although quantum systems are not ideally isolated, interactions with the environment cause them to lose coherence and information. Consequently, the dynamics of such systems can be categorized into two types: memoryless dynamics, known as Markovian, and dynamics with memory, known as non-Markovian. In Part II, I introduce a new witness of non-Markovianity and evaluate its validity and efficiency in various systems.

Identical particles (e.g., photons, electrons, atoms, qubits, and plasmons) are fundamental components of quantum networks due to their intrinsic features, such as having the same mass and charges. Consequently, determining the properties and quantum correlations of identical particles is essential for advancing quantum technology. The quantum correlations of identical particles are not solely intrinsic properties of the quantum state; they also depend on the performed measurements when the particles spatially overlap and become indistinguishable. Therefore, in Part I, I begin by introducing the spatial indistinguishability of identical particles as a resource for information processing and discuss how spatially local operations and classical communication, and the no-label approach, can be used to exploit and quantify the entanglement of indistinguishable identical particles.

Furthermore, it has been both experimentally and theoretically demonstrated that spatially overlapping two independent particles with opposite pseudospins that meet only at the detection stage can create entanglement [12, 13, 14]. Considering the importance of multipartite entangled states such as W, Dicke, GHZ, and cluster states in various areas of quantum technology, including quantum computing [15, 16], communication [17, 18, 19], and sensing [20]. During my Ph.D., I extended the given scheme in [12] to  $N$  identical particles. In Chapter 2, I provide a controllable and comprehensive scheme capable of generating, in three steps: preparation (starting with  $N$  independent particles), deformation (distributing and manipulating their spatial wavefunctions in  $N$  different spatial regions), and post-selection (performing single-photon counting), a broad class of multipartite entangled states for both bosonic and fermionic statistics. Using graph-based representations within the framework of spatially localized operations and classical communication (sLOCC), I mathematically translated the generation schemes of specific entangled states into colored, complex, and weighted digraphs, each corresponding to a given experimental setup. In this translation, the spatial overlaps appear as adjacency matrix elements of the digraph. This graph-theoretical approach facilitates the optimization of generation efficiency for specific multipartite entangled states by exploring various generation schemes. The presented theoretical method, already implementable with current linear optics architectures, offers significant advantages over existing technologies, such as in quantum computing search algorithms and the design of new experiments in quantum optics or other platforms.

In the following part of my Ph.D. thesis, I consider the concept of indistinguishability from an operational point of view. Particles are assumed to be indistinguishable when they share the same polarization, time, frequency, position, and momentum. In this context, the quantum correlation between  $N$  indistinguishable particles, which are in a superposition state between two paths (or two modes, any degrees of freedom) known as the NOON state (NOON-like state), play a crucial role in quantum sensing and quantum metrology. Therefore, in Chapter 3, I discuss phase estimation. By utilizing discrete-time quantum walks (DTQWs) to realize positive operator-valued measurements (POVMs) on particles, controlled quantum walks can be effectively implemented to achieve optimal measurements for phase sensing problems. In this work, I theoretically demonstrate quantum-enhanced phase estimation using a time-bin entangled state, specifically a NOON-like state with a two-photon number ( $N = 2$ ), as a probe. This is achieved by performing optimal measurements through a controlled quantum walk scheme on synthetic photonic lattices. The use of a coupled fiber-loop system with dynamic gates enables precise control over the quantum walk and facilitates the determination of quantum Fisher information through post-selection. I theoretically demonstrate that using a time-bin entangled state and performing optimal POVMs through the implementation of DTQW in two round trips leads to quantum enhancement by a factor of 2 compared to classical light. The related experiment was conducted by other group members; however, as the paper is not yet published, I present only the theoretical part without experimental results.

Understanding the behavior of quantum features based on indistinguishability within a dynamic framework is im-

portant. The presence of disorder and inhomogeneities in quantum networks has often been unexpectedly beneficial for both quantum and classical resources [84, 36, 85, 86, 87, 88]. In Chapter 4, I discuss how we experimentally realize controllable inhomogeneous quantum walk (QW) dynamics, which can be exploited to investigate the effect of coherent disorder on the quantum correlations between two indistinguishable photons. Through the imposition of suitable disorder configurations, we observe two-photon states that exhibit an enhancement in the quantum correlations between two selected modes of the network, compared to the case of an ordered QW. Different configurations of disorder can steer the system toward different realizations of such an enhancement, thus allowing spatial and temporal manipulation of quantum correlations between remote modes of QW networks [75]. Despite various studies [89], strategies for enhancing two-particle correlations in separate modes of a quantum walk network via disorder control remain unexplored. We bridge this gap by injecting two indistinguishable photons (a biphoton) into a one-dimensional inhomogeneous Discrete-Time Quantum Walk (1D DTQW) and experimentally controlling specific disorder configurations while ensuring the system remains isolated. As demonstrated experimentally in Fig. 4.4, it is feasible to tune the two-mode and total enhancement of non-classicality in position and intensity. This represents an adaptive network whose parameter evaluation determines the focusing of nonclassical resources into specific modes. Additionally, in Fig. 4.9, we numerically illustrate that averaging over randomized disorder configurations decreases the initial biphoton quantum correlations. Our findings clearly demonstrate that the presence of disorder facilitates the dynamic enhancement of the biphoton's quantum correlations in a controllable manner, paving the way to its potential application in quantum information scenarios.

Realistic quantum systems are inevitably influenced by their interactions with the environment, resulting in the loss of coherence and information. Consequently, the dynamics of the system can either be memoryless (known as Markovian) or associated with the backflow of information from the environment to the system (known as non-Markovian). In the next stage of my Ph.D., I introduce a new witness of non-Markovianity and examine its validity and efficiency through different examples. Therefore, in Chapter 6, inspired by the fact that non-Markovian effects can speed up the dynamics of systems while quantifiers of quantum statistical speed can derive the limit of the evolution time, we introduce a novel witness as a characterizer of non-Markovianity behavior of open quantum systems. This witness is based on the positive changing rate of Hilbert-Schmidt speed (HSS) as a special case of quantum statistical speed. The advantage of this witness is that it does not require the diagonalization of the evolved density matrix of the system. The efficiency of the witness is checked in several useful examples of open quantum systems: a qubit subject to phase-covariant noise and Pauli channel, two independent qubits locally interacting with leaky cavities,  $V$ -type and  $\Lambda$ -type three-level atoms (qutrits) in a dissipative cavity. We find that the HSS-based non-Markovianity witness identifies memory effects in total agreement with the trace distance-based BLP witness [120], thus detecting system-environment information backflows [144]. Subsequently, in Chapter 7, I investigate the sensitivity of this witness to detect non-Markovianity in various high-dimensional and multipartite open quantum systems. We illustrate that the time oscillations of the HSS-based witness are in excellent agreement with those of the quantum negativity or quantum correlation measure. Therefore, our proposed witness is a faithful identifier to characterize the memory effects appearing in the quantum evolution of a high-dimensional system [156].

Furthermore, in quantum information theory, quantum channels with (classical) memory are characterized by classical correlations between successive applications of the channel in a sequence of quantum systems. This concept completely differs from the non-Markovian memory effects. In correlated non-Markovian channels, the global memory effect refers to both the classical correlation between consecutive uses of the quantum channel on the sequence of quantum systems and the intrinsic memory effects of a single channel governed by its spectral density. The first interplay between correlation factors and non-Markovian memory effects in colored pure channels under a dephasing scenario has been discussed in [145]. Given the importance of correlated non-Markovian channels in enhancing the quantum speed limit [146], increasing channel capacity, and mitigating noise in quantum error correction [147], we extend our analysis in Chapter 8 to demonstrate that Hilbert-Schmidt speed (HSS), as an effective measure of non-Markovianity, is a reliable tool for evaluating correlations in both unital and non-unital channels with varying noisy spectral densities. For unital channels, we consider examples such as Pauli and depolarizing channels, while for non-unital channels, we focus on amplitude-damping channels. Additionally, we examine the influence of classical memory, arising from correlated applications of quantum channels, on the non-Markovian evolution of the system. Finally, we show that in unital noisy channels with a large number of qubits, the sensitivity of HSS to the strength of classical correlations between consecutive channel uses is significantly reduced. This finding suggests that larger quantum systems may exhibit diminished sensitivity to the accelerating effects of classical correlations on state evolution.

## **Part I**

# **Exploitation of quantum correlations of identical quantum systems in quantum information scenario**



# Introductory remarks I

In Part I of my PhD thesis, I explore the quantum correlations of identical quantum systems in quantum information scenarios. I begin by discussing the role of indistinguishability of identical particles as a resource for generating multipartite entangled states. Then, I examine indistinguishability from an operational point of view and consider another important class of multipartite entangled states, namely the N00N state, and its application in quantum-enhanced phase estimation. Furthermore, to investigate the quantum features based on indistinguishability behavior in a dynamic framework, I discuss discrete-time quantum walks.

## 1.1 Indistinguishability of identical particles as a resource for quantum information processing

### 1.1.1 Indistinguishability vs identity

Particles are identical when their intrinsic properties, including electric charge, mass, and total spin, are the same. Identical subsystems (photons, electrons, atoms, qubits, plasmons,...), whether bosonic or fermionic systems, are the main building blocks of quantum networks [1, 2, 3]. Generally, two types of physical quantities can be used to describe and distinguish identical particles: state-dependent quantities and state-independent quantities. State-dependent quantities refer to those that vary depending on the state of the particles (e.g., position and velocity), while state-independent quantities are the intrinsic properties of the particles (e.g., mass, electric charge, position, momentum, etc.).

In classical physics, identical particles are distinguishable and can be individually characterized through a set of unique step-independent physical quantities. Although two or more classical particles may share several physical properties, they still differ in their spatial positions. In other words, it is possible to give a physical identity to each classical particle by assigning it a unique name or label. These labels for classical particles include physical content such as position and trajectory. The physical concept of particle identity plays a role in the dynamics. This is required to guarantee that particles are individually addressable through observables. In classical mechanics, this situation is satisfied through detection techniques sensitive to spatial position, allowing the identification and tracking of identical particles trajectories without affecting their dynamics.

In quantum mechanics, elementary particles are described using both state-dependent and state-independent physical quantities. Unlike classical mechanics, the trajectories of identical particles cannot be observed without altering their dynamics and behavior; additionally, they cannot be uniquely assigned to specific particles. The wave-like and probabilistic description allows different particles wave functions to spatially overlap, resulting in a nonzero probability of simultaneously occupying the same region of space. In this situation, employing measurements that rely on state-dependent properties, such as those that depend solely on particles positions within the regions of overlap, does not allow the observer to determine which specific particle belongs to the measured outcome. Lets consider synchronized photon sources A and B, each emitting single photons that collide simultaneously in a restricted detecting spatial region with a certain probability. If a single photon detector clicks in that region, we cannot determine the source of the detected photon, and consequently, the particles are considered indistinguishable. In conclusion, while identity is an intrinsic property of particles, indistinguishability is related to the measurement process and to the eye of the detector.

### 1.1.2 No-label approach

The standard approach uses unphysical (unobservable) labels to distinguish identical particles, thereby modifying the description of their dynamics, which also requires the symmetrization postulate with respect to the permutation of the unphysical labels. According to the symmetrization postulate, the global state describing a collection of identical particles must remain unchanged when the roles of any pair of particles are exchanged. Bosonic particles exhibit symmetry under

particle swapping, whereas fermionic states are anti-symmetric under similar particle exchanges. Lets consider a simple example: a system consisting of two identical particles described in  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , only one of the following states is allowed accordingly,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle_1 |\psi\rangle_2 \pm |\psi\rangle_1 |\phi\rangle_2), \quad (1.1)$$

where the  $+$  sign refers to bosons and  $-$  sign to fermions. States in the form of  $|\Psi\rangle = |\phi\rangle_1 \otimes |\psi\rangle_2$  products are forbidden in the quantum mechanics of identical particles. Paulis exclusion principle is an immediate consequence of the symmetrization postulate, stating that two or more fermions cannot occupy simultaneously the same quantum state. Such a postulate carries significant consequences in quantum field theory, Bose-Einstein condensates, and particle physics. However, another consequence of the symmetrization postulate is that two identical particles, whether fermions or bosons are always entangled. This is the obvious mathematical conclusion. The standard approach gives rise to methodological and practical difficulties in the characterization of identical particle entanglement because it uses unphysical labels and requires the symmetrization postulate of the state. To deal with the challenges, an alternative approach known as the "no-label" approach to identical particles has been proposed [4, 5]. Instead of employing non-observable properties, this approach relies on observables to determine the state.

To briefly summarize the no-label approach, let us assume two identical particles in separate spatial regions, respectively, with  $|\xi\rangle$  and  $|\chi\rangle$  representing their state respectively. The single-particle measurements are implemented using localized devices in regions L and R, where the symmetrization postulate has no impact on physical predictions [6, 7, 8, 9]. In this scenario, the density matrix can be obtained by assuming that  $L$  and  $R$  as distinguishable real labels and without applying symmetrization to these labels. Additionally, the result of performing a measurement in region  $L(R)$  cannot originate from a particle in  $R(L)$  [9]. The global state thus can be factorized to  $|\Psi\rangle = |\xi\rangle_L \otimes |\chi\rangle_R$ , for both bosons and fermions and consequently identical particles behave like non-identical ones. Nonetheless, when particles are perfectly distinguishable, the global state can no longer be written as a product state.

A one-particle state can be characterized by a complete set of observables in the no-label approach. The state of two identical particles can be represented, without non-observable labels, as a list of one-particle states  $|\phi\rangle_1$  and  $|\phi\rangle_2$ , in the form of  $|\Psi\rangle = \frac{|\phi_1, \phi_2\rangle}{\mathcal{N}}$ , where  $\mathcal{N}$  is the normalization factor.

The subsequent step involves defining the action of operators within the no-label approach. Let's consider the one-particle operator  $O^{(1)}$  acting on the global state of two-particle one at the time, leading  $O^{(1)}|\Psi\rangle = |O^{(1)}\phi_1, \phi_2\rangle + |\phi_1, O^{(1)}\phi_2\rangle$ , similar to the label-based approach [8]. Within a single-particle state space with a basis  $\beta^{(1)} = \{|\psi_k\rangle, k = 1, 2, \dots\}$ , the operation of the single-particle operator is defined in general form as  $\hat{O}^{(1)} = \sum_{j,k} |\psi_j\rangle \langle\psi_k|$ . Thus, a symmetric inner product between state spaces of different dimensionality can be established as

$$\langle\psi_k|\phi_1, \phi_2\rangle := \langle\psi_k|\phi_1\rangle\langle\phi_2\rangle + \eta\langle\psi_k|\phi_2\rangle\langle\phi_1\rangle, \quad (1.2)$$

where  $\eta = 1(-1)$  denotes to the bosonic or fermionic statistics. The above equation illustrates the unnormalized reduced single-particle pure state resulting from the projection of a two-particle state on  $|\psi_k\rangle$  (one particle projective measurement). Likewise, we define the two-particle probability amplitude as a symmetric inner product between the state spaces of a two-particle vector state, expressed as a linear combination of products of one-particle amplitudes, defined as follows

$$\langle\Phi'_1, \Phi'_2|\Phi_1, \Phi_2\rangle := \langle\Phi'_1|\Phi_1\rangle\langle\Phi'_2|\Phi_2\rangle + \eta\langle\Phi'_1|\Phi_2\rangle\langle\Phi'_2|\Phi_1\rangle, \quad (1.3)$$

The equation above arises due to the indistinguishability effect, where the probability amplitude of finding a particle in state  $\Phi_1(\Phi_2)$  results from having a particle in  $\Phi_1(\Phi_2)$ .

We can define the one-particle partial trace  $Tr^{(1)}$  of a system, which is physically interpreted as the statistical ensemble of all normalized reduced states obtained after projective measurement on the basis states. This operation corresponds to measuring a subsystem particle without recording the outcome, as follows:

$$Tr^{(1)}[|\Psi\rangle\langle\Psi|] := \sum_k \langle\psi_k|\phi_1, \phi_2\rangle\langle\phi_1, \phi_2|\psi_k\rangle, \quad (1.4)$$

as a consequence, the one-particle reduced density matrix can be determined as

$$\rho^{(1)} = \frac{Tr^{(1)}[|\Psi\rangle\langle\Psi|]}{\mathcal{M}}, \quad (1.5)$$

where  $\mathcal{M}$  is a normalization constant such that  $Tr^{(1)}\rho^{(1)} = 1$ . According to the above definition, the partial trace demonstrates a physical operation on the system state based on effective projective measurements, unlike the partial trace performed with unphysical labels when the measurement settings are ambiguous.

As a result of fundamental preliminary findings from the no-label approach, we can define the entanglement  $E_M(|\Psi\rangle)$  of the global pure state with respect to measurements performed on localized region  $M$  [6, 10]. This is determined by the von Neumann entropy of the one-particle reduced density matrix given in Eq. 1.5. Thus we have

$$E_M(|\Psi\rangle) := S(\rho_M^{(1)}) = -\sum_j \lambda_j \log_2 \lambda_j, \quad (1.6)$$

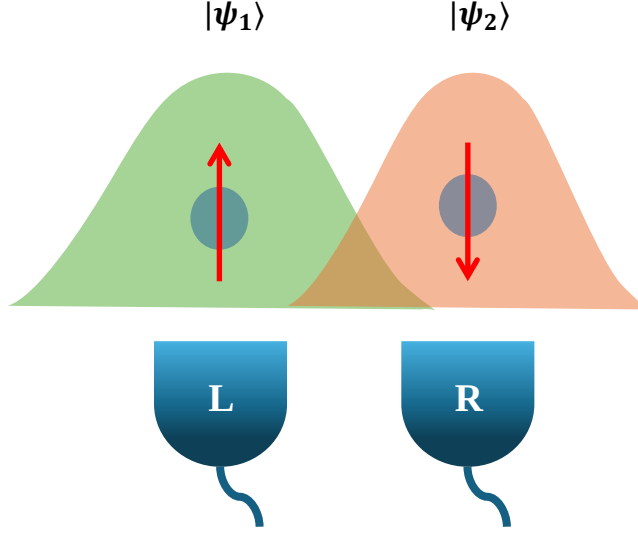


Figure 1.1: Operational entanglement. Two identical particles with opposite pseudospins (internal states) have spatial wave functions  $\psi_1$  and  $\psi_2$  with a given degree of overlap. The entanglement between pseudospins is operationally defined by local measurements in two separated localized spatial regions L and R.

where  $S(\rho_M^{(1)})$  is the Von-Numman entropy of  $\rho_M^{(1)}$  and  $\lambda_j$  are its eigenvalues. Because of spatial indistinguishability, the amount of  $E_M$  is anticipated to depend on  $M$ . Therefore, when the particles are either spatially separated or in the same mode, the localized mode where the measurement is performed does not affect the outcome, and we assume the obtained entanglement is intrinsic, absolute entanglement of the system.

In conclusion, the no-label approach enables the determination of the entanglement of identical particles using the same notions commonly adopted for distinguishable particles, specifically the von Neumann entropy of the reduced state after performing a partial trace and thus overcoming the drawbacks of relying on unphysical labels.

### 1.1.3 Spatially localized operations and classical communication

A well-established operational framework is employed to exploit quantum traits such as entanglement and coherence in composite systems with non-identical particles. This framework, within the context of local operations and classical communication (LOCC) [11], is based on addressing and performing local operations on orthogonal modes. However, since identical particles are individually unaddressable, especially when they spatially overlap, the idea of "particle locality" residing in LOCC becomes meaningless. Thus, a proper operational framework based on spatially localized operations and classical communication (sLOCC) [12] can in turn quantify the quantum features of composite systems of identical particles that spatially overlap.

Lets briefly summarize the sLOCC framework by starting with two independently prepared identical particles within the global state as follows:

$$|\Psi\rangle = |\psi_1 \uparrow, \psi_2 \downarrow\rangle, \quad (1.7)$$

where  $\psi_1$  and  $\psi_2$  are spatial wave functions illustrating external degrees of freedom, and  $\uparrow$  and  $\downarrow$  indicate pseudospin representing internal states. Accordingly, to activate and access the entanglement within this system, local measurements of single particles, which are insensitive to pseudospin states, are conducted in spatial regions L and R, as shown in Fig. 1.1. The projector used for the local particle counting process is defined as

$$\hat{\Pi}^{(2)} = \sum_{\sigma, \tau = \uparrow, \downarrow} |L\sigma, R\tau\rangle \langle L\sigma, R\tau|, \quad (1.8)$$

that projected the state  $|\Psi\rangle$  onto two-particle basis  $\beta_{LR} = \{|L \uparrow, R \uparrow\rangle, |L \uparrow, R \downarrow\rangle, |L \downarrow, R \uparrow\rangle, |L \downarrow, R \downarrow\rangle\}$ . The projected normalized pure state  $|\Psi_{LR}\rangle$  is then

$$|\Psi_{LR}\rangle = \frac{\hat{\Pi}^{(2)} |\Psi\rangle}{\sqrt{\langle \Psi | \hat{\Pi}^{(2)} | \Psi \rangle}} = \frac{lr' |L \uparrow, R \downarrow\rangle + \eta l' r |L \downarrow, R \uparrow\rangle}{\sqrt{|lr'|^2 + |l'r|^2}}, \quad (1.9)$$

$l = \langle L | \psi_1 \rangle$ ,  $r' = \langle R | \psi_1 \rangle$ ,  $l' = \langle L | \psi_2 \rangle$  and  $r = \langle R | \psi_2 \rangle$  are the probability amplitudes to find a particle in the sites L and R. Due to the postselection, the probability of success of sLOCC, represented as  $P_{LR}$ , is obtained as

$$P_{LR} = P_L P'_R + P'_L P_R, \quad (1.10)$$

where  $P_L = |l|^2$ ,  $P_R = |r|^2$ ,  $P'_L = |l'|^2$  and  $P'_R = |r'|^2$  are the probabilities of finding a particle in region L and R, respectively, within the wave functions  $\psi_1$  and  $\psi_2$ . After performing the partial trace on one region, such as L in the basis  $\{|L \downarrow\rangle, |L \uparrow\rangle\}$ , followed by projecting onto region R spanned by basis  $\{|R \downarrow\rangle, |R \uparrow\rangle\}$ , the result is reduced density as

$$\rho_R^{(1)} = \frac{1}{P_L P'_R + P'_L P_R} (P'_L P_R |R \uparrow\rangle \langle R \uparrow| + P'_L P_R |R \downarrow\rangle \langle R \downarrow|). \quad (1.11)$$

Hereafter, the von Neumann entropy  $S(\rho^{(1)}) = E_{LR}(|\Psi\rangle) = -\text{Tr}(\rho_{LR}^{(1)} \log_2 \rho_{LR}^{(1)})$  gives the operational entanglement as

$$\begin{aligned} E_{LR}(|\Psi\rangle) = & -\frac{P_L P'_R}{P_L P'_R + P'_L P_R} \log_2 \frac{P_L P'_R}{P_L P'_R + P'_L P_R} \\ & -\frac{P'_L P_R}{P_L P'_R + P'_L P_R} \log_2 \frac{P'_L P_R}{P_L P'_R + P'_L P_R}, \end{aligned} \quad (1.12)$$

that revealed quantum correlations between the pseudospins of particles as observed through local measurements. The degree of this entanglement depends on the probabilities of finding particles in the localized regions L and R.

Let's discuss the extent of entanglement in terms of the spatial overlap of the wave functions  $\psi_1$  and  $\psi_2$ . The first scenario is when there is no spatial overlap. In which the wave functions  $\psi_1$  and  $\psi_2$  are spatially separated and localized around sites L and R, respectively; thus,  $P'_L = P_R = 0$ . In this scenario, the state  $|\Psi_{LR}\rangle = |L \uparrow, R \downarrow\rangle$  is separable, resulting in zero entanglement  $E(|\Psi_{LR}\rangle) = 0$ . This is due to the spatial separation and local addressing of the particles. A further example is when the wave functions  $\psi_1$  and  $\psi_2$  partially overlap, different situations arise. If a local measurement on either L or R is performed outside the overlap region (e.g.,  $P'_L = 0$  or  $P_R = 0$ ), the entanglement is zero. In the case that local measurements are performed within the overlap region, the entanglement between the pseudospins of particles is conditionally obtained with a probability  $P_{LR}$  given in Eq. 1.10. Finally, both wave functions completely overlap, and this is when both particles can be found with the same nonzero probability in the spatial region. In such a scenario, for example, when  $P_L = P_R = P'_L = P'_R = 1/2$ , the maximum operational entanglement  $E(|\Psi_{LR}\rangle) = 1$  is obtained with a success probability  $P_{LR} = 1/2$ . This analysis reveals that the quantification of the operational entanglement of Eq. 1.12, is not only an intrinsic property but also depends on both the structure of the state and the method of measurement.

The sLOCC framework within the no-label approach allows for the identification, manipulation, and exploitation of the indistinguishability of identical particles. This makes the indistinguishability of identical elementary systems (particles) an inherent quantum feature that plays an important role in quantum information processing, including entanglement generation and phase estimation. In the following sections, I will briefly discuss these processes.

## 1.2 Entanglement generation

The indistinguishability of identical particles, whether achieved through spatial overlap or photonic path identity, is a true quantum resource that can be harnessed to generate both bipartite and multipartite entangled states. Despite the photonic path identity that relies on direct interaction between indistinguishable particles, the indistinguishability entangling gate depends solely on spatially overlapping two independent particles with opposite pseudospins that meet only at the detection level [12]. Here, we briefly discuss the indistinguishability of the entangling gate for generating bipartite entangled states, which has been demonstrated both experimentally [13, 14] and theoretically [12]. Let's consider two identical particles, originating from separate sources, that are in the initial uncorrelated form of  $|\psi_1 \uparrow, \psi_2 \downarrow\rangle$ , that is given in the no-label approach. Each particle's wave function is coherently and controllably distributed into two remote operational regions, L and R, using a beam splitter,  $|\psi_1\rangle \rightarrow |\psi_{1D}\rangle$ ,  $|\psi_2\rangle \rightarrow |\psi_{2D}\rangle$ , so the global state takes the form of  $|\Psi\rangle = |\psi_{1D} \uparrow, \psi_{2D} \downarrow\rangle$ , where  $|\psi_{1D}\rangle = l|\psi_L\rangle + r|\psi_R\rangle$  ( $|l|^2 + |r|^2 = 1$ ) and  $|\psi_{2D}\rangle = l'|\psi'_L\rangle + r'|\psi'_R\rangle$  ( $|l'|^2 + |r'|^2 = 1$ ), with  $|\psi_x\rangle$  and  $|\psi'_x\rangle$  denoting the two wavefunctions located in the spatial region  $x$ . The sLOCC measurement demonstrated in Fig 1.2 can be performed by single photon counting. As the particle paths do not share any past information and only meet at the detection levels, the detectors are insensitive to the integral degrees of freedom. Consequently, particles seem indistinguishable to the eyes of detectors. Therefore, the projector given in Eq. 2.1  $\hat{\Pi}^{(2)} = \sum_{\sigma, \tau=\uparrow, \downarrow} |L\sigma, R\tau\rangle \langle L\sigma, R\tau|$  to activate the desired state is applied on the  $|\Psi\rangle$  and leads the quantum state given  $|\Psi_{LR}\rangle = \frac{l r' |L \uparrow, R \downarrow\rangle + \eta l' r |L \downarrow, R \uparrow\rangle}{\sqrt{|l r'|^2 + |l' r|^2}}$  in Eq. 1.9 with the probability of  $P_{LR} = |l r'|^2 + |r l'|^2$ , where  $\eta$  denotes to the fermionic or bosonic statistics.

Given the importance of multipartite entangled states such as W, Dicke, GHZ, and cluster states in various areas of quantum technology, including quantum computing [15, 16], quantum communication [17, 18, 19], quantum metrology [20], and quantum teleportation [21], the key concern is whether we can generate these entangled states for both bosonic and fermionic systems using the spatial indistinguishability of identical particles. According to the previous finding described in [12], the sLOCC formalism can be extended to general scenarios involving N particles distributed across N regions. This extension facilitates the investigation of multipartite entanglement among multiple identical particles. In chapter 2, I will discuss this in detail, as it represents the main result of my PhD thesis: the general controllable scheme for generating multipartite entangled states. I also discuss how to represent this mathematical scheme in a more insightful and visual form. Using graph-based representation, we translate and map our scheme into colored, complex, and weighted directed graphs (digraphs), each corresponding to the experimental setup [22].

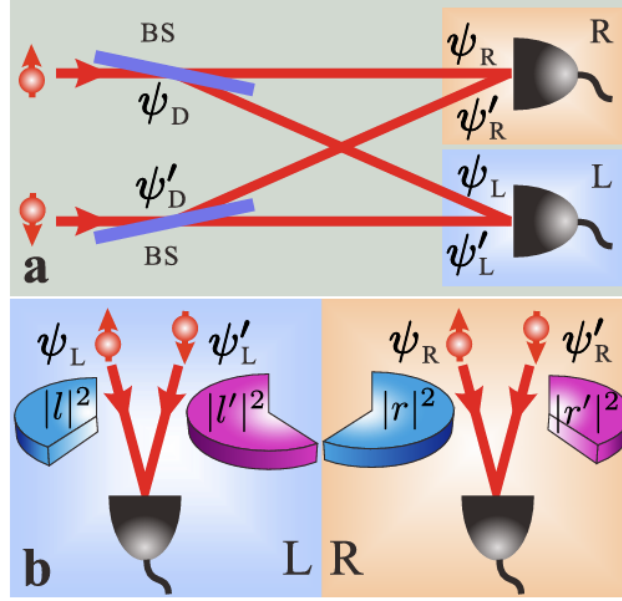


Figure 1.2: Theoretical schematic. a. The simplest setup is designed to produce entanglement via sLOCC due to spatial indistinguishability. b. The general spatial distribution of particles in the state  $|\Psi\rangle = |\psi_{1D} \uparrow, \psi_{2D} \downarrow\rangle$  before sLOCC, with  $|\psi_{1D}\rangle = l|\psi_L\rangle + r|\psi_R\rangle$ , and  $|\psi_{2D}\rangle = l'|\psi'_L\rangle + r'|\psi'_R\rangle$ . [13]

From an operational perspective, particles are also considered indistinguishable if they share the same mode in the electromagnetic fields such as polarization, time, frequency, position, and momentum when characterized by two-particle interference [23]. Accordingly, one can note that another important class of multipartite entangled states is the N00N state [24, 25]. In this example,  $N$  indistinguishable particles are in a superposition between two paths or modes, either all in one path or all in another mode. As N00N states enhance and improve spatial and phase resolution with factor  $N$  scaling  $\frac{1}{N}$ , they are at the heart of quantum sensing [26] and quantum metrology [27, 28], particularly quantum phase estimation [29]. In the next section, I will briefly discuss phase estimation.

### 1.3 Phase estimation theory

The precise measurement of physical quantities, such as the strength of a field, a force, or time, is one of the basic pillars of both fundamental physics and applications, including metrology, satellite, and geo-positioning.

Precise measurements are often achieved by mapping the physical quantities into phase shifts that are estimated by means of interferometric techniques [29, 30]. In the phase estimation scenario, the unknown parameter is a phase shift  $\phi$  between two different paths or modes (more in general, any degree of freedom) of photons. The ultimate achievable bound to estimate  $\phi$  is given by the Cramer-Rao bound [31],

$$\langle \Delta\phi \rangle^2 \geq \frac{1}{\mu F_c(\phi)}, \quad (1.13)$$

where  $\mu$  is an asymmetrically large number of trials. Indeed, the variance is an intrinsic error in estimation processes. Here,  $F_c(\phi)$  is the (classical) Fisher information, defined as

$$F_c(\phi) = \sum_i \frac{[\partial_\phi p_i(\phi)]^2}{p_i(\phi)}, \quad (1.14)$$

where the Born rule  $p_i(\phi) = \text{Tr}[\rho_\phi E_i]$  represents the probability distribution associated with a set of positive operator-valued measures (POVMs)  $\{E_i\}$  that satisfy  $\sum_i E_i = \mathbf{I}$ . In this Fisher information, the error on estimating a parameter  $\phi$  depends on both the initial state and the set of measurements. Therefore, one can optimize  $F_c(\phi)$  over all possible POVMs to maximize the classical Fisher information and evaluate the ultimate bounds with precision. This ultimate bound is known as the Quantum Cramér-Rao bound (QCRB)

$$\langle \Delta\phi \rangle^2 \geq \frac{1}{\mu F_c(\phi)} \geq \frac{1}{\mu F_Q(\phi)}, \quad (1.15)$$

where Quantum Fisher information (QFI) is given by  $F_Q(\phi) = \text{Tr}[\hat{L}_\phi^2 \hat{\rho}_\phi]$  [24], introducing an upper bound on classical Fisher information (precision) for any quantum measurement. Here,  $\hat{L}_\phi$  is the Symmetric Logarithmic Derivative (SLD)

self-adjoint operator satisfying the equation  $\frac{\partial \hat{\rho}_\phi}{\partial \phi} = \frac{1}{2}(L_\phi \hat{\rho}_\phi + \hat{\rho}_\phi \hat{L}_\phi)$ . One can rewrite the state  $\hat{\rho}_\phi$  in its eigenbasis as  $\rho_\phi = \sum_i \lambda_i |e_i\rangle \langle e_i|$ , and then obtain the quantum Fisher information in the following form

$$F_Q(\phi) = 2 \sum_{i \neq j} \frac{1}{\lambda_i + \lambda_j} \left| \left\langle e_i \left| \frac{\partial \hat{\rho}_\phi}{\partial \phi} \right| e_j \right\rangle \right|^2, \quad (1.16)$$

where, in the case of a pure quantum state, the expression above simplifies to [20]

$$F_Q(\phi) = 4 \left[ \langle \partial_\phi \psi_\phi | \partial_\phi \psi_\phi \rangle - |\langle \partial_\phi \psi_\phi | \psi_\phi \rangle|^2 \right]. \quad (1.17)$$

Until here, I talk about how indistinguishability plays an important role in generation entanglement. It is crucial to understand how quantum features based on indistinguishability behave in a dynamic framework, especially when considering bosons propagating through a non-homogeneous system. A proper theoretical platform for performing such a study is demonstrated by the quantum walk (QW), which offers a very general coherent propagation model. Hence, in the next section, I will discuss the quantum walk model.

## 1.4 Quantum walk for quantum information processing

Quantum walks are advanced tools for building quantum algorithms [32], quantum computation [33], and providing insight into complex phenomena in nature, such as the photosynthesis process [34, 35], and explaining the low loss during transport in light-harvesting systems [36]. In the following, I provide a brief introduction to the background of quantum walks.

**Classical random walk:** A random walk is the random movement of a walker on a line, where, by flipping a coin, the walker decides whether to move to the right or left. The notions of  $+$  and  $-$  are used to represent the result of heads or tails and accordingly move to the right or left. Likewise  $P_+$  and  $P_-$  are used to represent the probabilities of each outcomes, and for an ordinary unbiased coin, one can expect that  $P_+ = P_- = \frac{1}{2}$ . Before flipping the coin, there is a chance  $P_+$  that the walker would take a path right or, with  $P_-$  chance, would take the path left. However, after the coin flip, there is no ambiguity, and the walker makes a move to the right or left with absolute certainty. If the walker initially is at position  $x = 0$ , moving right and left corresponds to stepping to position  $x = +1$  and  $x = -1$ . The walker can continue using coin flips to move along a decision tree. This procedure is known as a classical random walk.

**Quantum walks (QWs)** the quantum-mechanical analogue of classical random walks, enable a variety of applications, including universal quantum computing models [37], universal quantum measurement devices [38], and quantum simulators [39].

### 1.4.1 Discrete time quantum walk (DTQW)

In general, the DTQW model is defined on the joint Hilbert space  $H = H_p \otimes H_c$  of the position ( $H_p$ ) and coin ( $H_c$ ) spaces of the walker. The state of the system is thus defined by  $|\psi\rangle_0 = \sum_k c_k |k\rangle$ , where  $c_k$  is the probability amplitude to find a photon in mode  $k$ . Each mode consists of two degrees of freedom  $|x, c\rangle$ , where the position of particles ( $|x\rangle = \dots, |-1\rangle, |0\rangle, |1\rangle, \dots$ ) and the coin state  $|c\rangle = |R\rangle, |L\rangle$ , where  $|R\rangle$  and  $|L\rangle$  stand for the moving particles to the right and left, respectively. The one-step unitary evolution operator is  $\hat{U}(t) = \hat{S} \cdot \hat{C}(x, t)$ . This operator includes the conditional shift operator [40]

$$\hat{S} = \sum_k |x+1, R\rangle \langle x, R| + |x-1, L\rangle \langle x, L|, \quad (1.18)$$

and the coin operator for the dynamical coupler in each synthetic position and time-step [40]

$$\hat{C}(x, t) = \sum_x \begin{pmatrix} \cos[\theta(x, t)] & -i \sin[\theta(x, t)] \\ -i \sin[\theta(x, t)] & \cos[\theta(x, t)] \end{pmatrix}, \quad (1.19)$$

where  $\theta(x, t)$  determines the transmission and reflection coefficients. Finally, the repetitive action of the unitary operator on the initial state at step zero  $t = 0$  will move the particle to the state at time  $t$ , given as

$$|\psi(t)\rangle = \hat{U}(t) \hat{U}(t-1) \dots \hat{U} |\psi(0)\rangle. \quad (1.20)$$

### 1.4.2 The role of disorder

Disorder in DTQWS refers to the introduction of randomness or irregularity in the walkers evolution. In a standard quantum walk, the evolution follows a fixed set of unitary operators typically the coin and shift operators. In an ordered quantum walk, the spreading behavior is ballistic, meaning that the variance of the walkers position grows quadratically with time,  $\sigma(x)^2 \propto t^2$ , in contrast to classical random walks, where the variance scales linearly. This quadratic scaling

indicates that the quantum walker spreads significantly faster in position space. However, when disorder is introduced, the evolution deviates from these standard rules, leading to various effects, including localization, decoherence, and modifications in transport behavior.

**Localization** One of the most notable consequences of disorder is Anderson localization, where the quantum walker becomes confined to a specific region of the lattice rather than spreading freely. This occurs due to static (time-independent) disorder, which disrupts interference patterns, leading to a probability distribution that remains localized around certain positions. This effect has been experimentally studied in photonic quantum walk platforms [41, 42]. In Chapter 4, I further explore how random static disorder influences disorder-induced localization in quantum walks.

**Decoherence:** Disorder can be introduced through random fluctuations, often arising from the walkers interaction with its environment. This disrupts constructive interference, leading to a gradual loss of coherence. In some cases, however, decoherence plays a useful role, such as in the development of quantum algorithms [43]. For example, decoherence via measurement or environmental interactions serves as a natural framework to study the transition from quantum walks to classical random walks [44, 45]. Notably, disorder can also be coherently imposed, a concept explored in Chapter 4.

**Anomalous and Normal Diffusion** In an ordered quantum walk, the mean square displacement of the walker follows a ballistic growth pattern. However, in the presence of disorder, the spreading behavior can change drastically. Anomalous diffusion refers to deviations from the standard linear scaling observed in classical transport. Quantum walks can exhibit superdiffusion, where the walker spreads faster than in a classical random walk, or subdiffusion, where disorder hinders efficient exploration of the lattice [44, 45, 46, 47].

**Non-Classical Enhancement:** Although disorder is often considered a source of noise, it can also be exploited for specific quantum applications. In quantum information processing, disorder can enhance entanglement between the walker and the coin states or introduce correlations between different parts of the system [48]. In Chapter IV of my Ph.D. thesis, we demonstrate how controlled disorder can optimize quantum correlations, particularly in the case of two indistinguishable photons propagating in separate modes of a quantum walk network. This highlights the potential of disorder as a resource for quantum technologies rather than merely an obstacle.

### 1.4.3 Discrete time quantum walk as a generalized device measurement

In the DTQW model, the photons evolution on the lattice depends on the quantum state of the coin (i.e., on the quantum mode). Controlling the coins quantum mode enables the control over the walkers (the photons) and, in turn, over the system's evolution, and as a consequence, allows the design of optimal positive-operator-valued measures (POVMs). Further, it has been shown that discrete QWs can be used to generalize POVMs. In this scenario, the measurement of a quantum walker at a certain position corresponds to an element of the POVM measurement on a certain coin state. Alternatively, a general POVM can be constructed by engineering coin operators  $\hat{C}(x, t)$  and then measuring the walkers positions after certain steps. Mathematically, the POVM elements  $E_i$  are given by  $E_i = \text{Tr}_{anc}[(1 \otimes \sigma)\pi_i]$ , where 1 is the identity operator,  $\sigma$  is the state of ancilla (additional system) and it corresponds to position of walker,  $\pi_i$  the von Neumann projector on the position-coin joint Hilbert space. In the QW theory, a single step of a quantum walker starting from a defined position can be considered as a projective von Neumann measurement of the coin state. Specifically, the projector  $\pi_i$  at position  $x = i$ , i.e.,  $\sigma_i = |x = i\rangle \langle x = i|$  is given by  $\pi_i = \hat{U}(t)^\dagger [|x = i\rangle \langle x = i| \otimes 1] \hat{U}(t)$  [49]. Finally, the probability of measuring the  $i$ -th POVM element of an initial state  $\rho$  is  $p_i = \text{Tr}(E_i \rho)$  and corresponds to the projective measurement  $M_i = |x, c\rangle \langle x, c|$  on the final state of the quantum walker after  $t$ -th step evolution  $\rho_t = \hat{U}(t)^\dagger \rho \hat{U}(t)$ , i.e.,  $p_i = \text{Tr}(M_i \rho_t)$ , where the  $t$ -th step is given by the roundtrip of the walker.

### 1.4.4 Implementation of discrete-time quantum walk in quantum optics

A broad range of physical quantum systems have been proposed for implementing quantum walks, including trapped atom [50], trapped ions [51, 52], superconducting qubits [53, 54, 55], nuclear magnetic resonance [56, 57], laser beam and bulk optics [58, 59, 60, 61, 62] and integrated photonic systems [63, 64, 65, 66, 67, 68]. Hereafter, as in my thesis, the focus will be on the implementation of DTQWs in optical systems, and I will discuss some examples of them.

#### *DTQW in optic systems*

In optical systems, various methods of encoding degrees of freedom, such as spatial such as spatial modes (or paths), polarization, time-bin, and orbital angular momentum (OAM), and frequency, different photonic platforms have been used for implementing DTQWs. The photon platforms consist of bulk optic systems, fiber systems, and integrated photonic systems, representing promising progress in the photonic implementation of DTQWs. In the bulk optic implementation, several different methods have been used, in which laser pulses are simulated as walkers. In one example, photon polarization and spatial mode are used as coin and position state accordingly, with wave plates and beam displacers (BDs) used for coin operation and the conditional shift operator, respectively [69, 62]. In another example, coin and position are encoded in the polarization and time-bin of photons, respectively, for implementing one- and two-dimensional DTQWs with controllable coin operations [58, 59].

Another possibility is using other degrees of freedom (DOFs) of photons, where spin angular momentum (SAM) and orbital angular momentum are used to encode the coin and position states to implement a one-dimensional discrete-time

quantum walk (DTQW)[70]. Alternatively, OAM modes as lattice sites and polarization as the coin state are used to implement a DTQW with a laser beam in a ring interferometer [71].

However, bulk optical systems have limitations and challenges in terms of scalability and stability, especially over a large number of steps. Fiber systems are promising candidates for simulating quantum walks, as they can easily use fiber loops to implement time-bin encoded quantum states. In the implementation of DTQW on the line in a fiber loop setup with adjustable coin operations, the coin and position states are encoded in the polarization of the photon and time domain [72]. Another example of using fibers involves a coupled fiber loop system as a synthetic lattice consisting of coupled fiber loops and a dynamical gate, where the synthetic dimension is given by the time modes [73]. Such setups have been used for tasks such as bi-photon state preparation, manipulation, and the realization of time-bin two- and four-level interference between two photons [74]. In Chapter 3, I will discuss how we use such a platform to simulate DTQW for quantum-enhanced phase estimation. Furthermore, another alternative approach involves using a bulk-optics multipass double Sagnac interferometer [75]. In Chapter 4 of my thesis, I will discuss the role of coherent disorder in photonic lattices through controllable inhomogeneous quantum walk dynamics.



# Generation of genuine multipartite entangled states via indistinguishability of identical particles

## Abstract

Indistinguishability of identical particles is a resource for quantum information processing and has been utilized to generate entanglement from independent particles that spatially overlap only at the detection stage. Here, we introduce a controllable scheme capable of generating via three steps, initialization, deformation, and post-selection, different classes of multipartite entangled states starting from a product state of  $N$  spatially distinguishable identical qubits. While our scheme is generalizable to any class of entangled bosonic and fermionic systems, we provide an explicit recipe for the generation of W, Dicke, GHZ, and cluster states, which are resource states for quantum information processing. Using graph-based representations within the framework of spatially localized operations and classical communication (sLOCC), we mathematically demonstrate a direct translation of the generation schemes of specific entangled states into colored, complex, and weighted digraphs, each corresponding to a given experimental setup. We also show that this graph-theoretical approach allows for the optimization of the generation efficiency of specific multipartite entangled states by exploring a variety of generation schemes. The presented theoretical approach, while already implementable with current linear optics architectures, has the potential to bring clear advantages over existing technologies, such as in quantum computing search algorithms and in the design of new experiments in quantum optics or other platforms.

This chapter is related to Publication 1 in the List of Publications.

## 2.1 Introduction

The outperformance of quantum technologies compared with classical ones resides in exploiting quantum effects, such as entanglement, for various applications, including computing [157, 158, 159], secure communication [160, 161], and sensing [29]. In addition, the indistinguishability associated with systems of identical quantum entities (e.g., photons, electrons, and atoms of the same species) is a unique notion in quantum mechanics. Particles are named indistinguishable when their wavefunctions become spatially overlapped and their global state does not allow to address them individually [162]. Consequently, these particles behave differently and combine together to form compound structures of matter and light, in which the states of identical bosons (fermions) are symmetric (antisymmetric) under the exchange of any pair of particles [162]. This intrinsic property is responsible for distinct phenomena such as electron orbital occupation [163] and photon bunching [164].

Beyond the fundamental aspects, indistinguishability can be a key resource for quantum-enhanced applications. For instance, spatially localized operations and classical communication (sLOCC) [12] have been proposed to exploit indistinguishability for tasks such as quantum teleportation [12, 13], quantum metrology [149, 150], and communication in multimode photonic quantum networks [151, 152, 165]. Also, indistinguishability offers a shield to protect quantumness against various noise sources [166, 167, 168, 169, 170, 171, 172, 173, 174]. An elementary entangling gate can be achieved by spatially overlapping two independent particles with opposite pseudospins that meet only at the detectors, as proven both theoretically [12] and experimentally [13, 14]. However, these generation schemes have been so far limited to bipartite [12, 13, 14] and tripartite cases [148].

Unlike bipartite systems, multipartite entangled systems exhibit much more complex correlation structures [175, 11]. Different inequivalent entanglement classes emerge in the multipartite setting, and they never convert to each other under

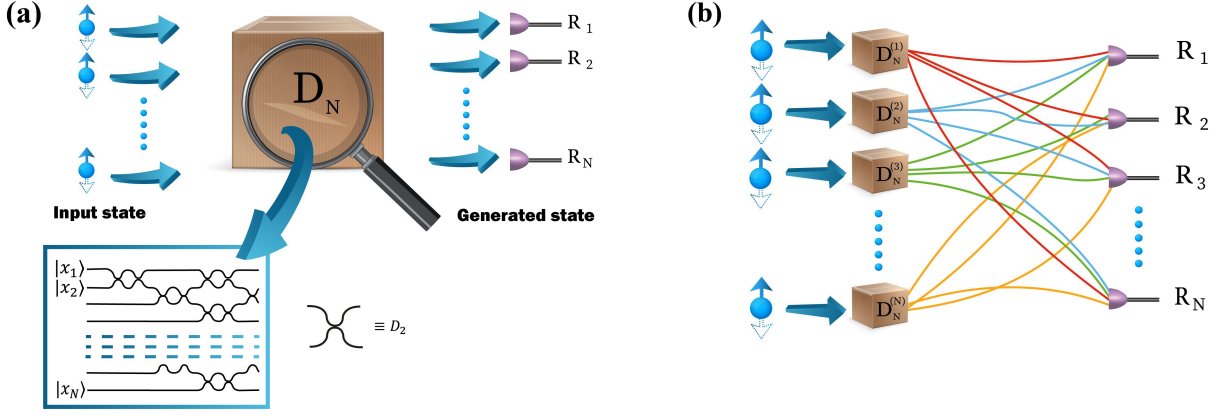


Figure 2.1: **Illustration of the deformation in both on-site and remote schemes.** (a) Illustration of an on-site designed  $N$ -mode deformation, denoted as  $\hat{D}_N$ , where  $N$  identical qubits, each possessing an internal degree of freedom  $\sigma_i = \{\uparrow, \downarrow\}$ , pass through  $N$  modes of this deformation. At the end, qubits arrive in  $N$  detection regions  $R_1, \dots, R_N$ , for sLOCC projection measurement to generate the output state. The inset illustrates the potential for implementing any  $N$ -mode deformation by combining  $D_2$  operators. (b) Illustration of remote deformation design using  $N$  deformations  $D_N^{(i)}$  ( $i = 1, \dots, N$ ) that send  $N$ -qubits into  $N$  individual regions for postselection, generating the output state.

stochastic local operations and classical communication [176]. Each multipartite entangled class has unique features and finds applications to specific quantum information tasks. For example, GreenbergerHorneZeilinger (GHZ) states [16] are used for ballistic quantum computing [15, 16] and quantum metrology [20], while cluster states are building blocks for various universal measurement-based quantum computers [177, 178, 179].  $W$  and Dicke states stand out because their entanglement is robust to particle loss [180, 181] and have applications in quantum communication, such as secret sharing [17, 18, 19] and quantum teleportation [21]. Several platforms have been proposed to generate multipartite entangled states, such as superconducting qubits [48, 182, 183, 184], trapped ions [185, 186, 187, 188, 189, 190, 191, 192], nuclear spins [193], and photonic qubits with both probabilistic [194, 184] and deterministic sources [195, 196, 197, 198, 199, 200].

Multipartite entangled states can be generated using both deterministic and probabilistic schemes. Deterministic schemes involve either global  $N$ -qubit [201] or  $\frac{N(N-1)}{2}$  pairwise [202] entangling gates. These schemes require precise interactions between qubits, making them unsuitable for photons, due to their weak interaction. In contrast, post-selective schemes, which rely on coherent transformations and post-selection measurements [203, 204], are more suitable for photonic and, more in general, remote (i.e., non-interactive) systems. Various probabilistic methods, each with different success probabilities, have been developed to generate multipartite entangled states. For example, multiple Bell pairs can be entangled to generate different classes of multipartite entangled states via exploiting indistinguishability of identical qubits, including GHZ states [203, 205]. Additionally, it has been shown that the indistinguishability of identical particles, whether through spatial overlap [204, 155, 148] or photonic path identity [206, 207] formalisms, is a genuine quantum resource for generating a wide class of multipartite entangled states. Unlike photonic path identity, which relies on direct interaction (or interference) of photons at a single beam splitter and nonlinear medium [207], the spatial indistinguishability approach involves independent bosonic or fermionic qubits meeting (spatially overlapping) only at the detection regions [12, 13, 76].

In this work, we go beyond the case-specific designs and provide a comprehensive theoretical method that exploits the indistinguishability of identical qubits to generate different classes of multipartite entangled states, including  $W$ , Dicke, GHZ, and cluster states. It has been shown that a graph-based representation of quantum optical experiments [208, 209, 154, 155, 210, 211, 212] can be introduced and utilized algorithmically to facilitate the generation scheme of multipartite entangled states. A graph-theoretic approach has been used to represent the process of post-selective generation of multipartite entangled states, exploiting photons indistinguishability [206, 155]. Leveraging this connection, we establish a direct mathematical correspondence between the general generation recipe in the no-label approach and a digraph (bigraph) adjacency matrix in the graph formalism for both bosonic and fermionic particles statistics. This work enables the formulation of the sLOCC framework [12] in the graph-based approach, where spatial overlaps appear as adjacency matrix elements of the digraph (bigraph), to exploit the resource of indistinguishability for the generation of multipartite entanglement. This approach opens a clearer perspective on automated machine learning assistance designs [213] in generation schemes for arbitrary multipartite entangled states in various experimental platforms, such as photonic and superconducting setups, towards the optimized use of the quantum resources, including particle statistics imprints.

## 2.2 Indistinguishability-based entangling gate

The indistinguishability of identical qubits provides a fundamental quantum resource that can be utilized within the sLOCC framework to generate entanglement in a conditional fashion. In general, such a probabilistic generation scheme comprises three steps: (i) the state initialization of  $N$  independent identical qubits; (ii) a general form of deformation that controllably distributes  $N$  qubits into  $N$  separated operational regions; (iii) a postselection process that detects one particle per region to activate the desired quantum state.

**Step I: Initialization.** The protocol starts with a  $N$ -particle pure product state  $|\varphi_1\rangle \otimes \cdots \otimes |\varphi_N\rangle = |\varphi_1, \dots, \varphi_N\rangle$ , characterized by a complete set of commuting observables of the single-qubit state  $|\varphi_j\rangle$ . Also, each single-qubit state  $|\varphi_i\rangle = |\psi_i, \sigma_i\rangle$  comprises the particle's degrees of freedom, that is, a spatial wave function  $|\psi_i\rangle$  and a pseudospin  $|\sigma_i\rangle = \{|\uparrow\rangle, |\downarrow\rangle\}$ . This means that the protocol needs  $N$  single identical qubits on demand.

**Step II: Deformation.** Deformations play a crucial role in generating entanglement through coherent manipulation of spatially separated particles and making them spatially indistinguishable [13, 12]. These deformations can be applied to both pure and mixed quantum states. Here, we are interested in the case of composite systems in pure states. One can apply a deformation on  $N$  identical qubits in both on-site and remote designs, as illustrated in Fig. 2.1 (a) and (b).

In the on-site design, as illustrated in Fig. 2.1 (a), the  $N$ -particle state evolves using a single unitary deformation operator  $\hat{D}_N$ . We assume that each particle is sent through one of the input modes, denoted as  $|x_j\rangle = |\varphi_j\rangle$ , as illustrated in Fig. 2.1 (a). Adopting the no-label formalism for identical particles [4, 214, 215], the action of the on-site deformation on the  $N$ -particle state can be written as  $\hat{D}_N |\varphi_1, \dots, \varphi_N\rangle = |\varphi_{1D}, \dots, \varphi_{ND}\rangle = |\Psi_D^{(N)}\rangle$ , where  $|\varphi_{iD}\rangle$  and  $|\Psi_D^{(N)}\rangle$  represent the single-qubit and  $N$ -qubit deformed states, respectively. The single-qubit deformed states  $|\varphi_{iD}\rangle$  can be given as  $|\varphi_{iD}\rangle = \sum_j r_{ij} |R_i \sigma_j\rangle$ , where  $r_{ij}$  are meant as the probability amplitudes that the qubit originally coming from the state  $|\varphi_i\rangle$  ends up in the spatial mode  $|R_j\rangle$ . Notice that, since particles spatially overlap at the detection regions, the global deformed state is not generally factorizable in terms of single-qubit states  $|\Psi_D^{(N)}\rangle \neq |\varphi_{1D}\rangle \otimes \cdots \otimes |\varphi_{ND}\rangle$  meaning that the state vector is a whole entity [215]. In passive deformation, which we name spatial deformation,  $N$  identical qubits are spatially distributed in  $N$  regions, and their pseudospin remains untouched. Differently, in active deformation, qubits are not only spatially distributed in  $N$  distinct regions, but their pseudospins are also controllably altered. As sketched in Fig. 2.1 (a), any  $N$ -mode unitary transformation  $\hat{D}_N$  can be constructed using a sequence of 2-mode unitary operators  $\hat{D}_2$  [216]. For example, one can observe the effect of a two-mode spatial deformation in an optical setup using a beam splitter.

In the remote design in Fig. 2.1 (b), there are  $N$  deformations  $\hat{D}_N^{(i)}$  ( $i = 1 \dots N$ ) that are independently applied to each qubit originally in the state  $|\varphi_i\rangle$ . Practically, each deformation acts on an  $N$ -mode input state consisting of  $N - 1$  vacuum mode  $|vac\rangle$  and a single-qubit state  $|\varphi_i\rangle$ , that is  $\hat{D}_N^{(i)} |X_N^i\rangle = \hat{D}_N^{(i)} |\varphi_i, vac, \dots, vac\rangle := \hat{D}_N^{(i)} |\varphi_i\rangle = |\varphi_{iD}\rangle$ , where  $|\varphi_{iD}\rangle = \sum_{j=1}^M r_{ij} |R_j^{(i)} \sigma_j\rangle$  is the single-qubit deformed state with different spatial modes  $R_j^{(i)}$ . The overall unitary  $N$ -mode deformations are now written as  $\hat{D}_N^{(N)} = \hat{D}_N^{(1)} \otimes \cdots \otimes \hat{D}_N^{(N)}$ . Its action on the initial pure product state of  $N$  identical qubits is  $\hat{D}_N^{(N)} |\varphi_1, \varphi_2, \dots, \varphi_N\rangle := |\hat{D}_N^{(1)} \varphi_1, \dots, \hat{D}_N^{(N)} \varphi_N\rangle = |\varphi_{1D}, \dots, \varphi_{ND}\rangle = |\Psi_D^{(N)}\rangle$ . Note that each deformation  $\hat{D}_N^{(i)}$  is a unitary operation, as any two single-qubit deformed states (for example, states  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$ ) are orthogonal. However, since the measurement is performed in the same spatial region, it cannot resolve different spatial modes when qubits reach one detector in the same region, as illustrated in Fig. 2.1 (b). This is why, for simplicity, we will write  $R_j$  instead of  $R_j^{(i)}$  in the deformed states to clarify that identical qubits will be detected in the same spatial regions. As a result of the detection process in the remote design, the  $N$ -qubit deformed state becomes unnormalized, and we need to renormalize the state as  $|\Psi_D^{(N)}\rangle \rightarrow \frac{1}{\sqrt{\nu}} |\Psi_D^{(N)}\rangle$ , where the normalization factor, given as  $\nu = |\langle \varphi_{1D}, \dots, \varphi_{ND} | \varphi_{1D}, \dots, \varphi_{ND} \rangle|^2$ , takes into account the spatial overlap between single-qubit states [215].

As a result of this deformation, in both on-site and remote designs, identical qubits are distributed over  $N$  separated measurement regions in a global deformed state  $|\Psi_D^{(N)}\rangle$ .

**Step III: Activation.** As illustrated in Fig. 2.1, after the qubits have been distributed by the deformation, the sLOCC measurement process is applied to activate the desired  $N$ -qubit state. The measurement can be realized via single-qubit counting, which is insensitive to the internal degrees of freedom, i.e., pseudospin.

Thus, the desired  $N$ -qubit state is activated through the sLOCC projection [167], which simultaneously performs single-qubit counting in the  $N$  remote spatial regions. This projection is given by the operator

$$\hat{\Pi}^{(N)} = \sum_k |R_1 \sigma_1^{(k)}, \dots, R_N \sigma_N^{(k)}\rangle \langle R_1 \sigma_1^{(k)}, \dots, R_N \sigma_N^{(k)}|, \quad (2.1)$$

where the index  $k := \{\sigma_1^{(k)}, \dots, \sigma_N^{(k)}; \sigma_j^{(k)} = \uparrow, \downarrow\}$  runs over all the  $2^N$  pseudospin combinations. Necessarily, the measurement device cannot access the other degrees of freedom (e.g., pseudospin), so the particles are indistinguishable to the eyes of the sLOCC projection. The action of projector  $\hat{\Pi}^{(N)}$  on the  $N$ -qubit deformed state  $|\Psi_D^{(N)}\rangle$  gives the final state

$$|\Psi^{(N)}\rangle = \frac{1}{\sqrt{N_g}} \sum_k S_k |R_1 \sigma_1^{(k)}, \dots, R_N \sigma_N^{(k)}\rangle, \quad (2.2)$$

where  $N_g$  is a normalization constant. The probability amplitudes  $S_k = \sum_P \eta^P \langle \varphi_{1D} | R_{P_1} \sigma_{P_1}^{(k)} \rangle \dots \langle \varphi_{ND} | R_{P_N} \sigma_{P_N}^{(k)} \rangle$  are the inner products between  $N$ -qubit states, defined in Ref. [167], where  $P = \{P_1, \dots, P_N\}$  runs over all the single-qubit state permutations. The statistics factor  $\eta = \pm 1$  for bosons and fermions, respectively: so,  $\eta^P$  is 1 in the case of bosons and  $1 (-1)$  for even (odd) permutations in the case of fermions.

Due to the postselection, the target state  $|\Psi^{(N)}\rangle$  is conditionally activated with a success (sLOCC) probability  $\mathcal{P}(|\Psi^{(N)}\rangle) = \text{Tr}(\hat{\Pi}^{(N)} |\Psi_D^{(N)}\rangle \langle \Psi_D^{(N)}|)$ . We highlight that the sLOCC probability, as a witness, can detect the existence of a spatial overlap between the qubits in the state when the condition of the sLOCC probability,  $\mathcal{P} \neq 1$ , is met, i.e., the qubits are spatially overlapping in  $N$  spatial regions. Depending on how the deformation distributes the qubits, the produced target state  $|\Psi^{(N)}\rangle$  can assume different entanglement structures, as we shall see in the following.

## 2.3 Graph picture of the multipartite entangling gate

It has been shown that graph theory representations offer a powerful interface for developing computer-designed quantum optics experiments, which can be applied to specific quantum information processing tasks [217, 211, 212]. We now provide a graph-theoretical depiction by directly translating the three steps of the mentioned indistinguishability-based entangling gate (initialization, deformation, and activation) into a weighted bipartite graph. A (bipartite) bigraph  $G_b$  consists of vertices from two disjoint sets,  $V$  and  $U$ , where each edge  $e_{ij} \in E$  connects a node  $u_i \in U$  and a node  $v_j \in V$  with a complex weight  $r_{ij}$  [218]. In our context, each disjoint set of (quantum) nodes consists of  $N$  single-qubit state vectors, given as  $V = \{|\varphi_1\rangle, \dots, |\varphi_N\rangle\}$  and  $U = \{|\chi_1\rangle, \dots, |\chi_N\rangle\}$ . Those single-qubit states belonging to set  $V$  ( $|\varphi_i\rangle \in V$ ) consist of a wave function and an initial pseudospin, that is,  $|\varphi_i\rangle = |\psi_i \tau_i\rangle$ . Instead, the single-qubit states of the set  $U$  ( $|\chi_j\rangle \in U$ ) are related to a detection region  $R_j$  where the qubit with original state  $|\varphi_i\rangle$  can be found with any pseudospin state, that is  $|\chi_j\rangle = |R_j \sigma_j\rangle$  ( $\sigma_j = \uparrow, \downarrow$ ).

To establish the connectivity of the bigraph, a deformation is performed, as detailed in the previous section. This deformation, acting on a node  $|\varphi_i\rangle$ , establishes connections from the node  $|\varphi_i\rangle$  in set  $V$  to all possible nodes in set  $U$  through a designated probability amplitude  $r_{ij}$ , as appears in the deformed single-qubit state  $|\varphi_{iD}\rangle = \sum_{j=1}^N r_{ij} |R_j \sigma_j\rangle$ . This scenario is equivalent to the possible detection of a single qubit in all various regions originating from the state  $|\varphi_i\rangle$ . Consequently, each edge in the bigraph  $G_b$  corresponds to the situation where a qubit is detected in the state  $|\chi_i\rangle$ , which originates from the state  $|\varphi_i\rangle$ .

A balanced bigraph can be decomposed as  $G_b = G_{b1} \cup \dots \cup G_{bN}$  into perfect matching sub-bigraph graphs  $G_{bi}$  where every two node of two sets (i.e.,  $U$  and  $V$ ) belongs to precisely one of the edges [219, 220, 221]. In this picture, performing the projective measurement of Eq. (2.1), which aims to find one qubit per region, resembles the well-known problem of finding all perfect matching sub-graphs [219]. Also, we can associate a weighted adjacency matrix,  $\mathcal{A}_{bi}$ , to each of these sub-graphs, which can be written in the form of a block matrix as

$$\mathcal{A}_{bi} = \begin{pmatrix} 0 & \mathcal{R}_{\sigma_1, \dots, \sigma_N} \\ \mathcal{R}_{\sigma_1, \dots, \sigma_N}^T & 0 \end{pmatrix}. \quad (2.3)$$

Here, the entry  $\mathcal{R}_{\sigma_1, \dots, \sigma_N}$  represents the (nonzero block) weight matrix, with  $\mathcal{R}_{\sigma_1, \dots, \sigma_N}^T$  denoting its matrix transpose conjugate. This block weight matrix  $\mathcal{R}_{\sigma_1, \dots, \sigma_N}$  indicates edges between two sets of nodes, namely  $U$  and  $V$ , with matrix elements  $\mathcal{R}_{\sigma_1, \dots, \sigma_N} = \{\langle \chi_i | \varphi_{jD} \rangle\}$  ( $i, j = 1, 2, \dots, N$ ). A zero matrix block element indicates the absence of an edge between nodes of each set. Using weight matrices corresponding to each of the perfect matches, we ultimately obtain the generated state

$$|\Psi^{(N)}\rangle = \frac{1}{\sqrt{N_g}} \sum_{\sigma_1, \dots, \sigma_N = \{\uparrow, \downarrow\}} |\mathcal{R}_{\sigma_1, \dots, \sigma_N}|_\eta |R_1 \sigma_1, R_2 \sigma_2, \dots, R_N \sigma_N\rangle, \quad (2.4)$$

where  $N_g$  is the normalization constant and the sum runs over all the  $2^N$  pseudospin configurations. Hence,  $|\mathcal{R}_{\sigma_1, \dots, \sigma_N}|_\eta$  recalls the determinant-like of the weight matrix of each perfect match, taking into account the particle statistics  $\eta$ . Based on the Laplace approach of matrix determinant [222], we can define the determinant-like as  $|\mathcal{R}_{\sigma_1, \dots, \sigma_N}|_\eta = \sum_{i,j=1}^N \eta^{(i+j)} r_{ij} M_{ij}$ , where  $\{M_{ij}\}$  are the elements of the minors of the matrix  $\mathcal{R}_{\sigma_1, \dots, \sigma_N}$ . As a matter of fact, the factor  $|\mathcal{R}_{\sigma_1, \dots, \sigma_N}|_\eta$  represents the inner products between the deformed state and the various states of the sLOCC projection measurement  $\hat{\Pi}^{(N)}$  of Eq. (2.1), as obtained previously. The output state of Eq. (2.4) in this graph picture is thus exactly equivalent to the target state of Eq. (2.2) obtained after the action of the projection  $\hat{\Pi}^{(N)}$ .

Also, one can show a one-to-one correspondence between a balanced bigraph and a directed graph (digraph) [219, 220, 221]. In this representation, the set of nodes  $U = \{u_1, \dots, u_n\}$  is a binary relation between the single-qubit state and the measurement region  $u_i = (|\varphi_i\rangle, |\chi_i\rangle)$ . This means that the detection of a qubit with  $|\varphi_i\rangle$  in the region  $|\chi_j\rangle$  corresponds to a directed edge from node  $u_i$  to  $u_j$  with weight  $r_{ij}$ . In the case where  $i = j$ , it indicates a self-loop. Looking at it from another angle, the spatial overlap between single-qubit states in a specific region determines the connectivity of the graph. An important fact is that qubit spatially overlaps if and only if one vertex of the corresponding digraph has more than one edge.

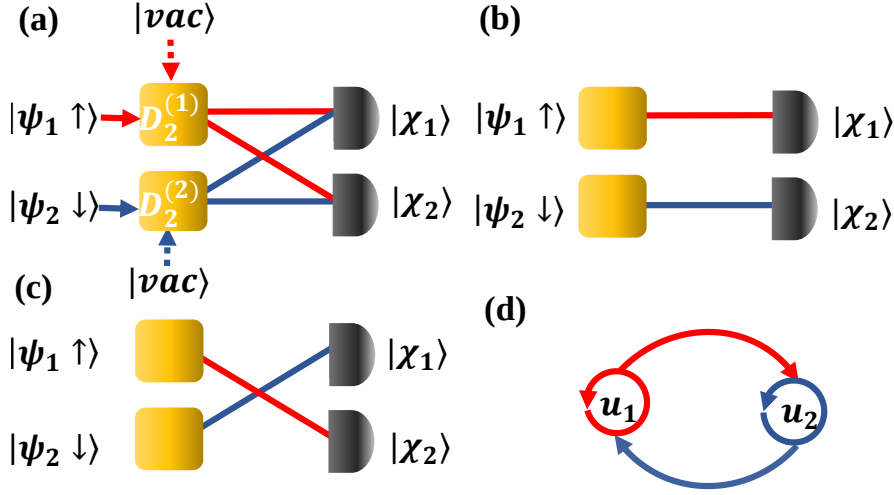


Figure 2.2: Graph representations of entangling protocol in the case of two particles. **(a)** bigraph with  $G_b = (V \cup U, E)$ , where  $V = \{|\psi \uparrow\rangle, |\psi \downarrow\rangle\}$  and  $U = \{|\chi_1\rangle, |\chi_2\rangle\}$ , edge  $e_{ij} \in E$  connects a node  $u_i \in U$  and a node  $v_j \in V$  with complex weight  $r_{ij}$ . Panels **(b)** and **(c)** display perfect matching sub-graphs that correspond to states  $|R_1 \uparrow, R_2 \downarrow\rangle$  and  $|R_1 \downarrow, R_2 \uparrow\rangle$ . **(d)** Colored digraphs (with self-loops) with nodes  $U = \{u_1, u_2\}$ , where each node is a binary relation between the single-qubit state and the measurement region  $u_i = (|\varphi_i\rangle, |\chi_i\rangle)$ . Blue and red correspond to down and up pseudospins, respectively.

### 2.3.1 Graph picture of bipartite entanglement generation

As a starting point for our work, we review the bipartite entanglement generation scheme, which has been theoretically and experimentally demonstrated in Ref. [12, 13], from a graph-theoretical perspective. Let us take two qubits with opposite pseudospins in a spatially separated configuration, represented by the state  $|\Psi_0\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \downarrow\rangle$ . This initial state establishes the first set of nodes,  $V = \{|\psi_1 \uparrow\rangle, |\psi_2 \downarrow\rangle\}$ . Two detection regions instead result in the second set of nodes, namely  $U = \{|\chi_1\rangle, |\chi_2\rangle\}$ , where  $|\chi_j\rangle = |R_j \sigma_j\rangle$  ( $j = 1, 2$ ). As described above, a deformation is applied to the initial states of the two qubits, distributing them into the separated measurement regions  $R_1, R_2$ . In the remote design, the overall two-mode deformation is given by  $|\varphi_{1D}\rangle = D_2^{(1)} |\psi_1 \uparrow\rangle = r_{11} |R_1 \uparrow\rangle + r_{12} |R_2 \uparrow\rangle$  and  $|\varphi_{2D}\rangle = D_2^{(2)} |\psi_2 \downarrow\rangle = r_{21} |R_1 \downarrow\rangle + r_{22} |R_2 \downarrow\rangle$ . An experimental realization of such spatial deformation has been achieved using two independent beam splitters, which distribute two photons to separate regions  $R_1$  and  $R_2$  without altering their polarization [13]. After performing the sLOCC measurement with  $r_{ij} = 1/\sqrt{2}$ , the desired Bell-state, denoted as  $|\Psi^{(2)}\rangle = (|R_1 \uparrow, R_2 \downarrow\rangle + \eta |R_1 \downarrow, R_2 \uparrow\rangle) / \sqrt{2}$ , which exhibits maximal entanglement for both fermionic and bosonic statistics, is generated. Furthermore, this state can also be generated using an on-site scheme with a single spatial deformation. In this approach, two qubits with opposite pseudospins pass through each mode of the deformation with different coefficients  $r_{ij}$  that should satisfy the unitarity constraint of the spatial deformation.

The graph representation in Fig. 2.2 (a) illustrates the outcome of the three-step generation scheme. Here, the two sets are connected and form a bigraph, where the edge colors denote the different pseudospins of the two qubits. In the two-qubit scenario shown in Fig. 2.2 (b), there are two perfect matching sub-graphs. The first perfect matching sub-graph involves the qubit with an up pseudospin  $|\psi_1 \uparrow\rangle$  detected in region one  $|R_1 \uparrow\rangle$  and the one with a down pseudospin  $|\psi_2 \downarrow\rangle$  detected in region two  $|R_2 \downarrow\rangle$ . The second matching involves the opposite, with the qubit having a down pseudospin  $|\psi_2 \downarrow\rangle$  detected in region one  $|R_1 \downarrow\rangle$  and the one with an up pseudospin  $|\psi_1 \uparrow\rangle$  detected in region two  $|R_2 \uparrow\rangle$  (Fig. 2.2 (c)). For each of these perfect matches, we can define the weight matrix adjacency as

$$\mathcal{R}_{\sigma_1, \sigma_2} = \begin{pmatrix} \langle \chi_1 | \varphi_{1D} \rangle & \langle \chi_1 | \varphi_{2D} \rangle \\ \langle \chi_2 | \varphi_{1D} \rangle & \langle \chi_2 | \varphi_{2D} \rangle \end{pmatrix}, \quad (2.5)$$

indicating the connectivity of the perfect match in the sub-graphs. Finally, we get the output state that is in the superposition of the two mentioned perfect matching, given by

$$|\Psi^{(2)}\rangle = \frac{1}{\sqrt{N_g}} \sum_{\sigma_1, \sigma_2 = \{\uparrow, \downarrow\}} |\mathcal{R}_{\sigma_1, \sigma_2}|_{\eta} |R_1 \sigma_1, R_2 \sigma_2\rangle, \quad (2.6)$$

where the determinant-like operation  $|\mathcal{R}_{\sigma_1, \sigma_2}|_{\eta} = \langle \chi_1 | \varphi_{1D} \rangle \langle \chi_2 | \varphi_{2D} \rangle + \eta \langle \chi_1 | \varphi_{2D} \rangle \langle \chi_2 | \varphi_{1D} \rangle$  is precisely the result that has been identified in Ref. [4] as the inner product between states of two identical qubit (two-qubit probability amplitude). One can easily see that the expansion of the above expression is equal to

$|\Psi^{(2)}\rangle = \frac{1}{\sqrt{N_g}} (r_{11}r_{22}|R_1 \uparrow, R_2 \downarrow\rangle + \eta r_{21}r_{12}|R_1 \downarrow, R_2 \uparrow\rangle)$  where  $N_g = |r_{11}r_{22}|^2 + |r_{21}r_{12}|^2$ . It can be observed that the Bell state can be generated with  $r_{ij} = 1/\sqrt{2}$  choices and the sLOCC probability  $\mathcal{P}(|\Psi^{(2)}\rangle) = 1/2$  sLOCC probability. Also, we illustrate the colored digraph of detecting two qubits in two regions in Fig. 2.2 (d), where red and blue edges represent the up and down pseudospins, respectively. In appendix A, we demonstrate the generation of another type of Bell state, namely  $|\Phi_+^{(2)}\rangle = \frac{1}{\sqrt{2}}(|R_1 \uparrow, R_2 \uparrow\rangle + |R_1 \downarrow, R_2 \downarrow\rangle)$ . This is accomplished using a general form of deformation that not only alters the spatial wave functions but also modifies the internal degrees of freedom.

## 2.4 Multipartite entanglement generation schemes

In the following, we provide indistinguishability-based generation schemes for genuine multipartite entangled states using a graph-theoretical representation approach. The multipartite states of interest are W, Dicke, GHZ, and cluster states.

### 2.4.1 W state

We consider a  $W$  state to be the first type of genuine multipartite entangled state. The  $W$  state is an equal combination of all potential pure states, where only one of the qubits has pseudospin  $\uparrow$  while the remaining qubits have pseudospins  $\downarrow$ . We assume that each identical qubit will ultimately occupy distinct spatial regions. Therefore, an  $N$ -qubit  $W$  state of identical qubits has the form

$$|W^{(N)}\rangle = \frac{1}{\sqrt{N}} \left( |R_1 \uparrow, R_2 \downarrow, \dots, R_N \downarrow\rangle + |R_1 \downarrow, R_2 \uparrow, \dots, R_N \downarrow\rangle + \dots + |R_1 \downarrow, R_2 \downarrow, \dots, R_N \uparrow\rangle \right). \quad (2.7)$$

To generate an  $N$ -qubit  $W$  state, we start with the pure product state of  $N$  qubits as  $|\Psi_0\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \downarrow\rangle \otimes \dots \otimes |\psi_N \downarrow\rangle$ , where  $N-1$  qubits have pseudospin down ( $\downarrow$ ) and one qubit has pseudospin up ( $\uparrow$ ). The next step involves designing the deformation, either in remote or on-site schemes, as illustrated in Fig. 2.1.

As a first remote design, we consider a deformation operation that equally distributes each qubit among  $N$  regions, expressed as  $\hat{D}_N^{(i)} |\psi_i \sigma_i\rangle = |\varphi_{iD}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^M |R_j \sigma_i\rangle$ . Subsequently, with the action of the projection  $\hat{\Pi}^{(N)}$  (Eq. (2.1)), the  $W$  state (Eq. (2.7)) is generated. Such a generation scheme can be interpreted using a complete digraph (digraph) with equal weight, denoted as  $G_C$  (where  $C$  represents the complete digraph). In a complete digraph, each pair of graph vertices is connected by a pair of edges with non-zero weights (one in each direction), as illustrated in Fig. 2.3 (a) for four qubits. The perfect matching can be found using Eq. (2.4), and the  $W$  state (Eq. (2.7)) is a superposition of these perfect matchings (see Appendix B for more details on three- and four-qubit states). The complete digraph with equal weight designed to generate the bosonic  $W$  state (Eq. (2.7)) has the success (sLOCC) probability given by

$$\mathcal{P}_C(|W^{(N)}\rangle) = \frac{(N-1)!}{N^{(N-1)}}. \quad (2.8)$$

However, generating the fermionic  $W$  state with the complete digraph  $G_C$  is impossible due to the zero sLOCC probability  $\mathcal{P}_C(|W^{(N)}\rangle) = 0$ . This zero probability can be attributed to the Pauli exclusion principle [163], which prohibits two fermions with the same pseudospin from occupying the same spatial wave function. Therefore, an alternative deformation scheme is needed to generate the fermionic  $W$  state.

An alternative remote approach to generating the  $W$  state involves employing the generation scheme based on the star digraph  $G_S$ . In this configuration, as illustrated in Fig. 2.3 (b) for four nodes, a qubit with the opposite pseudospin (i.e., up) resides in the center and spatially overlaps with all other qubits, while the remaining qubits with a down pseudospin only spatially overlap with the central one. To construct such a generation scheme, spatial deformation operations are defined as  $\hat{D}_N^{(1)} |\psi_1 \uparrow\rangle = |\varphi_{1D}\rangle = \frac{1}{\sqrt{N}} (|R_1 \uparrow\rangle + \eta \sum_{i=2}^M |R_i \uparrow\rangle)$  and  $\hat{D}_N^{(i)} |\psi_i \downarrow\rangle = |\varphi_{iD}\rangle = \frac{1}{\sqrt{2}} (|R_1 \downarrow\rangle + |R_i \downarrow\rangle)$ , for  $i = 2, \dots, N$ , where  $\eta = \pm 1$ . It is worth mentioning that the generation of the  $W$ -state depends not only on the form of deformation but also on particle statistics. When designing a generation scheme, one must consider the particle exchange phase  $e^{i\phi}$  ( $\phi = 0$  for bosons and  $\phi = \pi$  for fermions) that emerges when two single-qubit states are swapped [162]. Finally, after applying a projection measurement, we can generate the  $W$  state for both bosonic and fermionic statistics. Interestingly, the sLOCC probability for fermions is given by

$$\mathcal{P}_{star}(|W^{(N)}\rangle) = \frac{1}{N}, \quad (2.9)$$

which is significantly enhanced compared to the sLOCC probability in the complete digraph case for bosons.

Furthermore, one can employ on-site deformation, where each identical qubit is sent through one of the input modes, as illustrated in Fig. 2.1 (a). In this scenario, the initial state undergoes a spatial quantum Fourier transform (QFT) [106]. For each initial single-qubit state  $|\psi_j \sigma\rangle$ , we can express the QFT using the following map:

$$|\psi_j \sigma\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=1}^N \omega_N^{(j-1)(k-1)} |R_k \sigma\rangle, \quad (2.10)$$

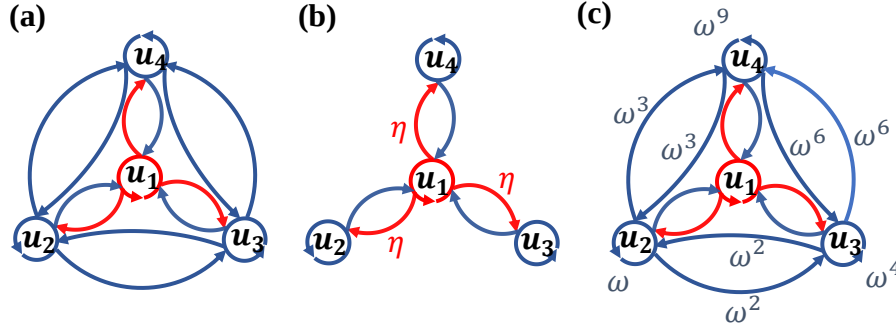


Figure 2.3: **Digraph representations of four-particle  $W$  state generation.** The digraph illustrates four quantum nodes  $u_j$  ( $j = 1, \dots, 4$ ), representing a binary state and measurement region. Red (blue) edges indicate the detection of a qubit with an up (down) pseudospin. The presence of  $\eta$  on red edges denotes a  $\pi$  phase shift between connected nodes for fermions. **(a)** The complete digraph illustrates mutually oriented edges between every pair of nodes, each with equal weighting. **(b)** The star graph displays mutually oriented edges only between the center node and the other nodes. **(c)** The QFT digraph exhibits mutually oriented edges between each pair with a specific weighting as  $\omega_N^n$  ( $\omega_N = e^{2i\pi/N}$ ).

where  $\omega_N = e^{2i\pi/N}$  is the relative phase. Here, the QFT deformation does not alter the pseudospin  $\sigma$ . This deformation represents a special form of a complete graph with specific complex weights relative phase  $\omega_N$  introduced by the QFT as illustrated in Fig. 2.3 (c) for four qubits. Finally, the  $W$  state can be generated for both fermions and bosons (only in the case of an odd number of qubits for bosons; see appendix B) using such QFT-based on-site design. The sLOCC probability in the generation of the  $W$  state with fermions using the QFT-based digraph representation is given by

$$\mathcal{P}_{QFT}(|W^{(N)}\rangle) = \frac{1}{N}, \quad (2.11)$$

which is equivalent to the sLOCC probability of  $W$  state generation using the star digraph representation in the remote design. It is worth mentioning that the star digraph representation for  $W$ -state generation, as illustrated in Fig. 3, has been demonstrated in Ref [155] via an optical linear on-site design. On-site design is based on the linear transformation of creation (annihilation) operators in the second quantization formalism.

In order to recap, we compare different possible schemes in terms of sLOCC probability to assess the generation scheme efficiency. We plot the sLOCC probability for generating an  $N$ -qubit  $W$  state as a function of the number of qubits  $N$  in Fig. 2.4 (a) using complete, star, and QFT digraphs with both particle statistics. As depicted in Fig. 2.4 (a), fermions exhibit better performance in both on-site and remote schemes. This advantage can be attributed to the Pauli exclusion principle, which prohibits fermions from coexisting in the same region with the same pseudospin. Moreover, different configurations allow for many different structures of multipartite (refer to appendix C for more information on the general form of generating three- and four-qubit states). Furthermore, the weight matrix adjacency in a star digraph can be adjusted to maximize the success probability, analogous to what was reported in Ref. [223] as an efficient optical setup to generate a  $W$  state. It is worth noting that the graph-theoretic approach enables the discovery of alternative generation schemes for  $W$  states, as can be seen in Refs. [154, 155] with bosonic statistics.

We have already demonstrated that the ideal scenarios of complete, star, and QFT digraphs can lead to the generation of the  $W$  state. However, it is essential to explore the imperfect spatial overlap configurations where probability amplitudes  $r_{ij}$  (weight of the edges in digraphs) are not exactly our desired ones. When we have an imperfect situation, the crucial question is whether the generated state is genuinely multipartite-entangled. A typical figure of merit is fidelity, which measures how close the generated state is to an ideal pure entangled state [224, 225, 226, 227]. Formally, fidelity is a measure of the distance from a general generated state  $\rho$  to the desired pure state  $|\psi\rangle$ , given by  $F_\psi = \langle\psi|\rho|\psi\rangle$ . Therefore, if the fidelity between the generated state and the multipartite  $W$  state violates the inequality  $F_W \leq \frac{N-1}{N}$ , the generated state is genuinely an  $N$ -partite (here  $N$ -qubits)  $W$  state [228, 229].

Besides the imperfections of the generation scheme, it is crucial to assess the maximum probability of success since our scheme is probabilistic. Such a consideration leads to an interesting trade-off between fidelity and the probability of success, in which one may sacrifice perfect entanglement generation for a higher probability of success. Here, the objective is to maximize the success probability, denoted as  $\mathcal{P}(|W^{(N)}\rangle)$ , by searching for an optimal digraph, i.e., a generation scheme, while ensuring that the fidelity does not fall below a certain threshold for the multipartite  $W$  state, specifically  $F_W \leq \frac{N-1}{N}$ . To solve the maximization problem, we first select the complete digraph configuration with bosonic statistics. Then, we randomly sample a large number of digraphs with positive weights and calculate both the fidelity and sLOCC probability. Subsequently, we divide the fidelity values exceeding the threshold  $F_W \leq \frac{N-1}{N}$  into small intervals and determine the maximum sLOCC probability for each of these intervals. As a result, we plot the fidelity as a function of its maximum probability of success for three and four bosons in Fig. 2.4 (b). Additionally, for a digraph with all possible edges, the highest sLOCC probabilities that guarantee the generation of genuine three- and four-boson  $W$  states, respectively, are  $\mathcal{P}_3 = 0.47$  and  $\mathcal{P}_4 = 0.20$ .



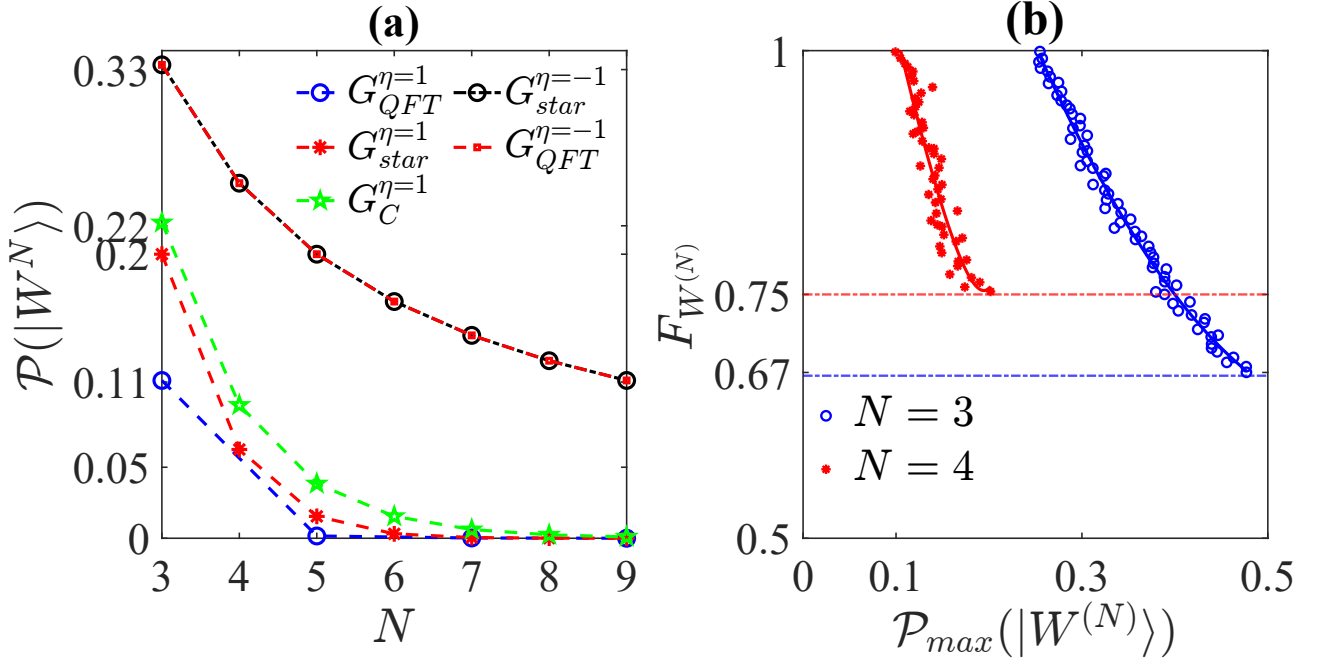


Figure 2.4: **Success probability and fidelity of generating  $W$  states with different schemes and particle statistics.** (a) Success probability  $\mathcal{P}(|W^{(N)}\rangle)$  as a function of the number of particles  $N$  for different  $W$  state generation schemes, including the complete digraph for bosons (green star), the star digraph for bosons (red asterisk), the QFT digraph for bosons (blue circle), the star digraph for fermions (black circle), and the QFT digraph for fermions (red square). (b) Fidelity as a function of the maximum probability of success  $\mathcal{P}_{max}(|W^{(N)}\rangle)$  for three bosons (blue circles) and four bosons (red stars) in the form of genuine  $W$  states in the complete digraph. The dashed lines correspond to the bound that guarantees genuine multipartite  $W$  entanglement.

### 2.4.2 Symmetric Dicke state

The Dicke state is another type of multipartite entangled state, which was first investigated concerning light emission from clouds of atoms [230]. We assume that each qubit will reside in different detection regions  $R_j$ . Thus, the  $N$ -qubit symmetric Dicke state with  $N/2$  down pseudospins can be defined as follows:

$$|D^{(N)}\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_{P_j} |R_1 \uparrow_{P_1}, \dots, R_{N/2} \uparrow_{P_{N/2}}, R_{N/2+1} \downarrow_{P_{N/2+1}}, \dots, R_N \downarrow_{P_N}\rangle, \quad (2.12)$$

where the summation goes over all distinct permutations of pseudospins while keeping the regions  $R_i$  fixed.

We start the generation process with the pure initial state of an  $N$ -qubit system as  $|\Psi_0\rangle = |\psi_1 \uparrow\rangle \otimes \dots \otimes |\psi_{N/2} \downarrow\rangle \dots |\psi_{N/2+1} \uparrow\rangle \otimes \dots \otimes |\psi_N \downarrow\rangle$ , which comprises half of the qubits with up pseudospin and the other half with down pseudospin. As the first generation scheme, we employ a remote design and apply a spatial deformation to each qubit as  $\hat{D}_N^{(i)} |\psi_i \sigma_i\rangle = |\varphi_{iD}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^M |R_j \sigma_i\rangle$ , which distributes each qubit equally among all detection regions. After applying the projection measurement, the bosonic symmetric-Dicke state is generated. Such a generation process is based on the complete digraph with equal weights, as illustrated in Fig. 2.5 (a) for four qubits. The associated sLOCC probability in generating the  $N$  bosonic symmetric-Dicke state is given by:

$$\mathcal{P}_C(|D^{(N)}\rangle) = \frac{(N-1)!}{N^{N-1}}. \quad (2.13)$$

Alternatively, one can find different digraphs that still lead to the generation of symmetric Dicke states. For example, by removing connections in complete graphs, one can check whether a symmetric Dicke state is generated. Notably, alternative remote designs for generating four qubits in bosonic symmetric Dicke states have been introduced in Ref. [155]. It is worth noting that the on-site approach with the QFT scheme is incapable of generating possible symmetric Dicke states, as reported in Ref. [231] [for details, refer also to appendix C].

In non-ideal situations, we can employ fidelity-based criteria to test the entanglement between  $N$  identical qubits. An entangled symmetric Dicke state is a genuine  $N$ -partite entangled state if and only if the inequality  $F_D \leq \frac{N}{2(N-1)}$  is violated [226]. This also allows us to explore the trade-off between fidelity and the probability of success in the genuine multipartite Dicke state. We sample a large number of random digraphs and maximize the sLOCC probability for each value of  $F_D$  over the threshold. As a result, the fidelity is plotted as a function of the maximum probability of success for



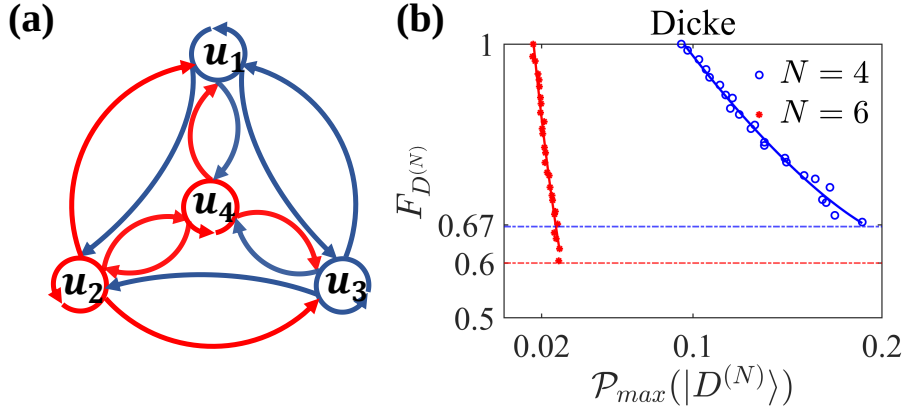


Figure 2.5: **Symmetric Dicke state generation.** (a) Colored complete digraph  $U = \{u_1, \dots, u_4\}$  for a four-qubit symmetric Dicke state. (b) Fidelity as a function of maximum probability of success  $\mathcal{P}_{max}(|D^{(N)}\rangle)$  for four (blue circles) and six (red stars) bosons in the form of a genuine symmetric Dicke state with two pseudospins down in the complete digraph. The dashed lines correspond to the bound that guarantees genuine multipartite symmetric Dicke entangled states.

four and six bosons in Fig. 2.5 (b). For a complete digraph with non-equal weighted edges, the maximum probability of success for genuine four-boson and six-boson Dicke states is  $\mathcal{P}(|D^{(4)}\rangle) = 0.19$  and  $\mathcal{P}(|D^{(6)}\rangle) = 0.03$ , respectively.

### 2.4.3 GHZ state

GHZ states play a central role in quantum computing [15, 16], and metrology [20]. A GHZ state is a superposition of all qubits being in either up or down pseudospin states. Assuming each qubit resides in a different spatial region, the  $N$ -qubit GHZ state is expressed as

$$|\text{GHZ}^{(N)}\rangle = \frac{1}{\sqrt{2}}(|R_1 \uparrow, \dots, R_N \uparrow\rangle + |R_1 \downarrow, \dots, R_N \downarrow\rangle). \quad (2.14)$$

To generate a GHZ state, we begin with  $N$  identical qubits in a pure product, represented as  $|\Psi^{(N)}\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \dots \otimes |\varphi_N\rangle$ , where  $|\varphi_i\rangle = |\psi_i \uparrow\rangle$ . Spatial deformation alone is insufficient for generating GHZ states, so we employ a more general form of deformation that modifies both the spatial wave function and the pseudospins of the initial qubits. In this deformation, each qubit is distributed among two regions, and its pseudospin is simultaneously altered if the qubit is detected in the nearest neighboring region. Specifically, the action of each deformation on the initial qubits is given by  $\hat{D}_N^{(i)} |\psi_i \uparrow\rangle = |\varphi_{iD}\rangle = \frac{1}{\sqrt{2}}(|R_i \uparrow\rangle + \eta |R_{i+1} \downarrow\rangle)$  for the first to  $(N-1)$  qubits. Additionally, the  $N$ -th qubit undergoes deformation as  $\hat{D}_N^{(N)} |\psi_N \uparrow\rangle = |\varphi_{ND}\rangle = \frac{1}{\sqrt{2}}(|R_N \uparrow\rangle + |R_1 \downarrow\rangle)$ . After selecting one qubit per region (which is the action of  $\hat{\Pi}^{(N)}$  (Eq. (2.1))), the GHZ state is generated.

In the digraph representation of the GHZ state generation scheme, as illustrated in Fig. 2.6 (a), we have  $N$  nodes with the same self-loop edges in red and outgoing directed edges in blue that connect the  $i$ -th node to the  $(i+1)$ -th node. The differences in color among the self-loops, along with their oriented edges, indicate that the same qubit with different pseudospins will be detected in the nearest neighboring regions. The digraph representation of the generation of the bosonic GHZ state, as shown in Fig. 6, was originally introduced in Ref. [155]. This graph representation corresponds to an experimental implementation of a three-photon GHZ state using a free-space linear optics system, as described in Ref. [148]. For the fermionic GHZ state, a  $\pi$ -phase is necessary for the generation of the GHZ state. Finally, the GHZ state is generated with the sLOCC probability of

$$\mathcal{P}(|\text{GHZ}^{(N)}\rangle) = \frac{1}{2^{N-1}}, \quad (2.15)$$

for both bosonic and fermionic statistics.

Additionally, we discuss the trade-off between the success probability  $\mathcal{P}(|\text{GHZ}^{(N)}\rangle)$  in the generation of the GHZ state and the fidelity-based entanglement criteria  $F_{\text{GHZ}} \geq \frac{1}{2}$  [224, 232]. The fidelity criteria ensure that the generated state is entangled in the form of GHZ. We plot the fidelity  $F_{\text{GHZ}}$  as a function of the maximum probability of success for three to six qubits (bosons and fermions) in Fig. 2.6 (b). The plot clearly demonstrates that one can generate a weak GHZ state with a high probability of success.

### 2.4.4 Cluster state

Cluster states are a type of multipartite entangled state that serves as a fundamental building block for measurement-based quantum computing and one-way quantum computing, in which measurements on qubits within the cluster state are used

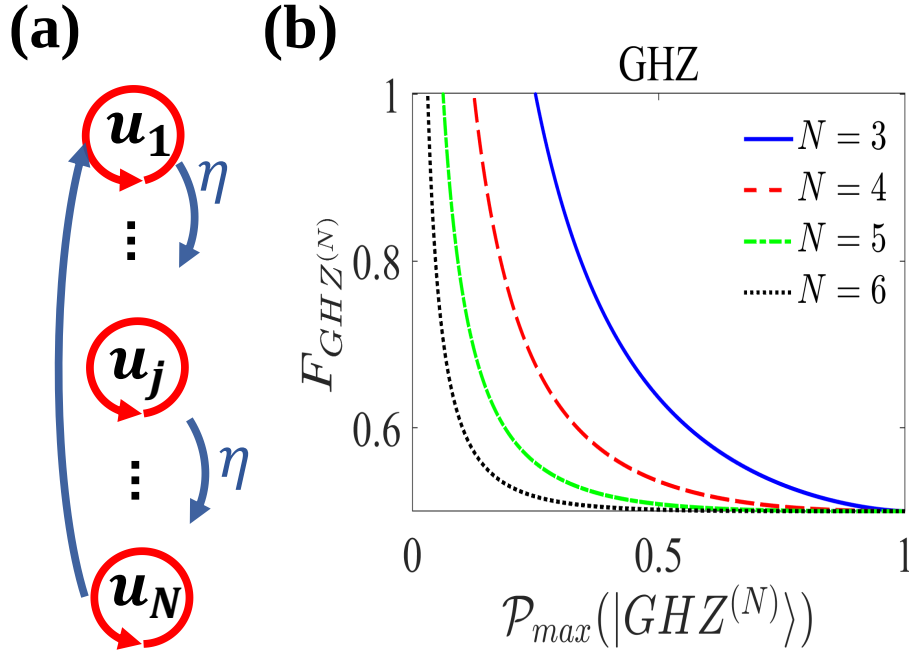


Figure 2.6: **(a) Digraph representations of  $N$ -qubit GHZ state.** Colored digraph (with self-loops) with nodes  $U = \{u_1, u_2 \dots u_N\}$ . Also, the self-loops and oriented edges have different colors. The presence of  $\eta$  shows a  $\pi$  phase shift between connected nodes for fermions. Indeed, this phase shift is not necessary when working with bosons. **(b) Trade-off between fidelity and sLOCC probability.** Fidelity as a function of maximum probability of success  $\mathcal{P}_{max}(|GHZ^{(N)}\rangle)$  for three (blue), four bosons (red), five (green) and six (black) bosons in the form of genuine GHZ state

to perform universal quantum computing [233, 178]. These classes of multipartite entangled states exhibit robustness against noise and decoherence [234, 233]. Historically, the generation of a cluster state has been based on next-neighbor interaction between qubits on a chain [233, 178]. In this work, we employ the indistinguishability-based entangling gate to generate the cluster state. By assuming that each qubit resides in a distinct spatial region, an  $N$ -qubit linear cluster can be expressed in the form

$$|C^{(N)}\rangle = \frac{1}{2^{\frac{N}{2}}} \bigotimes_{i=1}^N (|R_i \downarrow\rangle \sigma_z^{R_{i+1}} + |R_i \uparrow\rangle). \quad (2.16)$$

Here, the spatial Pauli operator is defined as  $\sigma_z^{R_i} = |R_i\rangle \langle R_i| \otimes \sigma_z$ , where  $\sigma_z = |\downarrow\rangle \langle \downarrow| - |\uparrow\rangle \langle \uparrow|$ , and  $\sigma_z^{R_{N+1}} = 1$ . The above state satisfies the cluster state condition, i.e.,  $\sigma_x^{R_i} \sigma_z^{R_{i+1}} |C^{(N)}\rangle = \pm |C^{(N)}\rangle$  with  $\sigma_x^{R_i} = |R_i\rangle \langle R_i| \otimes (|\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow|)$  under a unitary transformation on one or more of the qubits [233, 178]. Furthermore, one can consider an even number of qubits and arrive at the following state:

$$|C^{(N)}\rangle = \frac{1}{2} \left( |R_1 \uparrow, \dots, R_N \uparrow\rangle - |R_1 \downarrow, \dots, R_N \downarrow\rangle + |R_1 \uparrow, \dots, R_{\frac{N}{2}} \uparrow, R_{\frac{N}{2}+1} \downarrow, \dots + R_N \downarrow\rangle + |R_1 \downarrow, \dots, R_{\frac{N}{2}} \downarrow, R_{\frac{N}{2}+1} \uparrow, \dots, R_N \uparrow\rangle \right). \quad (2.17)$$

To generate the cluster state in the form described above, we consider an  $N$ -qubit pure product state of  $|\Psi^{(N)}\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \dots \otimes |\varphi_N\rangle$ , where  $|\varphi_i\rangle = |\psi_i \uparrow\rangle$ . The generation process involves placing the  $i$ -th qubit in a superposition of two spatial regions using the deformation operation, given as  $\hat{D}_N^{(i)} |\psi_i \uparrow\rangle = |\varphi_{iD}\rangle = \frac{1}{\sqrt{2}} (|R_i \uparrow\rangle + |R_{i+1} \downarrow\rangle)$ , with the exception of the middle  $\frac{N}{2}$  and last  $N$  qubits. Subsequently, the  $\frac{N}{2}$  th qubit undergoes the deformation operation as  $\hat{D}_N^{(N/2)} |\psi_i \uparrow\rangle = |\varphi_{\frac{N}{2}D}\rangle = \frac{1}{\sqrt{3}} (|R_{\frac{N}{2}} \uparrow\rangle + |R_{\frac{N}{2}+1} \downarrow\rangle \mp \eta^{\frac{N}{2}} |R_N \uparrow\rangle)$  where the positive (negative) sign is for bosons (fermions). Finally, the deformation on the  $N$ th qubit is given by  $\hat{D}_N^{(N)} |\psi_i \uparrow\rangle = |\varphi_{ND}\rangle = \frac{1}{\sqrt{3}} (\mp |R_1 \downarrow\rangle \pm \eta^{\frac{N}{2}} |R_{\frac{N}{2}+1} \downarrow\rangle + |R_N \uparrow\rangle)$ .

The digraph representation of cluster state generation, as illustrated in Fig. 2.7 (a), consists of  $N$  nodes with self-loop edges in red and outgoing directed edges in blue connecting the  $i$ -th node to the  $(i+1)$ -th node. Additionally, the node at the middle,  $R_{\frac{N}{2}}$ , has an additional outgoing red directed edge to the final node at  $R_N$  with weights of  $-1/\sqrt{3}$  for bosons and  $e^{iN\pi/2}/\sqrt{3}$  for fermions. The node at the last phase has two outgoing directed edges in blue to  $R_{\frac{N}{2}}$  and the first node  $R_1$  with weights of  $-1/\sqrt{3}$  for bosons and  $-e^{iN\pi/2}/\sqrt{3}$  for fermions. Subsequently, the ideal cluster state in Eq. (2.17)

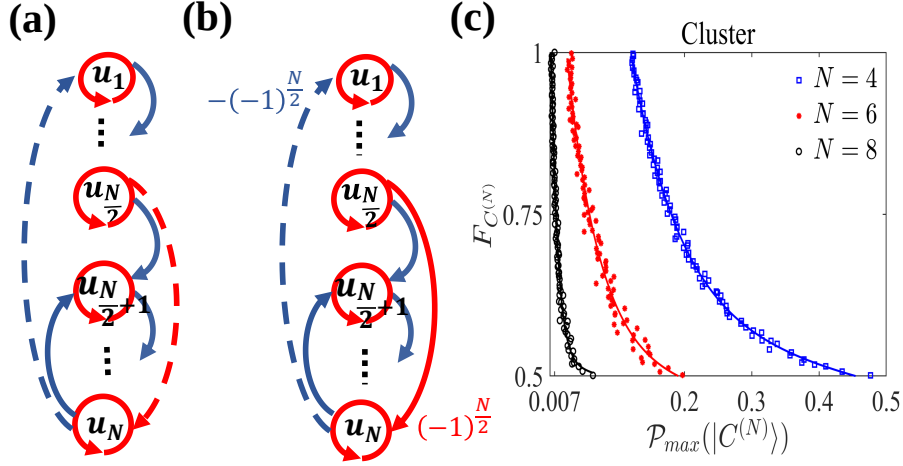


Figure 2.7: **Graph representations of  $N$ -qubit cluster states for (a) bosons and (b) fermions.** The colored digraph (with self-loops) has nodes  $U = \{u_1, u_2, \dots, u_N\}$ . (a) In the bosonic design, the dashed-oriented edge represents a  $\pi$  phase shift between connected nodes. (b) For fermions, the dashed-oriented edge represents a  $\pm(-1)^{N/2}$  phase shift between connected nodes, depending on the number of particles. (c) **Trade-off between fidelity and sLOCC probability.** Fidelity as a function of maximum probability of success  $\mathcal{P}_{\max}(|C^{(N)}\rangle)$  for four (blue square), six bosons (red stars), and eight (black circle) qubits in the form of a genuine cluster state. The solid lines correspond to the fitted data.

is generated for bosonic and fermionic statistics with the sLOCC probability given by

$$\mathcal{P}(|C^{(N)}\rangle) = \frac{1}{9} \frac{1}{2^{N-4}}, \quad (2.18)$$

for an even number of qubits larger than four.

Similar to previous sections, one might be interested in the trade-off between sLOCC probability and the amount of entanglement that can be generated using this design. Based on fidelity-based criteria, any generated state  $\rho$  that violates the inequality  $F_{C^{(N)}} \leq \frac{1}{2}$  is considered a genuine  $N$ -qubit cluster state [224, 232]. Additionally, to illustrate the trade-off between fidelity and the sLOCC probability, we maximize the probability of success  $\mathcal{P}(|C^{(N)}\rangle)$  subject to a fidelity constraint  $F_{C^{(N)}} \geq \frac{1}{2}$  by sampling random positive weights over the graph representation of the cluster state for bosonic and fermionic, respectively, as illustrated in Fig. 2.7 (a) and Fig. 2.7 (b). Moreover, we plot the fidelity as a function of the maximum sLOCC probability for four to eight qubits (both bosons and fermions) in Fig. 2.7 (c). We obtain the maximum sLOCC probabilities  $\mathcal{P}(|C^{(4)}\rangle) = 0.48$ ,  $\mathcal{P}(|C^{(6)}\rangle) = 0.20$ , and  $\mathcal{P}(|C^{(8)}\rangle) = 0.07$  for four, six, and eight qubits (for both bosonic and fermionic statistics), respectively.

## 2.5 Discussion on Experimental Feasibility

In the realm of free-space quantum optics, a few experiments have been conducted to exploit the indistinguishability of identical particles, specifically spatially overlapping photons, for generating entangled states. These experiments have successfully generated Bell states [13, 14], as well as three-photon W and GHZ states [148]. The internal degree of freedom, or pseudospin, of the photon is represented by the polarization state, while the spatial wave functions correspond to the regions where the photons are detected. In the following, we provide an overview of the experimental platforms needed for such experiments, as well as the potential experimental implementation of the three-step generation proposed in our work.

In the first step (initialization), one can prepare an initially separable state through various methods. One approach involves using multiple independent heralded single-photon sources, as used in the experimental verification of spatially overlapping photons for generating entangled states [13, 14, 148]. These probabilistic photon sources introduce additional costs in terms of success probability for generating the target state. Alternatively, a deterministic single-photon source, such as a quantum dot, can be employed [235, 236, 237], which can enhance the success probability. Despite this, implementing deterministic single-photon sources faces practical challenges, including limited spectral tunability, lossy device-to-fiber coupling, and cryogenic cooling requirements [238].

In the second step (deformation), one can control and adjust the single photons spatial wave functions and polarization (i.e., pseudospin) by using linear optics elements. These include beamsplitters, multimode beamsplitters [239], and half-wave plates. These optical components act as deformation operators on the initial state, as described in our theoretical framework.

In the third step (activation), that is at the detection and verification stage, one can use single-photon counting and quantum state tomography to verify the generated state, as in Refs. [13, 14, 148]. As a prospect, it may be valuable to explore the use of quantum witness operators within the sLOCC framework, which could offer a simplified alternative to quantum state tomography, especially in the case of increasingly complex multipartite systems [153].

The goal of this work is to encourage the development of new platforms that can utilize indistinguishability to prepare many-particle entangled states in various experimental configurations beyond free-space optics. For example, superconducting qubits present a promising platform for simulating both bosonic and fermionic systems with two internal and external degrees of freedom. Superconducting ring resonators can function as beam splitters and be used to deform initial states [240]. Alternatively, qubits in Bose-Einstein and Fermi-Dirac condensates can be prepared at distinct sites within a lattice. Possible internal degrees of freedom in these systems include electrons spin or energy levels. Experiments with Bose-Einstein and Fermi-Dirac condensates require suited techniques for single-particle trapping and control. As a deformation operation, particles can be tunneled to other sites similarly to a beam-splitting action. The probability amplitude of this tunneling effect can be adjusted by modifying external parameters such as voltages, magnetic fields, and laser beams [241, 242].

## Conclusions

The indistinguishability of identical particles serves as an exploitable resource for generating both bipartite [12, 13, 14] and multipartite [148] entangled states. After the sLOCC measurement, the resource state becomes available for various quantum information processing tasks [12, 13, 149, 150, 151, 152], because identical particles become distinguishable and individually addressable by their spatial regions. Based on spatial wave functions, one can devise experimentally feasible entanglement witness operators to test the generated states [153].

Although maximizing the post-selection probability was not our focus, our scheme for generating multipartite entanglement is more efficient than the heralded-based approach [243] and the path identity approach [206, 207], both utilizing the indistinguishability of photons. For example, our approach achieved a post-selection probability of  $\frac{1}{2^{(N-1)}}$  for the generation of the GHZ state, which is significantly higher than the probability of the path identity approach, given as  $\frac{2}{N(N/2)}$  [206]. Additionally, we report success probabilities for generating three-partite W states of  $\frac{1}{5}$  for bosons and  $\frac{1}{3}$  for fermions, improving upon the previously reported probabilities. Specifically, in a linear-optic setup using a three-port splitter, the probability is  $\frac{1}{9}$  [244], and in the no-touching paradigm, it is 0.15 [245]. Also, it is important to note that optimizing the weight matrix adjacency of a given digraph can maximize the post-selection probability. This is the case, for instance, of a bosonic W state [223] and cluster state [246]. Noteworthy, this probability of generation W state can further be improved using fermionic statistics, as emerges from our method.

In this work, we have proposed an algorithmic approach that uses the indistinguishability of identical particles to generate a broad class of multipartite entangled states, including W, Dicke, GHZ, and cluster states. We have represented each generation scheme using bigraphs and digraphs, aligning with the no-label approach for both fermionic and bosonic statistics. This facilitates the examination of the output state, the assessment of its genuine entanglement, and the evaluation of resources in terms of postselection probability. Our approach allows for the search for optimal graph configurations to maximize both the fidelity and postselection probability of the targeted multipartite entangled state. Different graphs may yield the maximum fidelity, such as complete and star digraphs, for the generation of the W state. The graph representation enables the search for arbitrary forms of multipartite state generation by optimizing resources, considering constraints, and using particle statistics. Thus, it serves as a proper interface for optimizing the generation of arbitrary multipartite entangled states with specific probabilities of success and maximum fidelity.

The generation of various multiqubit entangled states via our scheme relies on probabilistic conditions, assuming that a single particle is available upon request. Also, probabilistic sources can be employed for single-particle preparation in the initial generation stage, such as spontaneous parametric down-conversion (SPDC) and spontaneous four-wave mixing (SFWM) processes in second- and third-order nonlinear materials, respectively [247]. However, this source will add an additional cost to sLOCC probability, due to the probabilistic nature of these phenomena. This is why we explore the trade-off between fidelity and the probability of success, potentially sacrificing the amount of entanglement to achieve a higher sLOCC probability. In prospect, one can also consider experimental imperfections, limitations, and constraints across all three stages and optimize the generation schemes using this algorithmic approach.

As future perspectives, optimization methods such as simulated annealing could be employed to identify the optimal graph structure with higher success probabilities and fidelity. In conclusion, the exploitability of indistinguishable identical graph nodes in quantum networks offers innovative possibilities within a widely used technology, suggesting further avenues for exploration and study.

## Appendix

### A Bell State

As mentioned in the main manuscript, spatial deformations alone are insufficient for generating graph states within our formalism. The general deformation alters both the spatial wave function and the pseudospin states of the initial qubits. Here, we discuss the Bell state, represented as:

$$|\Phi_+^{(2)}\rangle = \frac{1}{\sqrt{2}}(|R_1 \uparrow, R_2 \uparrow\rangle + |R_1 \downarrow, R_2 \downarrow\rangle), \quad (2.19)$$

which requires the general deformation to generate.

First, we consider the initial state in the pure product state form as  $|\Psi_0^{(2)}\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \uparrow\rangle$ . The actions of deformations result in each qubit undergoing a transformation:  $|\varphi_{1D}\rangle = D_2^{(1)}|\psi_1 \uparrow\rangle = r_{11}|R_1 \uparrow\rangle + \eta r_{12}|R_2 \downarrow\rangle$ , and  $|\varphi_{2D}\rangle = D_2^{(2)}|\psi_2 \uparrow\rangle = r_{21}|R_1 \downarrow\rangle + r_{22}|R_2 \uparrow\rangle$  where  $r_{ij}$  represent the single-qubit probability amplitudes. As observed in these single-qubit deformed states, the deformation not only alters the spatial wave function but also changes the pseudospins of the initial qubits upon detection in the second region  $R_2$ . The corresponding digraph is similar to Fig. (2) of the main manuscript, with the difference that the colors of the self-loops differ from the colors of the edges oriented toward them. With probability amplitudes  $r_{ij} = \frac{1}{\sqrt{2}}$ , the ideal Bell state in (2.19) is generated with an sLOCC probability of  $\mathcal{P}(|\Phi_+^{(2)}\rangle) = \frac{1}{2}$ .

### B W state with three and four qubits

In the main part of the paper, we delve into indistinguishability-based generation schemes for the  $W$  state with  $N$  identical qubits. Here, we provide a detailed explanation of the formalism, illustrating it with examples involving three and four qubits. This exploration allows us to elucidate how the bigraph and digraph picture, employed as a generation scheme, can be utilized to generate the  $W$  state.

#### Three-qubit $W$ state

We begin with an initial state of a three-qubit system expressed in the pure product form as  $|\Psi_0^{(3)}\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \downarrow\rangle \otimes |\psi_3 \downarrow\rangle$ . Subsequently, we apply spatial deformation, placing each qubit in a superposition of three spatial regions, described by  $|\varphi_{iD}\rangle = \sum_{j=1}^M r_{ij} |R_j \sigma_i\rangle$ , where  $r_{ij}$  denotes tunable single-qubit probability amplitudes. As a result, the deformed state of the three-qubit system can be written as  $|\Psi_D^{(3)}\rangle = \frac{1}{\sqrt{\nu}} |\varphi_{1D}, \varphi_{2D}, \varphi_{3D}\rangle$ , with the normalization factor  $\nu = |\langle \varphi_{1D}, \varphi_{2D}, \varphi_{3D} | \varphi_{1D}, \varphi_{2D}, \varphi_{3D} \rangle|^2$ . It's worth noting that the three-qubit deformed state is not generally factorizable in terms of single-qubit states, given as  $|\Psi_D^{(3)}\rangle \neq \frac{1}{\sqrt{\nu}} |\varphi_{1D}\rangle \otimes |\varphi_{2D}\rangle \otimes |\varphi_{3D}\rangle$  [215]. Finally, the sLOCC projection measurement  $\hat{\Pi}^{(3)}$  [12, 167] is utilized to project the deformed state onto three separated spatial regions where a single qubit can be found.

From a graph theory perspective, we can associate a digraph with the three mentioned generation steps: initialization, deformation, and projection. This assignment is facilitated by the adjacency matrix of a digraph, with elements  $\langle R_i \sigma_i | \varphi_{jD} \rangle$ , as given by

$$\mathcal{R}_{\sigma_1, \sigma_2, \sigma_3} = \begin{pmatrix} r_{11} \langle \sigma_1 | \uparrow \rangle & r_{12} \langle \sigma_1 | \downarrow \rangle & r_{13} \langle \sigma_1 | \downarrow \rangle \\ r_{21} \langle \sigma_2 | \uparrow \rangle & r_{22} \langle \sigma_2 | \downarrow \rangle & r_{23} \langle \sigma_2 | \downarrow \rangle \\ r_{31} \langle \sigma_3 | \uparrow \rangle & r_{32} \langle \sigma_3 | \downarrow \rangle & r_{33} \langle \sigma_3 | \downarrow \rangle \end{pmatrix}, \quad (2.20)$$

which is written over all possible combinations of pseudospins  $\sigma_i = \{\uparrow, \downarrow\}$  for  $i = 1, 2, 3$ . The subsequent steps involve obtaining the determinant of the above adjacency matrix  $|\mathcal{R}_{\sigma_1, \sigma_2, \sigma_3}|_\eta$  and substituting it into the expression  $|\Psi_W^{(3)}\rangle = \frac{1}{\sqrt{N_3}} \sum_{\sigma_1, \dots, \sigma_3 = \{\uparrow, \downarrow\}} |\mathcal{R}_{\sigma_1, \sigma_2, \sigma_3}|_\eta |R_1 \sigma_1, R_2 \sigma_2, R_3 \sigma_3\rangle$  to generate the desired state  $|\Psi_W^{(3)}\rangle$ . The explicit expressions of the adjacency matrices with nonzero determinants are given

$$\mathcal{R}_{\uparrow, \downarrow, \downarrow} = \begin{pmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & r_{23} \\ 0 & r_{32} & r_{33} \end{pmatrix}, \quad \mathcal{R}_{\downarrow, \uparrow, \downarrow} = \begin{pmatrix} 0 & r_{12} & r_{13} \\ r_{21} & 0 & 0 \\ 0 & r_{32} & r_{33} \end{pmatrix}, \quad \mathcal{R}_{\downarrow, \downarrow, \uparrow} = \begin{pmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ r_{31} & 0 & 0 \end{pmatrix}. \quad (2.21)$$

To generate the state, we connect each of these matrices to perfect matching bigraphs [208]. For example, the first adjacency matrix corresponds to the scenario where we detect a qubit with an up pseudospin in the first region, while two qubits with down pseudospins are in the second and third regions, or vice versa. The other two adjacency matrices depict the other four perfect matches, as illustrated in Fig. 2.8). Finally, we can write the general form of the three-qubit state as follows:

$$|\Psi_W^{(3)}\rangle = \frac{1}{\sqrt{N_3}} \left( S_1 |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow\rangle + S_2 |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow\rangle + S_3 |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow\rangle \right), \quad (2.22)$$

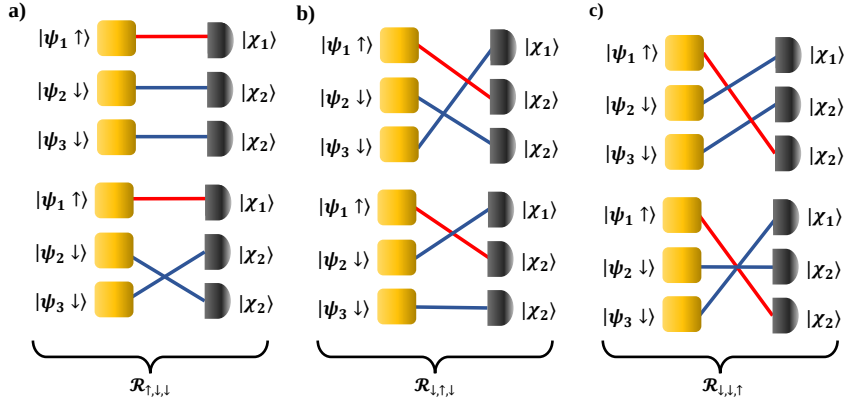


Figure 2.8: Perfect matching sub-bigraphs for generation of the three-particle  $W$  state [154, 155]. **a)** Represents the state  $|R_1 \uparrow, R_2 \downarrow, R_3 \downarrow\rangle$ . **b)** Represents the state  $|R_1 \downarrow, R_2 \uparrow, R_3 \downarrow\rangle$ . **c)** Represents the state  $|R_1 \downarrow, R_2 \downarrow, R_3 \uparrow\rangle$ .

where the probability amplitude coefficients  $S_1$ ,  $S_2$ , and  $S_3$  are obtained by determining the determinant of the above adjacency matrices and are specifically given as  $|\mathcal{R}_{\uparrow,\downarrow,\downarrow}|_\eta = S_1 = r_{11}r_{22}r_{33} + \eta r_{11}r_{23}r_{32}$ ,  $|\mathcal{R}_{\downarrow,\uparrow,\downarrow}|_\eta = S_2 = r_{13}r_{21}r_{32} + \eta r_{12}r_{21}r_{33}$  and  $|\mathcal{R}_{\downarrow,\downarrow,\uparrow}|_\eta = S_3 = r_{12}r_{21}r_{31} + \eta r_{13}r_{22}r_{31}$ , with the normalization factor  $N_3 = \sum_{k=1}^3 |S_k|^2$ .

Now, one can tune the digraph edge weights,  $r_{ij}$ , to generate the desired state. As discussed in the main manuscript, a specific choice of probability amplitudes is the complete digraph with equal weights, given as  $r_{ij} = \frac{1}{\sqrt{3}}$ , and bosonic statistics. This, in turn, results in the generation of the three-qubit  $W$  state given by  $|W^{(3)}\rangle = \frac{1}{\sqrt{3}}(|R_1 \uparrow, R_2 \downarrow, R_3 \downarrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow\rangle)$ , with the sLOCC probability of  $\mathcal{P}_C(|W^{(3)}\rangle) = 2/9$ . As discussed in the main manuscript, the complete graph is not the only structure that leads to the generation of the  $W$  state. Additionally, one can employ the star digraph to generate the  $W$  state. The star digraph configuration has an adjacency weight matrix with elements  $r_{1i} = \frac{1}{\sqrt{3}}$  (for  $i = 1, 2, 3$ ),  $r_{ii} = \frac{1}{\sqrt{2}}$  (for  $i = 2, 3$ ), and  $r_{i1} = \frac{\eta}{\sqrt{2}}$  (for  $i = 2, 3$ ), applicable to both fermions and bosons. The star digraph configuration leads to sLOCC probabilities  $\mathcal{P}_{star}(|W^{(3)}\rangle) = 1/5$  for bosons and  $\mathcal{P}_{star}(|W^{(3)}\rangle) = 1/3$  for fermions, respectively. Furthermore, one can adopt the Quantum Fourier Transform (QFT) deformation as well to generate the  $W$  state with sLOCC probabilities of  $\mathcal{P}_{QFT}(|W^{(3)}\rangle) = 1/9$  for bosons and  $\mathcal{P}_{QFT}(|W^{(3)}\rangle) = 1/3$  for fermions.

#### Four-qubit $W$ state

The next step involves considering a four-qubit system, which leads to diverse configurations in generations of quantum states. Similarly, we assume the initial state of a four-qubit in the pure product state as  $|\Psi_0^{(4)}\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \downarrow\rangle \otimes |\psi_3 \downarrow\rangle \otimes |\psi_4 \downarrow\rangle$ . Similar to the three-qubit case, we aim to come up with explicit expression of the adjacency matrices  $\mathcal{R}_{\sigma_1, \sigma_2, \sigma_3, \sigma_4}$ . The adjacency matrices, connected to perfect matching bigraphs with nonzero determinants are

$$\begin{aligned} \mathcal{R}_{\uparrow,\downarrow,\downarrow,\downarrow} &= \begin{pmatrix} r_{11} & 0 & 0 & 0 \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & r_{32} & r_{33} & r_{34} \\ 0 & r_{42} & r_{43} & r_{44} \end{pmatrix}, \quad \mathcal{R}_{\downarrow,\uparrow,\downarrow,\downarrow} = \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} \\ r_{21} & 0 & 0 & 0 \\ 0 & r_{32} & r_{33} & r_{34} \\ 0 & r_{42} & r_{43} & r_{44} \end{pmatrix}, \\ \mathcal{R}_{\downarrow,\downarrow,\uparrow,\downarrow} &= \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ r_{31} & 0 & 0 & 0 \\ 0 & r_{42} & r_{43} & r_{44} \end{pmatrix}, \quad \mathcal{R}_{\downarrow,\downarrow,\downarrow,\uparrow} = \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & r_{32} & r_{33} & r_{34} \\ r_{41} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.23)$$

After obtaining the determinant-like of the above adjacency matrices, we can the generated four-qubit state as

$$|\Psi_W^{(4)}\rangle = \frac{1}{\sqrt{N_4}} \left( S_1 |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \downarrow\rangle + S_2 |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle + S_3 |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + S_4 |R_1 \downarrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle \right), \quad (2.24)$$

where probability amplitude coefficients are  $S_1 = |\mathcal{R}_{\uparrow,\downarrow,\downarrow,\downarrow}|_\eta$ ,  $S_2 = |\mathcal{R}_{\downarrow,\uparrow,\downarrow,\downarrow}|_\eta$ ,  $S_3 = |\mathcal{R}_{\downarrow,\downarrow,\uparrow,\downarrow}|_\eta$ , and  $S_4 = |\mathcal{R}_{\downarrow,\downarrow,\downarrow,\uparrow}|_\eta$  with the normalization factor as  $N_4 = \sum_i |S_i|^2$ . Similarly, we can generate the four-qubit  $W$ -state  $|W^{(4)}\rangle = \frac{1}{2}(|R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle)$  using the complete (equal weights), star, and QFT digraph configurations with sLOCC probabilities of  $\mathcal{P}_C(|W^{(4)}\rangle) =$

| Graphs | Fidelity for bosons        | Fidelity for fermions      | Probability for bosons                          | Probability for fermions                        |
|--------|----------------------------|----------------------------|-------------------------------------------------|-------------------------------------------------|
|        | $F( W^{(3)}\rangle) = 1$   | $F( W^{(3)}\rangle) = 0$   | $\mathcal{P}_C( W^{(3)}\rangle) = 2/9$          | $\mathcal{P}_C( W^{(3)}\rangle) = 0$            |
|        | $F( W^{(3)}\rangle) = 1$   | $F( W^{(3)}\rangle) = 1$   | $\mathcal{P}_{star}( W^{(3)}\rangle) = 1/5$     | $\mathcal{P}_{star}( W^{(3)}\rangle) = 1/3$     |
|        | $F( W^{(3)}\rangle) = 1$   | $F( W^{(3)}\rangle) = 1$   | $\mathcal{P}_{QFT}( W^{(3)}\rangle) = 1/9$      | $\mathcal{P}_{QFT}( W^{(3)}\rangle) = 1/3$      |
|        | $F( W^{(4)}\rangle) = 1$   | $F( W^{(4)}\rangle) = 1$   | $\mathcal{P}_C( W^{(4)}\rangle) = 3/32$         | $\mathcal{P}_C( W^{(4)}\rangle) = 0$            |
|        | $F( W^{(4)}\rangle) = 1$   | $F( W^{(4)}\rangle) = 1$   | $\mathcal{P}_{star}( W^{(4)}\rangle) = 1/16$    | $\mathcal{P}_{star}( W^{(4)}\rangle) = 1/4$     |
|        | $F( W^{(4)}\rangle) = 0$   | $F( W^{(4)}\rangle) = 1$   | $\mathcal{P}_{QFT}( W^{(4)}\rangle) = 1/16$     | $\mathcal{P}_{QFT}( W^{(4)}\rangle) = 1/4$      |
|        | $F( W^{(4)}\rangle) = 3/4$ | $F( W^{(4)}\rangle) = 3/4$ | $\mathcal{P}_{chain}( W^{(4)}\rangle) = 0.1139$ | $\mathcal{P}_{chain}( W^{(4)}\rangle) = 0.1429$ |

Table 2.1: **Digraph configurations for generating  $W$  states with three and four qubits.** This table outlines diverse digraph configurations utilized to generate four-qubit  $W$  states involving identical bosonic and fermionic qubits. It includes their corresponding fidelities ( $F$ ) and sLOCC probabilities ( $\mathcal{P}$ ). The presence of  $\eta$ , highlighted in red, shows a  $\pi$  phase shift between interconnected nodes for fermions. Moreover, edge weights labeled with  $\omega_N$  denote  $e^{2i\pi/N}$ .

$3/32$  (for bosons),  $\mathcal{P}_{star}(|W^{(4)}\rangle) = 1/16$  (for bosons),  $\mathcal{P}_{star}(|W^{(4)}\rangle) = 1/4$  (for fermions), and  $\mathcal{P}_{QFT}(|W^{(4)}\rangle) = 1/4$  (for fermions). However, the state generated with QFT deformation for bosons is expressed as follows

$$|\Psi_{W-QFT}^{(4)}\rangle = \frac{1}{2} \left( |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \downarrow\rangle - |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle - |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle \right). \quad (2.25)$$

Although the state above is entangled among the four qubits, it does not have the form of the  $W$  state. One can explore alternative graph configurations, such as a closed-chained digraph. In this configuration, each node has three edges, including a self-loop. In this scenario, each qubit overlaps equally with its nearest neighbor nodes. Specifically, the qubit with a down pseudospins undergoes spatial deformation and takes the single-qubit state form  $|\varphi_{1D}\rangle = \frac{1}{\sqrt{3}}(|R_1 \uparrow\rangle + \eta|R_3 \uparrow\rangle + \eta|R_4 \uparrow\rangle)$ . The other qubits with an up pseudospins undergo deformation and take the form  $|\varphi_{2D}\rangle = \frac{1}{\sqrt{3}}(|R_2 \downarrow\rangle + |R_3 \downarrow\rangle + |R_4 \downarrow\rangle)$ ,  $|\varphi_{3D}\rangle = \frac{1}{\sqrt{3}}(|R_3 \downarrow\rangle + |R_2 \downarrow\rangle + |R_1 \downarrow\rangle)$ , and  $|\varphi_{4D}\rangle = \frac{1}{\sqrt{3}}(|R_4 \downarrow\rangle + |R_2 \downarrow\rangle + |R_1 \downarrow\rangle)$ . Within the choice of these  $r_{ij}$ , the generated state with closed-chain configuration takes the following form:

$$|\Psi_{W-chain}^{(4)}\rangle = \frac{1}{\sqrt{3}} \left( |R_1 \uparrow, R_2 \downarrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_4 \uparrow\rangle \right) \otimes |R_3 \downarrow\rangle, \quad (2.26)$$



| Graphs | Fidelity for bosons           | Fidelity for fermions      | Probability for bosons                          | Probability for fermions                        |
|--------|-------------------------------|----------------------------|-------------------------------------------------|-------------------------------------------------|
|        | $F( D^{(4)}\rangle) = 1$      | $F( D^{(4)}\rangle) = 1$   | $\mathcal{P}_C( D^{(4)}\rangle) = 0.0938$       | $\mathcal{P}_C( D^{(4)}\rangle) = 0$            |
|        | $F( D^{(4)}\rangle) = 4/9$    | $F( D^{(4)}\rangle) = 4/9$ | $\mathcal{P}_{star}( W^{(4)}\rangle) = 1/10$    | $\mathcal{P}_{star}( D^{(4)}\rangle) = 1/4$     |
|        | $F( D^{(4)}\rangle) = 0$      | $F( D^{(4)}\rangle) = 2/3$ | $\mathcal{P}_{QFT}( D^{(4)}\rangle) = 0.1250$   | $\mathcal{P}_{QFT}( D^{(4)}\rangle) = 1/4$      |
|        | $F( D^{(4)}\rangle) = 0.6429$ | $F( D^{(4)}\rangle) = 3/4$ | $\mathcal{P}_{chain}( D^{(4)}\rangle) = 0.1234$ | $\mathcal{P}_{chain}( D^{(4)}\rangle) = 0.1429$ |

Table 2.2: **Digraph configurations for generating four-qubit Dicke states.** The table summarizes digraph configurations utilized to generate four-qubit Dicke states, involving identical bosonic and fermionic qubits, along with their corresponding fidelities ( $F$ ) and sLOCC probabilities ( $\mathcal{P}$ ). The presence of  $\eta$ , highlighted in red, indicates a  $\pi$  phase shift between interconnected nodes for fermions. Moreover, edge weights labeled with  $\omega_N$  denote  $e^{2i\pi/N}$ .

The state mentioned above is a biproduct of a three-qubit state, in the form of a  $W$  state, and a single-qubit state. This indicates that only three qubits are entangled with a fidelity of  $F(|W^{(4)}\rangle) = 3/4$ , which does not violate the threshold of  $\frac{N-1}{N}$  for four qubits. In summary, we present the various generation configuration schemes as digraphs with their corresponding fidelity and success (sLOCC) probabilities for both bosonic and fermionic  $W$  states involving three and four qubits in Table B.

### C Symmetric Dicke state of four particles

We initially consider four identical qubits in product state form:  $|\Psi^{(4)}\rangle = |\psi_1 \uparrow\rangle \otimes |\psi_2 \downarrow\rangle \otimes |\psi_3 \uparrow\rangle \otimes |\psi_4 \downarrow\rangle$ . We can associate the adjacency matrix of a digraph with the three-generation steps: initialization, deformation, and projection, as follows:

$$\mathcal{R}_{\sigma_1, \dots, \sigma_4} = \begin{pmatrix} r_{11} \langle \sigma_1 | \uparrow \rangle & r_{12} \langle \sigma_1 | \downarrow \rangle & r_{13} \langle \sigma_1 | \uparrow \rangle & r_{14} \langle \sigma_1 | \downarrow \rangle \\ r_{21} \langle \sigma_2 | \uparrow \rangle & r_{22} \langle \sigma_2 | \downarrow \rangle & r_{23} \langle \sigma_2 | \uparrow \rangle & r_{24} \langle \sigma_2 | \downarrow \rangle \\ r_{31} \langle \sigma_3 | \uparrow \rangle & r_{32} \langle \sigma_3 | \downarrow \rangle & r_{33} \langle \sigma_3 | \uparrow \rangle & r_{34} \langle \sigma_3 | \downarrow \rangle \\ r_{41} \langle \sigma_4 | \uparrow \rangle & r_{42} \langle \sigma_4 | \downarrow \rangle & r_{43} \langle \sigma_4 | \uparrow \rangle & r_{44} \langle \sigma_4 | \downarrow \rangle \end{pmatrix}, \quad (2.27)$$

resulting in the general state of  $|\Psi_D^{(4)}\rangle = \frac{1}{\sqrt{N_4}} \sum_{\sigma_1, \dots, \sigma_4 = \{\uparrow, \downarrow\}} |\mathcal{R}_{\sigma_1, \dots, \sigma_4}|_{\eta} |R_1 \sigma_1, R_2 \sigma_2, R_3 \sigma_3, R_4 \sigma_4\rangle$  with  $N_4 = \sum_{\sigma_1, \dots, \sigma_4 = \{\uparrow, \downarrow\}} |\mathcal{R}_{\sigma_1, \dots, \sigma_4}|_{\eta}$  as a normalization factor. The non-zero determinants occur with six different combinations of pseudospin  $\{\sigma_1, \dots, \sigma_4\} = \{\uparrow, \downarrow\}$ , which result in the general form of the generated state, given in the following expression:

$$|\Psi_D^{(4)}\rangle = \frac{1}{\sqrt{N_4}} \left( S_1 |R_1 \uparrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + S_2 |R_1 \uparrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle + S_3 |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle + S_4 |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \uparrow\rangle + S_5 |R_1 \downarrow, R_2 \uparrow, R_3 \uparrow, R_4 \downarrow\rangle + S_6 |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \uparrow\rangle \right), \quad (2.28)$$

where probability amplitudes are given by the determinants of digraph adjacency matrices as  $S_1 = |\mathcal{R}_{\uparrow, \downarrow, \uparrow, \downarrow}|_{\eta}$ ,  $S_2 = |\mathcal{R}_{\uparrow, \uparrow, \downarrow, \downarrow}|_{\eta}$ ,  $S_3 = |\mathcal{R}_{\uparrow, \downarrow, \downarrow, \uparrow}|_{\eta}$ , and  $S_4 = |\mathcal{R}_{\downarrow, \uparrow, \downarrow, \uparrow}|_{\eta}$ ,  $S_5 = |\mathcal{R}_{\downarrow, \uparrow, \uparrow, \downarrow}|_{\eta}$ , and  $S_6 = |\mathcal{R}_{\downarrow, \downarrow, \uparrow, \uparrow}|_{\eta}$ .

As mentioned in the main part of the manuscript, the complete digraph is when qubits are distributed over all spatial regions with equal probability amplitudes of  $r_{ij} = \frac{1}{2}$  ( $i, j = 1, 2, 3, 4$ ). With such a generation setting, it leads to a symmetric Dicke state with a sLOCC probability of  $\mathcal{P}_C(|D^{(4)}\rangle) = 0.0938$  for bosons. However, this configuration does not lead to any Dicke state for fermions.

Alternatively, one may explore the on-site design utilizing the QFT operator to generate the symmetric Dicke state with four qubits. This involves defining  $r_{ij} = e^{i(i-1)(j-1)\frac{\pi}{2}}$  in the matrix adjacency as shown in Eq. 2.27. With this



choice, the resulting state is expressed as:

$$|\Psi_{D-QFT}^{(4)}\rangle = \frac{1}{2} \left( |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \uparrow, R_4 \downarrow\rangle - |R_1 \uparrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle - |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \uparrow\rangle \right). \quad (2.29)$$

As can be seen, the state above is missing the terms  $|R_1 \uparrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle$  and  $|R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \uparrow\rangle$ , thus not matching the symmetric Dicke state.

Also, let us consider the star digraph configuration as another example. With the star digraph, the generated state takes the following form:

$$|\Psi_{D-star}^{(4)}\rangle = \frac{1}{\sqrt{6}} \left( 2 |R_1 \uparrow, R_2 \downarrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \uparrow, \downarrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_4 \uparrow\rangle \right) \otimes |R_3 \uparrow\rangle, \quad (2.30)$$

with a sLOCC probability of  $\mathcal{P}_{star}(|D^{(4)}\rangle) = 0.1$  for bosons and  $\mathcal{P}_{star}(|D^{(4)}\rangle) = 0.25$  for fermions. However, as observed, the state is not a four-partite entangled Dicke state.

Furthermore, we can consider a closed chain configuration in digraph, which results in the following states for bosons and fermions, respectively:

$$|\Psi_{D-chain}^{(4)}\rangle = \frac{1}{\sqrt{21}} \left( 4 |R_1 \uparrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \uparrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle + |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \downarrow, R_4 \uparrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \uparrow\rangle \right), \quad (2.31)$$

$$|\Psi_{D-chain}^{(4)}\rangle = \frac{1}{\sqrt{5}} \left( |R_1 \uparrow, R_2 \downarrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \uparrow, R_2 \uparrow, R_3 \downarrow, R_4 \downarrow\rangle + |R_1 \uparrow, R_2 \downarrow, R_3 \downarrow, R_4 \uparrow\rangle + |R_1 \downarrow, R_2 \uparrow, R_3 \uparrow, R_4 \downarrow\rangle + |R_1 \downarrow, R_2 \downarrow, R_3 \uparrow, R_4 \uparrow\rangle \right). \quad (2.32)$$

Here, bosonic and fermionic states have fidelity values of  $F(|D^{(4)}\rangle) = 0.6429$  and  $F(|D^{(4)}\rangle) = 3/4$  and sLOCC probabilities of  $\mathcal{P}_{D-chain}(|D^{(4)}\rangle) = 0.1243$  and  $\mathcal{P}_{D-chain}(|D^{(4)}\rangle) = 0.1429$ , respectively. Finally, we summarize all generation settings and their associated fidelities and sLOCC probabilities for both bosons and fermions in Table C.

# Synthetic photonic lattices for single parameter phase estimation

## Abstract

In this project, I theoretically investigate time-bin entangled states for quantum metrology and the design of optimal measurements that saturate the Cramér-Rao Bound (CRB) in synthetic photonic lattices (SPLs) by implementing controlled discrete-time quantum walks (DTQWs). An SPL is experimentally realized using coupled fiber loops and a dynamical gate, where the synthetic dimension is defined by the time modes. Within this SPL, controlled DTQWs are employed to perform optimal positive operator-valued measurements (POVMs) for single-parameter quantum phase estimation. Through theoretical analysis, I identify and propose the optimal measurements necessary to experimentally demonstrate quantum-enhanced phase estimation using time-bin entangled states in SPLs.

It's worth mentioning that, since the paper for this part has not yet been published, I have included a report of the work here without the experimental results and figures.

This chapter is related to Publication 2 in the List of Publications.

## 3.1 Introduction

Accurately measuring physical quantities like field strength, force, or time is a key foundation of fundamental physics and practical applications such as metrology, satellites, and geopositioning.

Precise measurements are often made by converting physical quantities into phase shifts, which are then estimated using interferometric techniques [29, 30]. In phase estimation problems, the goal is to determine an unknown phase  $\theta$  embedded in a physical system  $\rho$  with the highest possible precision  $\Delta\theta$  [248]. Phase estimation is typically performed by making the system under analysis interact with a probe, in which phase information is encoded [24]. In the classical domain, the highest possible precision for classical light or  $N$  uncorrelated probes is set as  $N^{-\frac{1}{2}}$  [249], which is known as the shot noise limit (SNL) or standard quantum limit [29]. The accuracy of phase estimation can be enhanced by an extra factor  $N^{-\frac{1}{2}}$  when quantum mechanics is used, i.e., when non-classical systems are used as probes. In ideal conditions (i.e., in the absence of noise [? ]), the highest possible precision enabled by quantum mechanics is called the Heisenberg limit (HL) [29, 249]. A wide range of fields can benefit from quantum-enhanced phase estimation, including biology [250], atomic clocks [251, 252, 253], imaging [254, 49], and sensing [255, 256]. The most suitable and explored quantum states to implement quantum phase estimation are N00N states, which have the potential of saturating the HL [249, 257, 258]. N00N states are a class of entangled states in which all the  $N$  indistinguishable particles composing it can be found either in one mode (typically, the path) or in the other, but not in both. Photons are ideal systems to realize N00N states, as they feature several quantum modes (including path, orbital angular momentum, time, and frequency), which further facilitate access to higher dimensionalities per mode. Quantum phase estimation has been demonstrated thus far by using N00N photon states exhibiting entanglement in the path [259], polarization [260], and orbital angular momentum [261]. Yet, time-bin entangled states offer the advantage of implementing high-dimensional N00N states and quantum phase estimation in optical fibers, as they are robust against decoherence, noise, and polarization mode dispersion mechanisms in optical fibers [262, 263, 264, 265].

In addition to the realization of the probe state, it is necessary to optimize the set of quantum measurements, which in general is described by quantum mechanical positive-operator-valued measures (POVMs), leading to the most accurate and precise estimation of, e.g., unknown phases. Given a set of POVMs, the lower bound for the highest achievable precision for the selected probe state is provided by the Cramer-Rao bound (CRB) [31]. Optimizing over the set of measurements ultimately reduces the CRB to quantum Cramer-Rao bound (QCRB) [24].

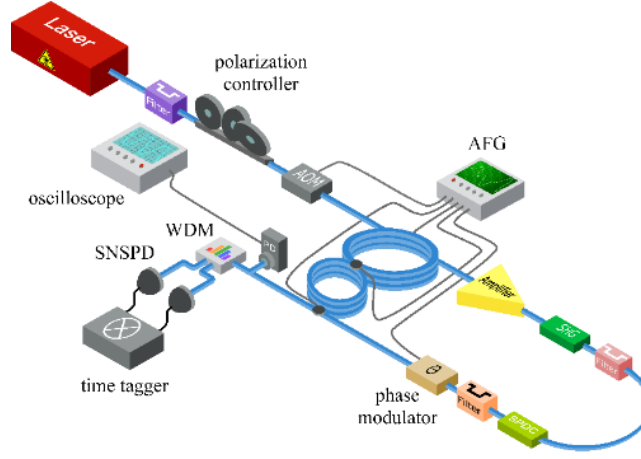


Figure 3.1: A coupler connects two polarization-maintaining fiber loops of lengths 120 m (600 ns) for the long loop and 100 m (500 ns) for the short loop. An arbitrary function generator (AFG) controls the couplers transmittance dynamically. The setup also includes a pulsed laser, periodically poled lithium niobate (PPLN) waveguides to generate photons via cascaded second harmonic generation (SHG) and spontaneous parametric down-conversion (SPDC) processes, a phase modulator, a wavelength division multiplexer (WDM), an oscilloscope, a photodetector (PD), and two superconducting nanowire single-photon detectors (SNSPDs). The oscilloscope and PDs are used in the classical part of the experiment [74].

Quantum walks (QWs), the quantum-mechanical analogue of classical random walks, enable a variety of applications, including universal quantum computing models [37], universal quantum measurement devices [38], and quantum simulators [266]. An interesting scenario is given by discrete-time quantum walks (DTQWs) [40] on a line, where the photons evolution on the lattice depends on the quantum state of the coin (i.e., on the quantum mode). Controlling the coins quantum mode enables the control over the walkers (the photons) and, in turn, over the system's evolution, and as a consequence, allows the design of optimal POVMs. Therefore, QW has been used to implement POVM measurements with both two-[38] and high-[266, 40] dimensional quantum systems (qubits and qudits, respectively), as well as collective POVMs for quantum state tomography [267, 268] and for unambiguous state discrimination problems [269]. However, the theoretical study of quantum phase estimation based on time-entangled photon states under DTQW evolution, as well as their combined use for optimal POVMs, is still missing and challenging. In this PhD project, I theoretically demonstrated that DTQW can provide a suitable framework to perform optimal POVMs, which can saturate the QCRB bound and enhance the sensitivity and accuracy in quantum phase estimation problems.

### 3.2 Temporal synthetic photonic lattice using a coupled-fiber loop system

The experimental scheme for realizing the SPL is presented in Fig. 3.1. The setup consisted of two modules: one module for biphoton state preparation, comprising the coupled-fiber loops and a cascade of two periodically poled lithium niobate (PPLN) waveguides; one module for quantum state measurements, comprising the same coupled-fiber loop and two superconducting nanowire single-photon detectors (SNSPDs). The fiber loop consisted of two polarization-maintaining fibers of lengths 120 m (600 ns) and 100 m (500 ns) for the long and the short loops, respectively. Due to this length difference, a time delay of 100 ns occurs when a pulse sequence propagates within the system. The systems also consisted of three programmable couplers, two of which control the light flow in and out of the short and long loops, while the third (henceforth referred to as the dynamical coupler) controls the light flow between the two loops. In the pulse preparation stage, a pulse laser (10 MHz repetition rate), centered at 1549.7 nm, was sent into the fiber-loop system. To have only a single pulse in the fiber-loop, its repetition rate and bandwidth were reduced to 181.8 kHz and 0.208 nm using an acousto-optic modulator (AOM) and a tunable spectral filter, respectively. Afterward, the pulse injected into one loop was split by the dynamical coupler into two pulses: one propagating through the long loop and the other through the short loop. The pulses were then split again at the dynamical coupler, with a 100 ns time delay, after a second roundtrip. Continuing this process eventually leads to the generation of a burst of time-delayed pulses. In the next step, the burst was spectrally shaped by utilizing a 120 dB short-pass filter before pumping the PPLNs. The first PPLN is responsible for the second harmonic generation (SHG). The second PPLN generates time-bin entangled photons at a central wavelength of 1549.7 nm (that is, in the telecommunication C-band) through spontaneous parametric down-conversion (SPDC). In second quantization notation, the generated two-photon time-bin entangled photon state can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|2, 0\rangle_{t_1, t_2} + |0, 2\rangle_{t_1, t_2}). \quad (3.1)$$

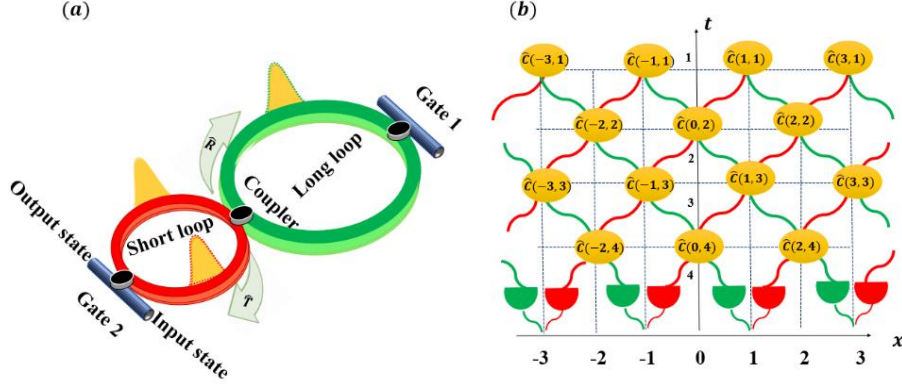


Figure 3.2: The coupled fiber-loop system consists of long and short loops (shown by green and red colors, respectively) connected by an ultra-fast optical dynamical coupler and  $\tau$  is the time delay among adjacent pulses. Two dynamical gates, Gate 1 and Gate 2, control the in-out flow of light in the long loop and short loop. (b) Typical representation of fully controllable QW network with dynamical coupler  $\hat{C}(x, t)$  where is a function of the discrete synthetic position  $x$  (arrival time of photons) in the lattice and the number of round trips  $t$ .

Here, the quantum state  $|2, 0\rangle_{t_1, t_2}$  ( $|0, 2\rangle_{t_1, t_2}$ ) describes a situation in which 2 photons (signal and idler) are in the time mode  $t_1(t_2)$  and 0 photons in time mode  $t_2(t_1)$ .

### 3.3 Realization of discrete-time quantum walk in synthetic photonic lattice

In this project, we experimentally realized a one-dimensional SPL via the coupled fiber-loop system (see Fig. 3.2). Due to the length difference  $\delta L$  between the two loops, a time delay occurs when a burst of pulses propagates within the system. Each round trip traveled by the pulse is equivalent to a discrete-time step  $t$  and the different arrival time of the pulse/photon is the synthetic position  $x$  [270]. The SPL provides a suitable platform for implementing controlled discrete-time quantum walks (CDTQWs) on the line [40, 271]. By employing a dynamic light-guiding operation via a programmable coupler, we can indeed control the evolution of the measured system. In general, the DTQW model is defined on the joint Hilbert space  $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$  of the position  $\mathcal{H}_p$  and coin  $\mathcal{H}_c$  spaces of the walker. The state of the system is thus defined by  $|\psi_0\rangle = \sum_k c_k |k\rangle$ , where  $c_k$  is probability amplitude to find a photon in mode  $k$ . Each mode consists of two degrees of freedom  $|x, c\rangle$ , the synthetic position of particles ( $|x\rangle = \cdots | -1\rangle, |0\rangle, |1\rangle, \cdots$ ), and the coin ( $|c\rangle = |L\rangle, |S\rangle$ ), where  $|L\rangle$  and  $|S\rangle$  stand for the short and the long loops, respectively. The one-step unitary evolution operator is  $\hat{U}(t) = \hat{S} \cdot \hat{C}(x, t)$ . The conditional shift operator in this case is given by [40]

$$\hat{S} = \sum_k |x+1, L\rangle \langle x, L| + |x-1, S\rangle \langle x, S| \quad (3.2)$$

and the coin operator for the dynamical coupler in each synthetic position and time-step [40]

$$\hat{C}(x, t) = \sum_x \begin{pmatrix} \cos[\theta(x, t)] & -i \sin[\theta(x, t)] \\ -i \sin[\theta(x, t)] & \cos[\theta(x, t)] \end{pmatrix} \quad (3.3)$$

which  $\theta(x, t)$  determines the transmission and reflection coefficients. By setting  $\theta(x, t) = \frac{\pi}{2}$ , the coin operator reduces to full reflection  $\hat{C}(x, t) = \hat{R}$ , and corresponds to sending the photon from the short to the long loop (or vice versa). Setting  $\theta(x, t) = 0$  the coin operator reduces to full transmission  $\hat{C}(x, t) = \hat{T}$  and corresponds to keeping the photon in the same loop. By setting  $\theta(x, t) = \frac{\pi}{4}$ , the coin operator results in the Fourier operator and corresponds to 50 : 50 splitting ratio. In the case that two quantum walkers (here, signal and idler photons) are involved, the single-step unitary evolution is given by  $\hat{U}_t = \otimes_{i=1}^2 \hat{U}_i(t)$ , where  $\hat{U}_i(t)$  is the unitary operator for the  $i$ -th step.

### 3.4 Phase estimation in synthetic photonic lattice

#### 3.4.1 Light pulse

In this project, a SPL based on a coupled-fiber loop system was initially used to implement phase estimation with classical light, which can be described by a coherent state in a similar manner as a single photon propagating in the coupled fiber-loop system. The measured light intensities of two outputs are proportional to the detection probabilities of a single photon at the different output ports of the coupled fiber-loop interferometer [272]. Using the second quantization notation, the

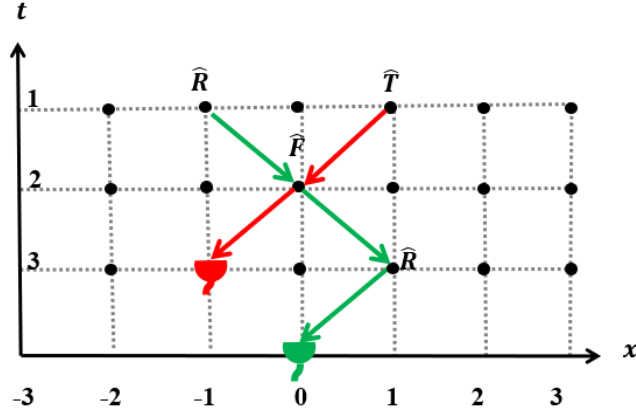


Figure 3.3: The schematic of controlled DTQW, which corresponds to performing optimal POVM measurements for single parameter phase estimation. In the first round trip, the coin operator is set to act as transmission  $\hat{T}$  and reflection,  $\hat{R}$  respectively, in the synthetic positions 1 and  $-1$   $\hat{T}\hat{R}$ . In the second-round trip, the coin operator is set to act as Fourier coin  $\hat{F}$  in the synthetic 0. In the detection level (third-round trip), the coin operator is set as reflection  $\hat{R}$  in the synthetic position  $x = 1$ , and the synthetic position  $x = -1$  is set as the identity  $\hat{I}$ .

coherent state of classical light entering the coupled-fiber loop system reads  $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle_{t_1, t_2} + |0, 1\rangle_{t_1, t_2})$ . The classical light pulse undergoes electro-optic phase modulation (PM) to encode the unknown phase  $\phi$  between the two-time bins, as described by the unitary evolution  $\hat{U}_{PM} = e^{i\phi\hat{n}_{t_2}}$ , where  $\hat{n}_{t_2}$  is the number operator along the mode  $t_2$ . After phase encoding, the light pulse can be written as

$$|\psi_1\rangle = \hat{U}_{PM} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle_{t_1, t_2} + e^{i\phi} |0, 1\rangle_{t_1, t_2}) \quad (3.4)$$

If the state remains pure (i.e., it does not interact with the environment), the Fisher information reaches the maximum value  $F_Q = 1$ . To achieve the highest precision with classical light, optimal POVMs must be performed on the probe. The phase-shifted state,  $|\psi_1\rangle$  which corresponds to the state  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |S\rangle + e^{i\phi} |1\rangle \otimes |S\rangle)$  in QW evolution, is then sent into the coupled fiber-loop to perform optimal POVMs using a controlled DTQW scheme. As illustrated in Fig. 3.3, in the first roundtrip, we set the coin operator as  $\hat{C}(-1, 1) = \hat{R}$ , and  $\hat{C}(1, 1) = \hat{T}$ . These describe a situation in which the photon in the earlier time bin ( $t_1$ ) is reflected in the position  $x = -1$ , while the photon in the later time bin ( $t_2$ ) is transmitted in the position  $x = 1$ . In the second roundtrip, we set the coin to perform the Fourier  $\hat{C}(0, 2) = \hat{F}$  at the position  $x = 0$ . At the detection level, light in the short loop travels half the length of the loop and is then extracted through gate 2. The light in the long loop travels the entire length of the loop. Afterward, the coupler acts as a reflector, i.e.,  $\hat{C}(1, 3) = \hat{R}$  for the light in the long loop, and then travels half the length of the short loop and is then extracted and subsequently detected. The evolved state after two roundtrips reads

$$|\psi_2\rangle = \frac{1}{2}((e^{i\phi} - 1) |1\rangle |L\rangle + i(1 + e^{i\phi}) |-1\rangle |S\rangle) \quad (3.5)$$

According to the optimal design shown in Fig. 3.3, the evolved state at the detection level becomes

$$|\psi_3\rangle = \frac{1}{2}(-i(e^{i\phi} - 1) |0\rangle |S\rangle + i(1 + e^{i\phi}) |-1\rangle |S\rangle) \quad (3.6)$$

Hereafter, with a set of projectors  $M_i = |0\rangle |S\rangle \langle 0| \langle S|, |-1\rangle |S\rangle \langle -1| \langle S|$ , satisfied the condition  $\sum_i M_i = 1$ , I derived the probabilities  $P_S^{(0)} = \frac{1-\cos\phi}{2}$ , and  $P_S^{(-1)} = \frac{1+\cos\phi}{2}$  detected a single photon in the short and long loops, respectively. Using the expression for the Fisher information  $F_c(\phi) = \sum_i \frac{[\partial_\phi p_i(\phi)]^2}{p_i(\phi)}$ , it results in  $F_c = F_Q = 1$ , which achieves the maximum value allowed by the QCRB and verifies the performance of the implemented SPL.

### 3.4.2 Time-bin entangled state

In this project, I theoretically demonstrated how phase estimation can be enhanced via quantum mechanics using a time-bin entangled state. To this end, I considered a two-photon number state (i.e.,  $N = 2$ ) as a quantum probe propagating in the SPL. The time-bin entangled state can be defined as in Eq. (2), with  $|\psi_0\rangle = |\psi\rangle$ . Similar to the classical case, the quantum state after phase encoding takes the form

$$|\psi_1\rangle = \hat{U}_{PM} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|2, 0\rangle_{t_1, t_2} + e^{2i\phi} |0, 2\rangle_{t_1, t_2}) \quad (3.7)$$

phase estimation is thus enhanced by a factor of two in comparison to the SNL. Such accuracy can be achieved when optimal POVMs (corresponding to the optimal design in Fig. 3.3 ) are performed on the state  $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|-1, -1\rangle \otimes |S, S\rangle + e^{2i\phi} |1, 1\rangle \otimes |S\rangle)$ . The evolved state after two roundtrips is

$$|\psi_2\rangle = \frac{1}{2\sqrt{2}}(-(1+e^{2i\phi})|1, 1\rangle|L, L\rangle + (1+e^{2i\phi})|-1, -1\rangle|S, S\rangle - i(1-e^{2i\phi})|1, -1\rangle|L, S\rangle - i(1-e^{2i\phi})|-1, 1\rangle|S, L\rangle) \quad (3.8)$$

At the detection level, the evolved state becomes

$$|\psi_3\rangle = \frac{1}{2\sqrt{2}}((1+e^{2i\phi})|0, 0\rangle|S, S\rangle + (1+e^{2i\phi})|-1, -1\rangle|S, S\rangle - (1-e^{2i\phi})|0, -1\rangle|S, S\rangle - (1-e^{2i\phi})|-1, 0\rangle|S, S\rangle) \quad (3.9)$$

where both photons (signal and idler) can be found in the short loop. Afterward, we consider the set of projective measurements  $M_1 = |0, 0\rangle|S, S\rangle\langle 0, 0| \langle S, S|$ ,  $M_2 = |-1, -1\rangle|S, S\rangle\langle -1, -1| \langle -S, -S|$ ,  $M_3 = |0, -1\rangle|S, S\rangle\langle 0, -1| \langle S, S|$ ,  $M_4 = |-1, 0\rangle|S, S\rangle\langle -1, 0| \langle S, S|$ , satisfying the condition  $\sum_i M_i = I$ . These projective measurements will lead to the probabilities  $P_{(S,S)}^{(|0,0\rangle)} = P_{(S,S)}^{(|-1,-1\rangle)} = (1 - \cos(2\phi))/4$  of detecting both photons in the short loop in synthetic positions 0 and -1 and  $P_{(S,S)}^{(|0,-1\rangle)} = P_{(S,S)}^{(|-1,0\rangle)} = (1 + \cos(2\phi))/4$  of detecting one photon in synthetic position 0 and the other photon in synthetic position -1. The Fisher information associated with these probabilities in the lossless case is obtained, and it equals the maximum value of  $F_C = F_Q = 4$ , which meets the QCRB limit and confirms the effectiveness of the implemented SPL.

### 3.5 Conclusion

Recently, a synthetic photonic lattice (SPL) based on a dynamically coupled fiber-loop system has been used as a platform for generating and processing tasks such as bi-photon state preparation, manipulation, and the realization of time-bin two- and four-level interference between two photons [74]. In this project, I provided a theoretical model for implementing quantum phase estimation using an SPL with time-bin entangled states as probe states. Exploiting the flow of light in an SPL to simulate controlled quantum walks (QWs), I demonstrated their use for optimal positive operator-valued measurements (POVMs). Our model ensures that these POVMs meet the quantum Cramér-Rao bound (QCRB). This result conclusively proves that our SPL is a promising platform for various quantum information protocols, such as multi-parameter phase estimation [77], quantum state tomography [78], and other phenomena related to QWs in quasi-PT-symmetry, including quantum interference [79, 80], quantum state discrimination [81], and ultrafast quantum transformation [82, 83].

# Enhancing nonclassical bosonic correlations in a quantum walk network through experimental control of disorder

## Abstract

The presence of disorder and inhomogeneities in quantum networks has often been unexpectedly beneficial for both quantum and classical resources. Here, we experimentally construct a controllable inhomogeneous quantum walk dynamics in which we can investigate the effect of coherent disorder on the quantum correlations between two indistinguishable photons. Through a suitable configuration of the disorder, we observe that quantum correlations of biphotons are enhanced between two modes of the network, compared to an ordered quantum walk. Different configurations of disorder can steer the system towards different realizations of such an enhancement, thus allowing spatial and temporal manipulation of quantum correlations.

This chapter is related to Publication 3 in the List of Publications.

## 4.1 Introduction

A thorough characterization of genuine quantum traits is crucial to understand the boundary between classical and quantum phenomena [273], and to perform quantum information tasks [274]. To this aim, several quantification methods have been introduced to faithfully identify the presence of quantum (nonclassical) resources such as entanglement [11], coherence [275], discord [276], joint measurability [277], steering [278], or thermal operations [279] exhibited in composite systems.

Indistinguishability of quantum identical particles [280, 12] has also revealed as a useful nonclassical resource. Particles are so-called indistinguishable if they are in the same mode characterized via two-particle interference [281]. From a broader viewpoint, the indistinguishability concept is related to a given set of quantum measurements [167], which in fact indistinguishability plays a direct role in activating quantum processing such as: many-body interference [281, 282], entanglement generation [280, 12, 283, 284, 14, 13], quantum teleportation [13], quantum metrology [25, 29], quantum coherence [285, 286, 149], quantumness protection [166, 167, 168], quantum key distribution [287, 288], and high complexity exploited by Boson Sampling algorithms [289, 159].

In this context, it is important to understand how quantum features based on indistinguishability behave in a dynamical framework, specifically in the case of bosons propagating through a non-homogeneous system. A suitable theoretical platform to perform such a study is represented by Quantum Walk (QW), which provides a very general coherent propagation model: at variance with classical Random Walks, QWs are able to preserve genuine nonclassical features such as superposition, interference, and entanglement [271, 290, 40]. QWs provide powerful models to describe energy transport phenomena in different types of systems like, photosynthetic complexes [36, 35], or solid state ones, as in the case of Luttinger Liquids [291].

It has been shown that adjustable disorder plays a significant role in the evolution of quantum walkers in which the ballistic growth can become anomalous, classical, or localized [292, 293, 294, 295]. The dynamics of a quantum walker is intimately connected to its nonclassical features. This has also highlighted the role of quantum coherence in biological systems [34, 296]. The way quantum-correlated walkers, realized by photon pairs, evolve in a homogeneous optical lattice has been studied, in which distinguishable or indistinguishable photons exhibit different behavior [297, 298, 299, 300, 301]. Moreover, the spreading pattern of the quantum walker(s) can be modified through various types of disorders

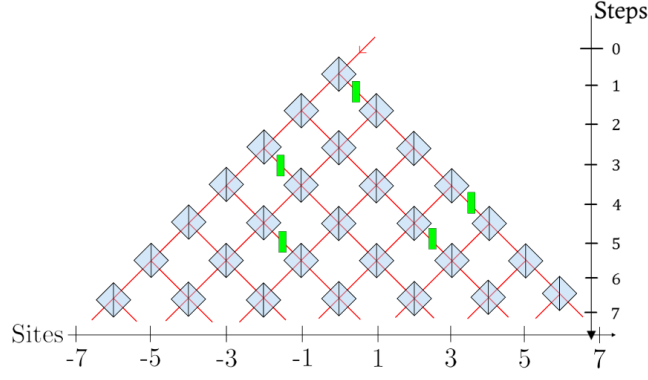


Figure 4.1: **Network representation of a disordered QW according to  $p$ -diluted model.** The green bars stand for the presence of  $\pi$  phases on the path which can be added to or removed from the network.

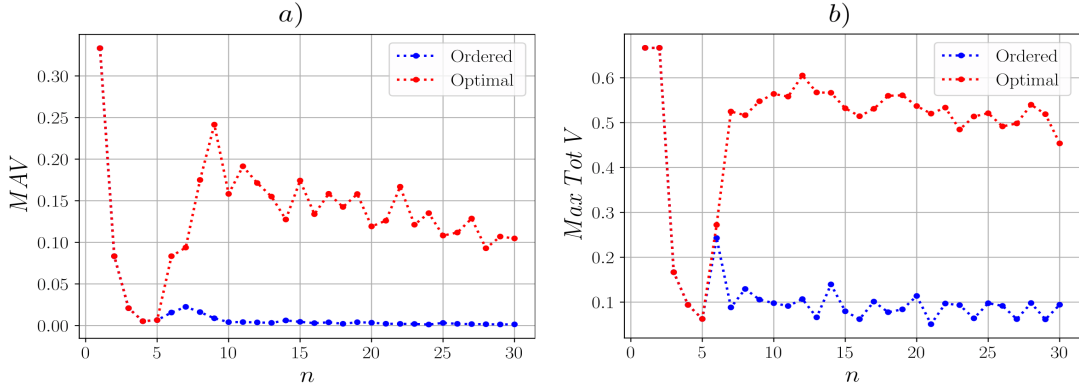


Figure 4.2: **Numerical simulation compression between order and optimal disordered QW.** **a)** Maximum Achievable Violation (MAV) and **b)** maximum achievable Total Violation versus the number of steps (discrete time)  $t$  for the ordered (blue circles) and optimal disordered (red squares) QW.

[302, 293, 295, 303]. For a large variety of systems, the disorder plays a detrimental role in Quantum Walk dynamics because it drives the system into decoherence [84, 292]. Contrarily, for some systems, the disorder can enhance physical properties such as coherent transport [36], quantum algorithm speedup [85], and quantum correlations [86, 87, 88, 89].

At a variance with previous studies so far [89], it remains to find strategies via disorder control for enriching two-particle correlations in separated modes of a quantum walk network. Here we fill this gap by injecting two indistinguishable photons (biphoton) in a one-dimensional inhomogeneous Discrete Time Quantum Walk (1D DTQW) and experimentally adjusting specific disorder configurations, while keeping the system isolated. Moreover, we numerically show that averaging over randomized disorder configurations results in diminishing the initial biphoton quantum correlations (details are reported in the Supplementary Material (SM)). Our findings unambiguously prove that the presence of disorder assists in the dynamical enhancement of biphoton quantum correlations in a controllable fashion, paving the way to its potential employment in quantum information scenarios.

## 4.2 Theoretical model

The QW model consists of walker(s) that move along the discretized sites of a line, as can be seen in Fig. 4.1. In general, the state of the system can always be written as a superposition of the QW modes  $|\Psi(t)\rangle = \sum_k \alpha_k(t) |k\rangle$ , where each mode  $|k\rangle := |x\rangle |\sigma\rangle$  is defined by both position  $|x\rangle$  and its coin  $|\sigma\rangle = \{|L\rangle, |R\rangle\}$ , and amplitudes  $\alpha_k(t)$  depend on the past evolution of the walker. Therefore, the single step evolution can be written as

$$|\Psi(t+1)\rangle = \sum_k e^{i\phi_k(t)} \hat{U} \alpha_k(t) |k\rangle, \quad (4.1)$$

where  $\hat{U} = \hat{S} \cdot (\hat{I} \otimes \hat{C})$  is the one-step evolution operator on mode  $|k\rangle$  with  $\hat{S} = \sum_x |x+1\rangle \langle x| \otimes |L\rangle \langle L| + |x-1\rangle \langle x| \otimes |R\rangle \langle R|$  being the shift operator, that moves the walker according to the coin state,  $\hat{I}$  the position identity operator and  $\hat{C} = \frac{1}{\sqrt{2}} (|L\rangle \langle L| + |L\rangle \langle R| + |R\rangle \langle L| - |R\rangle \langle R|)$  the Hadamard coin operator. Here, step-position dependent phases  $\phi_k(t)$ , out of two choices 0 or  $\pi$ , are responsible for the dynamical disorder that the quantum walker experiences. Now,



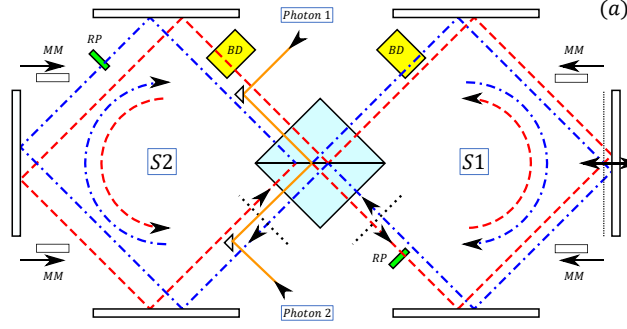


Figure 4.3: **2D sketch of the experimental setup.** BS: beam splitter, BD: beam displacer, RP: rotating glass plates, MM: moving mirror, S1: Sagnac Interferometer for odd steps, S2: Sagnac Interferometer for even steps. Blue and red beams circulate in opposite directions and impinge on the BS in the same horizontal point but at different heights along the  $z$  direction, due to the effect of BDs.

the question arises on when and where we should apply the phase shift  $\pi$ , represented by green bars in Fig. 4.1, that enhances the quantum correlations of biphoton.

To study the effect of inhomogeneous disorder on nonclassical bosonic correlations in a QW dynamics, we consider two indistinguishable photon walkers as input. This choice is strategical in that, unlike states of distinguishable photons, the state of an indistinguishable photon pair (biphoton)  $|\Psi^{(2)}\rangle = |k_1, k_2\rangle (|k_i\rangle = |x_i\rangle |\sigma_i\rangle)$  is not separable and exhibits intrinsic quantum correlations [4, 5]. The two-particle evolution is then obtained by applying  $\hat{U} \otimes \hat{U}$  to the state expressed in the standard symmetrized first quantization formalism. Inspired by the classical intensity correlation of light, one way to measure the nonclassicality of the correlation between two detected outputs is by violating the Cauchy-Schwarz inequality [297, 298]

$$V_{ij} = \frac{2}{3} \sqrt{\Gamma_{ii}\Gamma_{jj}} - \Gamma_{ij} < 0, \quad (4.2)$$

where  $\Gamma_{ij}$  is the probability of finding a photon in mode  $i$  and the other one in  $j$ , namely the probability of measuring a coincidence between modes  $i$  and  $j$ . Inequality (4.5), in fact, stands for classically correlated light and its violation is assumed to witness the presence of quantum correlations and a sign of photon bunching [297, 298, 301].

### 4.3 Numerical Results

Preliminary simulations were carried on in an ideal  $p$ -diluted framework: two indistinguishable photons travelling a bulk-optics 1D DTQW, provided with space-time disorder. To find the optimal phase map, we simulate the evolution with  $10^4$  different phase maps for each step number  $t$  up to 30 steps, for a total amount  $3 \cdot 10^5$  of explored configurations. We compute the  $V_{ij}$  between each pair of output modes for each simulated probability distribution, obtaining the corresponding violation matrices. Then, the entire set of  $V_{ij}$  elements corresponding to any combination of  $i$  and  $j$  modes are reported. Now, we can compare the values for any pair  $(i, j)$  and each phase map at a given step, to find the Maximum Achievable Violation (MAV), i.e. the maximum positive value of  $V_{ij}$  which could be achieved at that given step (we name this as optimal two-mode quantum correlations). Simulation results are shown in Fig. 4.2 **a)** and **b)**, respectively. We report the MAV as a function of the number of steps, and the step-wise trend of the maximum achievable Total Violation, defined as the sum of all the positive values  $V_{ij}$  of the considered violation matrix (named here as total quantum correlations). The analysis highlights that the disorder helps to retrieve quantum correlations after a specific step of QW. Hence, the MAV starts growing compared to the standard ordered case, reaching a peak at the 9th step in our framework. Similar, but not identical, results were obtained for the system's maximum Total Violation, suggesting that the two quantities are related but not bound to be maximized together. From numerical results, we can conclude that disorder, acting through mere interference, significantly modifies the evolution of the walker, not only reshaping the probability distribution but also affecting the amount of quantum correlations between the photon. As a consequence, disorder may enable enhancement of the quantum correlations of a bipartite system. It is worth noting that no violations are observed, whatever the phase map, when a state of distinguishable photons is employed.

### 4.4 Experimental Setup

The experimental setup, designed to test the theoretical predictions, consists of a bulk-optics multipass double Sagnac Interferometer (SI), already exploited in a previous experiment [295], in which inhomogeneities, described by the phases  $\phi_k(t)$ , can be addressed independently both in step number and evolution mode. The bulk-optics setup shown in Fig. 4.3

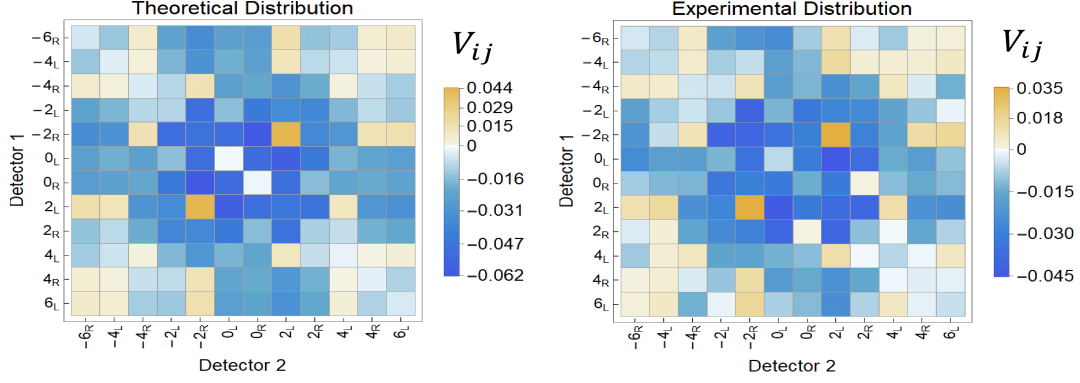


Figure 4.4: **Comparison between the theoretical and experimental violation matrices at the 6th step for an enhancing disorder configuration.** Numerical simulations are performed taking into account experimental parameters. The expected peak in the value of  $V$  is experimentally found, while the measured output coincidence distributions reach globally a similarity value of  $97.5(\pm 0.2)\%$ .

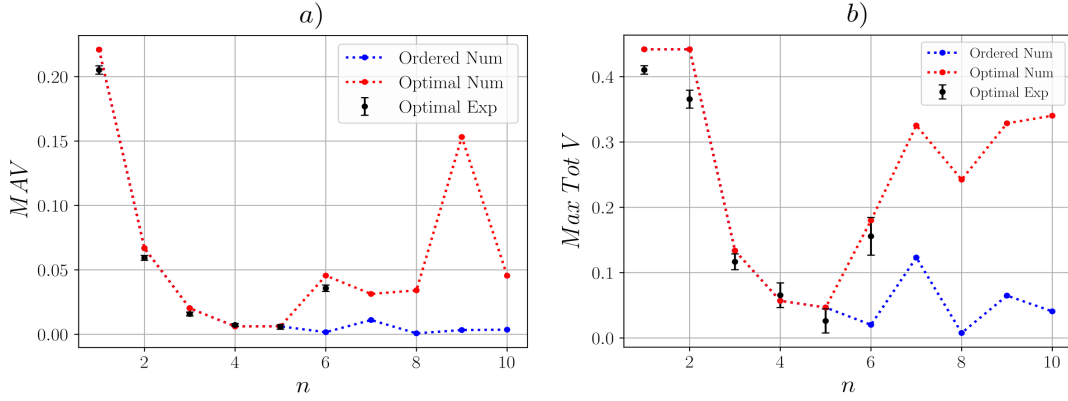


Figure 4.5: **Experimental results compression between the order and the enhanced disordered configuration QW.** Experimental results for (dark dot) **a)** Maximum Achievable Violation (MAV) and **b)** maximum achievable Total Violation versus the number of steps. The trends are compared with the simulation for optimal disorder (red squares) and ordered evolution (blue circles). The expected results are obtained by numerical simulations performed accounting for experimental parameters, so that the theoretical trends show some discrepancies with respect to the ones in Fig. 4.2.

is analogous to a chain of intrinsically phase-stable Mach-Zehnder Interferometers (MZIs), each of them provided with an individually tunable phase shifting. The additional exploitation of the  $z$  direction allows to effectively realize a Beam Splitter (BS) network, which reproduces a 1D DTQW dynamics. Thanks to the particular geometry of the implementation, each propagation mode of the QW at the given step has a specific position in the plane transverse to the propagation direction. Therefore, the phase shifts can be independently addressed in each mesh of the QW by the simple insertion of Rotating Glass Plates (RP) along the propagation path. The output state can be measured through a set of Moving Mirrors (MMs), intercepting, and extracting from the setup only modes of the selected step  $t_j$ . Also, the previous propagation steps  $t < t_j$  are not affected in any way by the measurement procedure. The extracted radiation is then coupled on a single-mode fiber and measured (for further details on the setup see Ref. [304, 295]). Using couplers to collect the extracted photons, we can measure coincidences between all possible modes at each step and experimentally reconstruct the two-photon probability distribution.

Photon pairs are made indistinguishable outside the setup up to a HOM visibility  $v = 85\%$  and then injected together in the network through the two different input ports of the BS. In this way, there is no need for heralding procedures, so losses are crucially reduced, making it possible to measure the two-particle distribution while keeping a high tunability of network parameters, even up to the 6th steps.

## 4.5 Experimental Results

To experimentally verify disorder-induced changes in the violation matrix, we measure ordered and disorder evolution QW output distributions, despite the absence of any unambiguous increase in the amount of violation (see SM). However,

based on the simulation study displayed in Fig. 4.2, the first quantum correlation enhancement, due to disorder, shall occur at the 6th step of QW. Therefore, the output violation matrices for optimal phase maps are measured up to the 6th step. The phase map with phase shifters at step  $t = 5$ , position  $x = -2$  with coin  $\sigma = L$  and  $x = 2$  with coin  $\sigma = R$  are  $\pi$  and all the others set to zero are realized and the measurements are finally performed; the corresponding experimental output violation matrix is shown in Fig. 4.4, compared with the expected one, where the mode  $|k\rangle = |x\rangle |\sigma\rangle$  is indicated by  $x_\sigma$ . A strong quantum correlation peak appears at modes  $(2_L, -2_R)$  and  $(-2_R, -2_L)$  confirming the expectation. Besides, it suggests that maximum quantum correlations can be induced by disorder only in correlations between "central" output mode pairs: in fact, photons emerging from these modes will have the most interfering paths, getting to be more affected by inhomogeneities along the evolution to improve more powerful and effective quantum correlation of pair (see SM for further numerical analysis on the subject).

As a further relevant result, the experimental step-wise trend for MAV is shown in (Fig. 4.5a comparison with the expected optimal one obtained by numerical analysis, taking into account the experimental limitations. They are plotted together with the ordered case trend to provide a clear display of the beneficial effect of the non-homogeneous evolution. Theoretical patterns are shown up to the 10th step, as a reference. The corresponding trends for the Total Violation computed over the same output distributions are also reported in Fig. 4.5b. Simulations of the MAV values in the ordered case show that the quantum correlation spread in a homogeneous network, so that the values of  $V_{ij}$  and its Total Violation are going to decline as the propagation proceeds. However, as can be seen in Fig. 4.5a, the inhomogeneity enriches the quantum correlation between two indistinguishable photons at the given modes. Experimental evidence, reported in Fig. 4.5a, show that the very same configuration enhances the total quantum correlation of the quantum walk. Discrepancies between theoretical trends shown in Fig. 4.5 and in Fig. 4.2 are due to asymmetries in the experimental setup, specifically the exploitation of an unbalanced BS ( $R = 45/T = 55$ ). Experimental errors are derived from the Poissonian statistics of the measured coincidences. Deviations from the expected results are mainly due to small drops in photons indistinguishability along the evolution, which slightly affect the exact violation values, while not changing the overall trend.

## 4.6 Conclusion

The presented numerical and experimental analysis demonstrates that two-mode quantum correlations due to particle indistinguishability, which disperses through the lattice and rapidly decays in an ordered evolution, can be retrieved by inserting suitable inhomogeneity patterns in the system after a minimum evolution time. By changing the disorder configurations, one can tune the two-mode and total enhancement of non-classicality in position and intensity; this corresponds to an adaptive network whose parameters evaluation determines the focusing of nonclassical resources in selected modes. Also, we show that the two-mode quantum correlation diminishes due to random phase disorder in the system (for details, see SM). Nevertheless, this quantum correlation degradation can be challenged by single realizations of disorder. These results supply a conceptual and practical advance compared to previous studies limited to single-photon disorder-assisted quantum correlation enhancement between two degrees of freedom of the photon [89]. In fact, since violations of Eq. 4.5 indicate biphoton quantum correlations between two modes, our method can be especially promising for quantum metrology issues. It is yet to be understood whether this enhancement procedure can be generalized to systems with  $N > 2$  photons or not, which could result in a benchmarking outcome in the context of Quantum Resource Theories.

## Supplementary material

### A Quantitative analysis of violation

The violation of the inequality:

$$\frac{2}{3}\sqrt{P_{i,i}P_{j,j}} - P_{i,j} < 0 \quad (4.3)$$

allows to point out the presence of non-classical correlations in photonic systems [297, 302, 299, 301]. The meaning of this simple relation can be traced back to the result of an elemental boson bunching phenomenon, i.e. the HOM effect [164], for the case of non-perfectly indistinguishable photons. The initial state of two partially indistinguishable photons entering a BS can be written as:

$$\rho_0 = (1 - q) |1\rangle_1 |1\rangle_2 \langle 1|_1 \langle 1|_2 + q |1\rangle_1^a |1\rangle_2^b \langle 1|_1^a \langle 1|_2^b$$

where  $\{1, 2\}$  are the input ports of the BS, and  $q$  can represent the probability of bad generation of the input state or rather the actual overlap of the two photons wave functions, such that the photons are distinguishable with classical probability  $q$ . The resulting state after the action of a supposedly balanced BS shall be:

$$\rho_1 = (1 - q) \left( \frac{|2\rangle_3 |0\rangle_4 + |0\rangle_3 |2\rangle_4}{\sqrt{2}} \right) \left( \dots \right)^T + q \left( e^{i\phi} \frac{|1^a, 1^b\rangle_3 |0\rangle_4 + |0\rangle_3 |1^a, 1^b\rangle_4}{2} + \frac{|1\rangle_3^a |1\rangle_4^b}{\sqrt{2}} \right) \left( \dots \right)^T$$

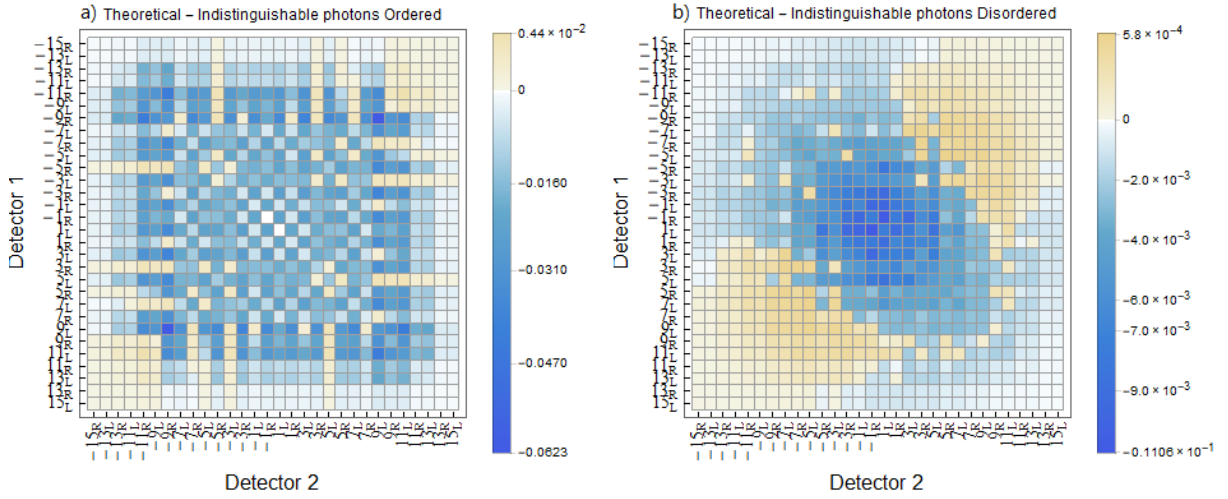


Figure 4.7: **Numerical simulation of Violation matrix.** Violation matrix at step 15 for indistinguishable photons in the **a)** ordered and **b)** completely disordered case  $p = 1$ . The disordered matrix has been computed by averaging over 10000 disorder configurations.

where  $\{3, 4\}$  stand for the output ports of the BS. The inequality 4.3 can be evaluated with the output probabilities of the process, so that  $P_{3,3} = P_{4,4} = \frac{1-q}{2} + \frac{q}{4}$  and  $P_{3,4} = \frac{q}{2}$ . The inequality then has the form:

$$\frac{2}{3} \left( \frac{1}{2} - \frac{q}{4} \right) - \frac{q}{2} < 0 \quad (4.4)$$

which, in order to be violated, requires a value  $q > \frac{1}{2}$ , corresponding to photons that are more likely to be indistinguishable than distinguishable. Hence, the inequality 4.3 provides a straightforward quantifier of the effective indistinguishability of photons, in the operative context of boson bunching occurrence. In the general case of a BS network, a value of  $V_{ij} > 0$  can be subject to multiple interpretations. In the general case for a pure initial state, after a  $n$  step propagation, the system will be in a superposition state, which can be written highlighting the modes of interest:

$$|\Phi\rangle = \sqrt{1 - \Pi}(\dots) + \sqrt{\Pi}(\alpha_{11} |2\rangle_1 |0\rangle_2 + \alpha_{12} |1\rangle_1 |1\rangle_2 + \alpha_{22} |0\rangle_1 |2\rangle_2)$$

where  $\Pi$  is the overall probability of having both photons in the selected modes, which normalizes the  $\alpha_{ij}$  coefficients, it is possible, obviously, also to have single-photon states of the two modes, but they would be invisible to coincidence-like measurements. In this case, the violation between modes 1 and 2 can be computed as:

$$V_{12} = \Pi * \left( \frac{2}{3} \sqrt{|\alpha_{11}|^2 |\alpha_{22}|^2} - |\alpha_{12}|^2 \right) \quad (4.5)$$

Therefore, the violation value depends on two factors:

- the actual non-classicality of the correlation determining a positive or negative value
- the global probability of the selected output modes (given by  $\Pi$ )

The first factor is the one pointing out the form of a hypothetically post-selected state of the two photons emerging from the considered modes. The higher this factor, the cleaner is the distillation of NOON states by post-selection, since it necessarily corresponds to a low  $|\alpha_{12}|^2$ . The second factor is an amplification parameter, which gives the probability of actually finding two photons in the two-modes selected subsystem, hence it gives the efficiency of the NOON states distillation. In conclusion, the violation value gives an indication over the composite effect of the two parameters, hence its maximization can be related to either one or the other. Hence, in a hypothetical application of this protocols, that needs to be taken into account. For instance, the most external output modes will provide the most pure NOON states, since they are the mere propagation of the first HOM resulting state, but with a very low probability. On the other hand, by means of disorder, it is possible to manipulate the probability for central modes, in order to get a higher efficiency, at the cost of a non-zero chance of extracting a useless state.

## B Numerical results

Here, we study the average behavior of the violation matrix by averaging over many different evolution realizations with  $p$  randomness. The average probability distribution is used to obtain the output of the violation matrices. After 15 step, We

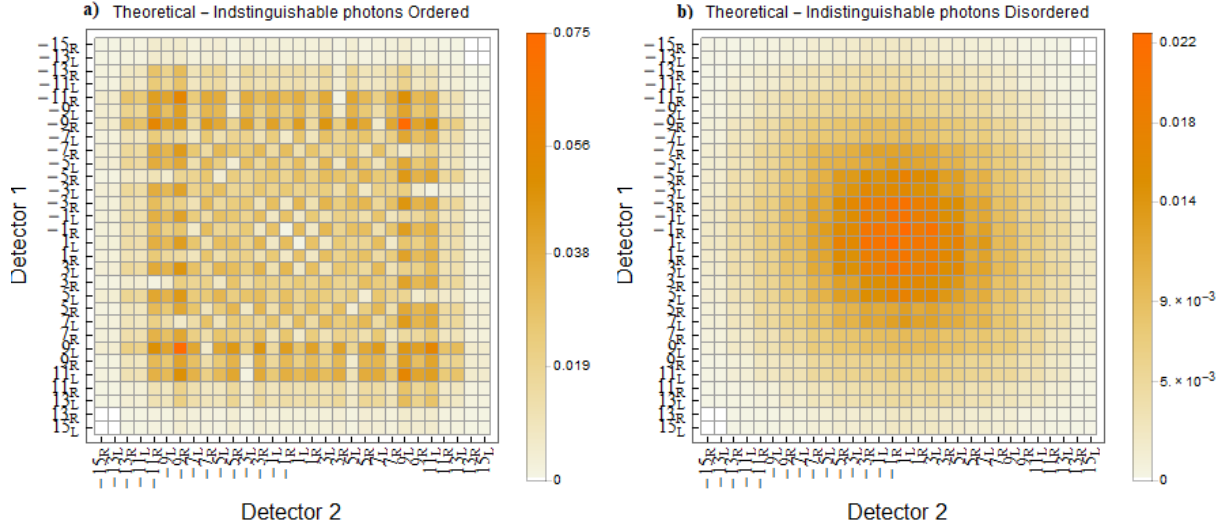


Figure 4.8: **Numerical simulation of coincidences matrix.** Step 15 output distributions of indistinguishable photons in the **a)** ordered and completely **b)** disordered case. The disordered matrix has been computed by averaging over 10000 disorder configurations.

show the violation matrix of indistinguishable pair in Fig. 4.7 **a** and the completely disordered case in Fig. 4.7 **b** with the maximal randomness  $p = 1$ . In this case, the disorder level  $p$  is a relevant quantity, since it indicates the average quantity of disorder imposed on the evolution. As can be seen, the violations are present both in the ordered and in the disordered case, though there is an evident migration of the violating values towards the matrix tails.

In (Fig. 4.8), we report the values of probability  $P_{ij}$  of finding a coincidence between two photons emerging from mode  $i$  and  $j$ . By comparing the violation and coincidence matrix, quantum correlations appear between modes with low coincidences and less populated. This consideration nourishes the idea that correlations specifically generated by the QW dynamics are classical and may even smother the underlying non-classical correlations. Therefore, the latter can be detected only where the evolution funnels low population and correlations.

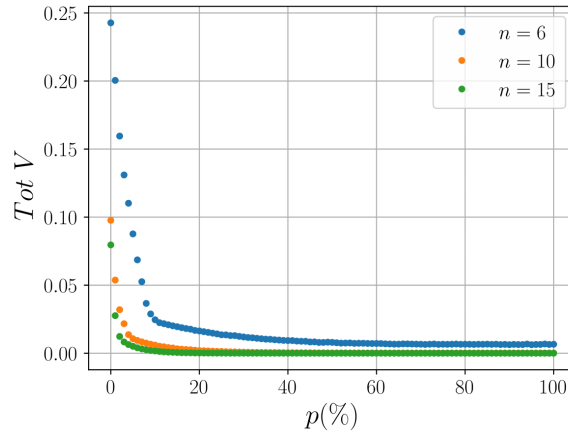


Figure 4.9: **Numerical simulation of coincidences matrix.** Step 15, step 10 and step 6 plot of the average Total Violation, computed over 500 disorder configurations, as a function of the disorder level  $p$ .

The dynamical average behavior of quantum correlation has been studied through Total Violation, defined as the sum of all the  $V_{i,j}$  of the violating coincidences. It has been considered as a measure of the total quantum correlation present in the system. In Fig. 4.9 the normalized trend in function of the disorder increase is reported, for different evolution time lengths. Total Violation has a decreasing trend as  $p$  increases, which can be seen as a consequence of the migration picture described above. Since violations are bound to appear only between scarcely populated modes, the global quantum correlations diminishes. Nevertheless, this non-classicality degradation can be challenged by specific single realizations of disorder. The average detrimental effect of disorder has a lower bound, independent of the time step. Besides, the evolution appears naturally featured by a decrease of global non-classicality in time.

## C Numerical results for Single realization

The highest MAV, besides the first step, is achieved at the output of the 9th step: the MAV for each modes pair  $(i, j)$  of the 9th step output distribution was computed, by analyzing  $10^6$  different phase maps each. The resulting landscape in Fig. 4.10 shows that this maximum can be achieved in different positions, depending on the chosen optimal disorder configuration; in particular, it confirms that the proper MAV can be achieved only in "central" modes pairs. The MAV can be induced between different mode pairs by imposing different (yet optimal) phase maps, although that is not possible in modes which have not interfered enough. As discussed in the main text, this phenomenon can be related to the network structure of the evolution, explicitly shown in Fig. 4.1.

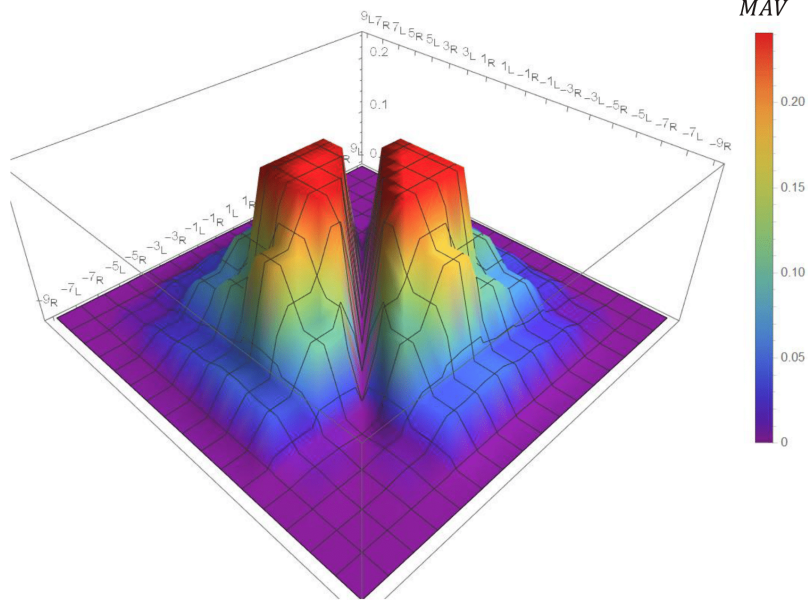


Figure 4.10: **Numerical simulation of violation landscape.** Plot of the maximum violation achievable for each output modes pair at step 9, obtained by comparing  $10^6$  different disorder realizations.

## D Experimental Results

The output violation distributions for the 5th step are shown in Fig. 4.11. They provide a preliminary demonstration of the dependence of quantum correlation distributions from the disorder pattern imposed on the evolution. The experimental results appears to be not in perfect agreement with the expected ones, since, in this case, the error on  $V_{i,j}$  values is big with respect to the measured absolute values of violation.

It is useful to observe the corresponding coincidence matrices for the ordered 5th step and the optimal 6th step (Fig. 4.12). Disorder, as demonstrated in many previous works [305, 293, 306, 295], has primarily an effect of spread hindering, also for multiparticle systems. That can be noticed even in the case of a single disorder configuration (Fig. 4.12). The manipulation of this localization effect can change the non-classicality pattern in the output distribution in many different configurations, changing the probability of finding coincident photons between the output modes. Indeed, the corresponding experimental coincidences distributions result in good agreement with the expected ones.

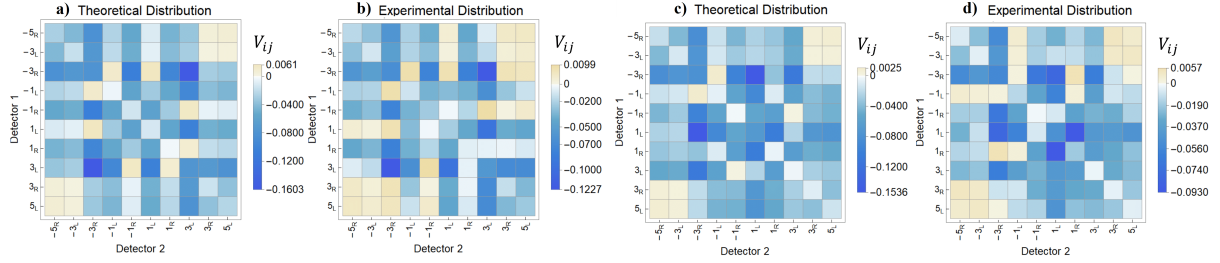


Figure 4.11: **violation matrices at the 5th.** Theoretical and experimental violation matrices at the 5th step for **a) and b)** ordered evolution and **c) and d)** disordered evolution. The disorder configuration has been chosen randomly.

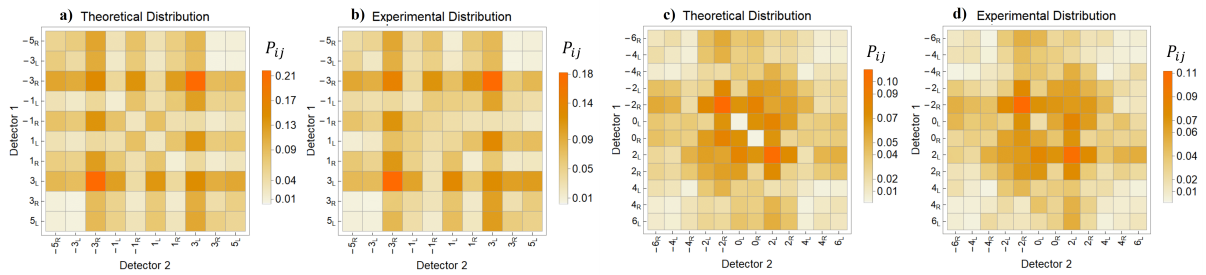


Figure 4.12: **Numerical simulation of coincidences matrix.** 5th step coincidence of **a) theoretical b) experimental** measurement outputs for the ordered quantum walk. 6th step output coincidence matrix **c) theoretical d) experimental** measurement outputs for the optimal disordered. The similarities between theoretical and experimental coincidences distributions are  $98.2(\pm 0.2)\%$  for case (a) and  $97(\pm 5)\%$  for case (b).

## **Part II**

# **Characterizing memory effects in open quantum systems**



## Introductory remarks II

In Part II of my PhD thesis, I focus on the characterization of memory effects in open quantum systems. This part consists of chapters based on previously published papers in reputable scientific journals. However, before delving into the results, I provide a brief introduction to the background.

The unavoidable interaction of quantum systems with their environments induces decoherence and dissipation. The system's dynamic can be memoryless, the so-called Markovian regime, or it can be associated with the back-flow of information from the environment to the system called the non-Markovian regime. Non-Markovian effects are present in various realistic systems such as quantum optical systems [90], solid systems [91, 92, 93], superconducting flux qubits and quantum control, quantum biology [94, 95] and chemical physics [96, 97], quantum optics: photonic band gap materials. Moreover, it is shown that non-Markovianity can be a resource for quantum information tasks [98, 99, 100, 101, 102].

### 5.1 Open quantum system and dynamical maps

An open quantum system  $S$  [103, 104, 105] can be considered a subsystem of a larger system comprising  $S$  and another subsystem  $E$ , its environment, as depicted in Fig. 5.1. The Hilbert space of the combined system  $S + E$  is represented by the tensor product of their individual Hilbert spaces as

$$\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E, \quad (5.1)$$

where  $\mathcal{H}_S$  and  $\mathcal{H}_E$  indicate the Hilbert space of the system and environment, respectively. The physical state of the total system is demonstrated by  $\rho_{SE}$  a Hilbert space  $\mathcal{H}_{SE}$  that satisfy  $\rho_{SE} \geq 0$  and  $\text{Tr}\rho_{SE} = 1$ . Accordingly, the corresponding state of  $S$  and  $E$  can be obtained by a partial trace over Hilbert space  $\mathcal{H}_E$  and  $\mathcal{H}_S$ , i.e., respectively, as  $\rho_S = \text{Tr}_E \rho_{SE}$  and  $\rho_E = \text{Tr}_S \rho_{SE}$ . We assume the total system  $S + E$  is a closed system governed by the following Hamiltonian,

$$H = H_S \otimes I_E + I_S \otimes H_E + H_I, \quad (5.2)$$

where  $H_S$  and  $H_E$  indicate the free Hamiltonian of the system and environment, respectively,  $H_I$  is the interaction Hamiltonian. Thus the corresponding unitary time evolution is given by

$$U(t) = \exp(-iHt), \hbar = 1. \quad (5.3)$$

The dynamics of the total system can be determined from the von Neumann equation in the following form:

$$\frac{d\rho_{SE}(t)}{dt} = -i[H, \rho_{SE}(t)], \quad (5.4)$$

that results in the general solution

$$\rho_{SE}(t) \equiv U(t)\rho_{SE}(0)U^\dagger(t). \quad (5.5)$$

We assume that the initial state of the total system is uncorrelated with the tensor product state, i.e.,  $\rho_{SE}(0) = \rho_S \otimes \rho_E(0)$ . Taking a partial trace over the degrees of freedom of the environment results in the following reduced density matrix of the system at any time  $t \geq 0$

$$\rho_S = \text{Tr}_E(U(t)\rho_S(0) \otimes \rho_E(0)U^\dagger(t)) \quad (5.6)$$

The above equation defines a linear map by considering a fixed initial environmental state and any fixed time in the form of

$$\Phi_t : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S). \quad (5.7)$$

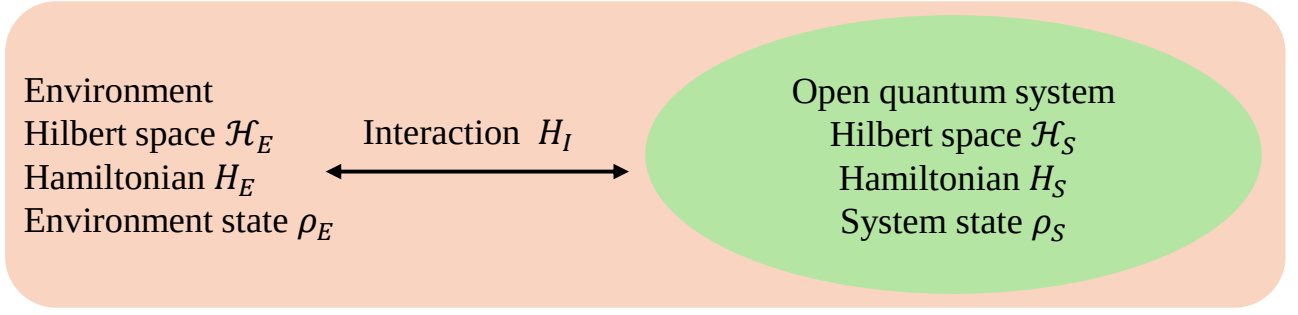


Figure 5.1: An open quantum system is described by the Hilbert space  $\mathcal{H}_S$  with Hamiltonian  $H_S$  being coupled to an environment with the Hilbert space  $\mathcal{H}_E$  with Hamiltonian  $H_E$  through an interaction Hamiltonian  $H_I$ .

This map takes any initial state of the open system  $\rho(0)$  and maps it to the corresponding state of the open system  $\rho_S(t)$  at time,  $t$  as

$$\rho_S(0) \rightarrow \rho_S(t) = \Phi_t \rho_S(0), \quad (5.8)$$

$\Phi_t$  is called a quantum dynamical map. This is a physical operation that maps the physical state to other physical states. It is both positive and trace-preserving.

One can consider the general scenario in which the system  $S$  is initially correlated with an ancillary system  $A$ , which is not affected by the evolution governed by the interaction between  $S$  and  $E$ . Therefore, a physical transformation should also preserve the physicality of the  $S - A$  states. To do so for any ancillary system with Hilbert space  $\mathcal{H}_A$ , we need

$$\rho_{SA}(0) \in S(\mathcal{H}_{SA}) \rightarrow \Phi_t \otimes I_A(\rho_{SA}(0)) = \rho_{SA}(t) \in S(\mathcal{H}_{SA}) \quad \text{for any } t \geq 0, \quad (5.9)$$

to extend the condition of Eq. 5.8 which requires it  $\Phi_t$  to be completely positive (CP) and trace-preserving (CPTP). In quantum information theory, maps with this property are also known as quantum operations or quantum channels [106].

The CP feature can be expressed as

$$\Phi(t) \text{ is CP} \iff \Phi(t) \otimes I(X_{SA}) \geq 0 \quad \text{for any } X_{SA} \geq 0, \quad (5.10)$$

where,  $X_{SA} \in S(\mathcal{H}_{SA})$  and the ancillary  $A$  has the same dimension as  $S$ . So this dynamical map describes the evolution of an open quantum system between the initial state and the final state. Thus, the stronger property of a completely positive map is that it transforms both the physical state  $S$  to another physical state  $S$  and all physical states  $S - A$  to physical states of  $S - A$ .

For the case where the time parameter  $t$  changes over an interval from 0 to  $T$ , which may be finite or infinite, a one-parameter family of dynamical maps can be described as

$$\Phi = \{\Phi_t | 0 \leq t \leq T, \Phi_0 = I\}, \quad (5.11)$$

and  $I$  indicate to the unit map; also, the initial environment state  $\rho_E(0)$  remains fixed to form the  $\Phi_t$ . This family includes complete information on the dynamic of all open systems over the interval  $[0, T]$ .

A more detailed form of the dynamical is achievable by examining the diagonal decomposition of the environments initial state  $\rho_E(t_0) = \sum_i p_i |E_i\rangle \langle E_i|$ , where  $|E_i\rangle$  represents the orthonormal basis in the Hilbert space of  $\mathcal{H}_E$  and  $\sum_i p_i = 1$ . One can assume that the Hamiltonian of system-environment  $\mathcal{H}_{SE}$  is time-independent; therefore, using the Eq. 5.6, the reduced density matrix of system can be described by

$$\rho_S(t) = \sum_{ij} p_i \langle E_j | U(t) | E_i \rangle \rho_S(0) \langle E_i | U^\dagger(t) | E_j \rangle. \quad (5.12)$$

By introducing the Kraus operators  $K_{ij} := \sqrt{p_i} \langle E_j | U(t) | E_i \rangle$  [107], the above equation can be represented in the form

$$\rho_S(t) = \sum_{ij} K_{ij} \rho_S(0) K_{ij}^\dagger. \quad (5.13)$$

As global evolution is unitary, the Kraus operators satisfy the completeness constraint, i.e.,  $\sum_{ij} K_{ij} K_{ij}^\dagger = I_S$ .

## 5.2 Markovian and non-Markovian evolution

Quantum Markovian evolutions, similar to classical Markov processes, are not affected by the past history of the system, and thus lost information is not recoverable. Therefore, quantum Markovian evolutions exhibit memoryless phenomenological properties, meaning no information backflow from the environment  $E$  to the open quantum system can occur. Before proceeding, I will discuss some properties of evolution.

Let's dive into the concept of divisibility [108]. An evolution, given by the family of CPTP maps  $\Phi = \{\Phi_t\}_{t \geq 0}$ , describes the evolution of the system at any final time  $t$  and does not provide any operator to evolve the system in time domain  $0 < s < t$ . Therefore, the evolution  $\Phi = \{\Phi_t\}_{t \geq 0}$  is defined as divisible if and only if for any  $0 \leq s \leq t$  there exists a linear map that  $V_{t,s} : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S)$  that

$$\Phi_t(\cdot) = (V_{t,s} \circ \Phi_s)(\cdot) \equiv V_{t,s}(\Phi_s(\cdot)), \quad (5.14)$$

where  $V_{t,s}$  is called an intermediate map of  $\Phi$  between  $s$  and  $t$ . It should also be noted that any invertible evolution is divisible, meaning an intermediate map always exists and is given by

$$V_{t,s} = \Phi_t \circ \Phi_s^{-1}. \quad (5.15)$$

The family of dynamical maps is called P divisible if  $V_{t,s}$  is positive, and CP divisible if  $V_{t,s}$  is completely positive for all  $t \geq s \geq 0$ .

Furthermore, based on the definition of the semi-group property, which indicates that the map can be divided into infinitely many time steps, each one identical and independent of past and future steps [109], the dynamical map represents memoryless dynamics. Consequently, the evolution  $\Phi = \{\Phi_t\}_{t \geq 0}$  is regarded as a dynamical semi-group if and only if, for any  $t_1, t_2 \geq 0$ , the following relation holds true

$$\Phi_{t_1+t_2}(\cdot) = \Phi_{t_1} \Phi_{t_2}(\cdot). \quad (5.16)$$

It should also be noticed that any evolution that satisfies the semi-group property is CP-divisible; however, the inverse is not true.

## Markovian evolutions

To describe quantum Markovianity, it is necessary first to understand the concept of the Markov process in the classical domain. Based on classical probability theory [110, 111], a stochastic process  $X(t)$ ,  $x \geq 0$ , which takes values in the discrete set  $\{x_i\}_{i \in N}$  can be described by a hierarchy of joint probability distributions  $P_n = P_n(x_n, t_n; x_{n-1}, t_{n-1}; \dots, x_1, t_1)$  for all  $n \in N$  and times  $t_n \geq t_{n-1} \geq \dots \geq t_1 \geq 0$ , known as the Kolmogorov hierarchy. The distribution  $P_n$  results in the probability the process takes on the value  $x_1$  at time  $t_1$ , the value  $x_2$  at time  $t_2$  and...  $x_n$  at time  $t_n$ . A hierarchy can represent a stochastic process when the Kolmogorov consistency conditions, along with other conditions such as positivity and normalization, are satisfied, as

$$\sum_{x_m} P_n(x_n, t_n; x_m, t_m; \dots, x_1, t_1) = P_{n-1}(x_n, t_n; x_{n-1}, t_{n-1}; \dots, x_1, t_1). \quad (5.17)$$

That connects the  $n$ -point probability distribution  $P_n$  to the  $n-1$ -point probability distribution  $P_{n-1}$ . A stochastic process  $X(t)$  called Markovian when the conditional probability is described as

$$P_{1|n}(x_{n+1}, t_{n+1} | x_n, t_n; \dots, x_1, t_1) = \frac{P_{n+1}(x_{n+1}, t_{n+1}; \dots, x_1, t_1)}{P_n(x_n, t_n; \dots, x_1, t_1)}, \quad (5.18)$$

satisfying

$$P_{1|n}(x_{n+1}, t_{n+1} | x_n, t_n; \dots, x_1, t_1) = P_{1|1}(x_{n+1}, t_{n+1} | x_n, t_n). \quad (5.19)$$

The condition for the Markov process emphasizes that the probability of the stochastic process taking the value  $x_{n+1}$  at the time  $t_{n+1}$ , given that it has taken values  $x_i$  at previous times  $t_i$ , depends only on the most recent value  $x_n$  at time  $t_n$ . In this context, the Markov process is considered memoryless, as the past history before  $t_n$  is irrelevant for predicting the future once the value  $x_n$  at time  $t_n$  is known.

Assume a finite-dimensional classical system, where the stochastic variable  $t$  and the stochastic variable  $X$  can take one value from  $\{i\}_{i=1}^d$ . Thus the probability vector of the starting time is  $\mathbf{p}(0) = (p_1(0), p_2(0), \dots, p_d(0))$ , where  $p_i(0)$  represents the initial probability that  $X$  is in  $i$ -state. The subsequent evolution over time  $t$  is thus given by stochastic matrix  $T(t, 0)$ , as  $\mathbf{p}(t) = T(t, 0)\mathbf{p}(0)$ . The  $ij$ -element  $T(t, 0)$  shows the transition probability from  $i$ -state at starting time to  $j$ -state at time  $t$ . Markovian processes are characterized by P-divisible stochastic matrices, which can be broken down into intermediate stochastic matrices  $T(s, t)$  [112]

$$T(t, 0) = T(s, t)T(s, 0) \quad \text{for any } s \leq t. \quad (5.20)$$

These processes are considered stochastically P-divisible because the matrices  $T(s, t)$  are needed to maintain the physical meaning of evolving probability distribution. The term stochastically is employed to differentiate this characteristic from the P-divisibility found in quantum channels. For the generalization of Eq. 5.20 to the quantum domain, it is required that  $V(s, t)$  in Eq. 5.14 be considered CPTP, which leads to the definition of quantum Markovian evolution based on CP-divisibility.

To illustrate this definition, consider the CP-divisible evolution  $\Phi_t(\cdot)$  and its intermediate  $V(t, s)$ , which are CPTP maps. The density matrix of the system can be defined as the consecutive application of two maps in the form that

$$\rho_s(t) = V_{t,s}(\Phi_s(\rho_s(0))), \quad (5.21)$$

when CP-divisibility is satisfied, no backflows of information can occur between any two times  $s$  and  $t$ . Additionally, evolutions that met the semi-group property have intermediate maps  $V_{t,s}$  that only rely on the duration of time interval  $t - s$  rather than specific time  $s$  and  $t$ . In this case, because of the consistent rate of information loss, these evolutions are regarded as most memoryless.

Therefore, based on the definition of the Markovian process, the quantum evolution is defined as non-Markovian if and only if the dynamical map  $\Phi_t(\cdot)$  is not divisible and the CP-divisibility condition is violated. This interpreted that information backflow can happen.

### 5.3 Master equation

Another approach to describing the evolution of the open quantum system is using the differential equation for  $\frac{d}{dt}\rho_S(t)$ , known as the master equation. The solution of such an equation results in the same time evolution of the reduced density matrix as obtained with dynamical maps. The master equation has the general form

$$\frac{d}{dt}\rho_S(t) = \kappa_t(\rho_S(t)), \quad (5.22)$$

where the  $\kappa_t$  is the generator operator [113]. The generator operator for any evolution that satisfies the semi-group property and is time-independent, i.e.,  $\kappa_t = \kappa$  is given by [114, 113]

$$\kappa(\rho_S(t)) = -i[H_S, \rho_S(t)] + \sum_i \gamma_i \left[ A_i \rho_S(t) A_i^\dagger - \frac{1}{2} \{A_i^\dagger A_i, \rho_S(t)\} \right], \quad (5.23)$$

where  $\gamma_i$  are the non-negative rates and  $A_i$  are called Lindblad or Jump operators. This equation has been extended to CP-divisible, namely Markovian, differentiable evolutions. In this scenario, the master equation, known as the Lindblad equation, and its generator take on the following form:

$$\kappa_t(\rho_S(t)) = -i[H_S(t), \rho_S(t)] + \sum_i \gamma_i(t) \left[ A_i(t) \rho_S(t) A_i^\dagger(t) - \frac{1}{2} \{A_i^\dagger(t) A_i(t), \rho_S(t)\} \right], \quad (5.24)$$

where all rates and operators are generally time-dependent. The  $\kappa_t$  is in the Lindblad form when  $\gamma_i(t) \geq 0$  for all  $i$ , and is generalized Lindblad form when at least one of rates  $\gamma_i(t)$  is negative in one or more time intervals.

A key property of the rates  $\gamma_i(t)$  is that, if they are finite,  $\kappa_t$  results in a Markovian evolution if and only if it can be expressed in a form where  $\gamma_i(t) \geq 0$  for all  $i$  and  $t \geq 0$ . The corresponding dynamical maps  $\Phi_t$  and intermediate maps  $V_{t,s}$  of the Lindblad master equation can be written through time-ordered exponential representations as follows:

$$\begin{aligned} \Phi_t &= T \left[ \int_0^t \kappa_\tau d\tau \right], \\ V_{t,s} &= T \left[ \int_s^t \kappa_\tau d\tau \right]. \end{aligned} \quad (5.25)$$

So the evolution  $\Phi_t$  is assumed to be Markovian if and only if the corresponding generator  $\kappa_t$  is in the Lindblad form. However, the above statement is true only for invertible evolutions [108]. Non-invertible CP-divisible (Markovian) dynamics with generators in the form of generalized Lindblad form exist. Moreover, any dynamic in the form of a generalized Lindblad master equation where one or one more rate  $\gamma_i(t)$  is negative for some times is non-Markovian; not all non-Markovian dynamics can be described by the solution of the generalized master equation.

In summary, the Lindblad master equation is widely regarded as the standard model for Markovian, memoryless dynamics. This is due to its semi-group property, which ensures that the future evolution of the state is independent of its past states. In contrast, when the time-dependent rates in the generalized Lindblad master equation temporarily take negative values, the divisibility of the corresponding dynamical map is broken. Therefore, the intermediate map  $V_{t,s}$  in Eq. 5.14 is no longer positive, and the dynamical map is known as non-Markovian.

### 5.4 Quantum non-Markovianity witness

Up to this point, I have discussed both the dynamics of quantum Markovianity and non-Markovianity. In this section, I will explore two approaches to quantifying non-Markovianity. The first focuses on the indivisibility and violation of the dynamic map. The second approach is based on the memory effect in the dynamics of open quantum systems, which relates to the exchange of information between quantum systems and their environment.

## Quantum non-Markovianity and Indivisibility

There are several approaches to quantifying and detecting non-Markovianity, depending on whether they propose different properties related to the divisibility of the dynamic map. These approaches include geometric measures [115], the optimization of the Helstrom matrix norm [116], and the RHP measure [117]. While the first two methods are complicated to compute in practice, the RHP is computationally simple. In the following, I will address this measure in more detail.

The measure of RHP, introduced by Rivas, Huelga, and Plenio, is based on the notion of divisibility: a completely positive trace-preserving map is divisible if it can be defined as

$$\Phi_{t+\tau,0} = \Phi_{t+\tau,t} \Phi_{t,0}, \quad (5.26)$$

and  $\Phi_{t+\tau,t}$  is completely positive for any time  $t, \tau \geq 0$  and satisfies the composition law

$$\Phi_{s,t} = \Phi_{s,u} \Phi_{u,t}, \quad (5.27)$$

for all  $s \geq u \geq t$ . After right-multiplication with the inverse  $\Phi_{t,0}^{-1}$  on both sides of above equation, we have

$$\Phi_{t+\tau,t} = \Phi_{t+\tau,0} \Phi_{t,0}^{-1}. \quad (5.28)$$

If these maps remain completely positive at all times  $t$ , the time evolution is considered Markovian. However, for the non-Markovian evolution, there must exist some time  $t$  at which  $\Phi_{t+\tau,t}$  is not completely positive. Since the basic idea behind the RHP is to quantify how much the intermediate dynamics  $\Phi_{t+\tau,t}$  are not completely positive for each time  $t$ , measuring the deviation of the intermediate map from completely positive maps characterizes the extent to which the time evolution is non-Markovian.

To do so, to assess the degree of non-complete positiveness of the intermediate maps  $\Phi_{t+\tau,t}$ , one can utilize the Choi-Jamiokowski isomorphism [118, 119]. Let's assume the maximally entangled state in the form of  $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{n=d-1} |n\rangle \langle n|$  (where  $d$  is the dimension of the system.); we link the  $\Phi(t+\tau, t)$  to a (ChoiJamiokowski) matrix constructed according to the role

$$[\Phi(t+\tau, t) \otimes \mathbb{I}] (|\psi\rangle \langle \psi|). \quad (5.29)$$

Based on Chojs theorem,  $\Phi(t+\tau, t)$  is completely positive if and only if the matrix equation in the above equation is semidefinite. Additionally, since  $\Phi(t+\tau, t)$  is trace-preserving, the trace norm of matrix Eq. 5.29 provides a metric for the non-completely positive nature of  $\Phi(t+\tau, t)$ . To put it more specifically

$$\|[\Phi(t+\tau, t) \otimes \mathbb{I}] (|\psi\rangle \langle \psi|)\|_1 = \begin{cases} = 1 & \text{iff } \Phi(t+\tau, t) \text{ is CP,} \\ > 1 & \text{otherwise} \end{cases} \quad (5.30)$$

According to [117], one can define the function  $g(t)$  via the right derivative of the trace norm as

$$g(t) := \lim_{\epsilon \rightarrow 0^+} \frac{\|[\Phi(t+\tau, t) \otimes \mathbb{I}]\|_1 - 1}{\epsilon}, \quad (5.31)$$

therefore,  $g(t) > 0$  for some time,  $t$  if and only if the evolution is non-Markovian. Thus, the total amount of non-Markovianity over time interval  $t \in I$  can be expressed as

$$\mathcal{N}_{RHP}^I = \int_I g(t) dt. \quad (5.32)$$

## Quantum non-Markovianity and information flow

Another criterion for discussing quantum non-Markovianity is based on the presence of memory effects in the dynamics of an open quantum system, which are related to the exchange of information between the open system and its environment. In Markovian evolution, the open system continuously loses information to the environment. However, in non-Markovian evolution, the process is characterized by the backflow of information from the environment to the open system [120, 121]. In this approach, quantum non-Markovianity relies on the notion of quantum memory. Specifically, it refers to the information transferred to the environment in the form of system-environment correlations or changes in the environmental states, which are later recovered by the system.

In this regard, the BLP approach, proposed by Breuer, Laine, and Piilo in [121, 120], identifies non-Markovian dynamics with specific physical features of the system-environment interaction. Within this framework, the non-Markovian dynamic is described as a time evolution for the open system determined through the temporary backflow of information from the environment to the system. In this scenario, the backflow of information may be displayed as an increase in the distinguishability of pairs of evolving quantum states. This indistinguishability can be quantified through the trace distance between two quantum states  $\rho_1$  and  $\rho_2$  for time  $t > 0$ , in the form of,

$$D(\rho_1, \rho_2) = \frac{1}{2} \|\rho_1 - \rho_2\|_1, \quad (5.33)$$

where  $0 \leq D(\rho_1, \rho_2) \leq 1$ ,  $D(\rho_1, \rho_2) = 0$  if and only if  $\rho_1 = \rho_2$  and  $D(\rho_1, \rho_2) = 1$  if and only if  $\rho_1, \rho_2$  are orthogonal. For the divisible dynamical process, the distinguishability of the states decreases monotonically, which can be interpreted as a continuous loss of information from the system to the environment. However, according to BLP, the dynamical map  $\Phi_t$  is considered non-Markovian if there exists a pair of quantum states  $\rho_i(t) = \Phi_{t,0}\rho_i$  ( $i = 1, 2$ ), such that the distinguishability between them increases, which is expressed in the term of information flow as

$$\sigma(\rho_1(t), \rho_2(t)) = \frac{d[D(\rho_1(t), \rho_2(t))]}{dt}. \quad (5.34)$$

Where negative information flow indicates the information loss from the system to the environment and positive flux denotes a reversed flow of information. Consequently, based on the definition of BLP, the integration over all positive fluxes assembles all the information that reverses to the system for a given initial pair; therefore, the degree of memory effect can be defined as

$$\mathcal{N}(\Phi) = \max_{\rho_1, \rho_2} \int_{\sigma > 0} \sigma(t) dt. \quad (5.35)$$

Given the simplicity and intuitive physical interpretation of trace distance, it has been employed to detect non-Markovian properties in the dynamics of qubit [122, 123, 124, 125] and qutrit systems [126] coupled to bosonic environments. Additionally, the crucial feature of the BLP measure makes it possible to directly experimentally assess non-Markovianity through tomographic measurements of different initial states at various times during the evolution [127, 128, 90, 129].

Quantifiers of quantum statistical distance and speed play a vital role in quantum information theory [106, 130]. They are used to evaluate the distinguishability of quantum states [106, 130, 131, 132], quantify initial correlations, information flow, and non-Markovian effects in quantum evolutions [132, 133, 134], and derive limits on evolution time [135, 136, 137, 138]. Among these, the aforementioned witness, trace distance, is a well-defined quantum statistical distance that provides an operational interpretation of the indistinguishability of two quantum states and is used to observe non-Markovianity.

Building on this understanding and inspired by the fact that non-Markovian effects can speed up the dynamics of systems [134, 139, 140, 141, 142, 143], while quantifiers of quantum statistical speed can derive the limit of the evolution time, in chapter 6 we introduce a novel witness as a characterizer of the non-Markovian behavior of open quantum systems. This witness is based on the positive changing rate of Hilbert Schmidt Speed (HSS) as a special case of quantum statistical speed. The advantage of this witness is that it does not require the diagonalization of the evolved density matrix of the system.

In Chapter 6, we examined the efficiency of the witness in several practical examples of open quantum systems: a qubit subject to phase-covariant noise and Pauli channels, two independent qubits locally interacting with leaky cavities, and V-type and  $\Lambda$ -type three-level atoms (qutrits) in a dissipative cavity. We found that the HSS-based non-Markovianity witness identifies memory effects in total agreement with the trace distance-based BLP witness [120], thus detecting system-environment information backflows.

Further, in Chapter 7, we assessed the validity of HSS-based non-Markovianity witnesses in high-dimension systems. To this end, we calculated the HSS for one qutrit (three-level system) and one qudit ( $d$ -level system, such as a spin coherent state) interacting with thermal and squeezed vacuum reservoirs. The results demonstrate that our proposed measure of non-Markovianity, based on the HSS, is effective for systems of dimension  $d > 2$  as well as for qubit systems ( $d = 2$ ). Consequently, these findings enable us to investigate our measure for multipartite systems, such as qubit-qudit or qudit-qudit systems, interacting with various environments.

In Chapter 8, we extend our analysis to demonstrate that Hilbert-Schmidt speed (HSS), an effective measure of non-Markovianity, serves as a reliable tool for evaluating correlations in both unital and non-unital channels with varying noisy spectral densities. For unital channels, we consider cases such as Pauli and depolarizing channels, while for non-unital channels, we focus on amplitude-damping channels. Furthermore, we examine the influence of classical memory arising from correlated applications of quantum channels on the non-Markovian evolution of the system. Additionally, we show that in unital noisy channels with a large number of qubits, the sensitivity of HSS to the strength of classical correlations between consecutive channel uses is significantly diminished. This observation suggests that larger quantum systems may exhibit reduced sensitivity to the accelerating effects of classical correlations on state evolution.

# Witnessing non-Markovian effects of quantum processes through Hilbert-Schmidt speed

## Abstract

Non-Markovian effects can speed up the dynamics of quantum systems, while the limits of the evolution time can be derived by quantifiers of quantum statistical speed. We introduce a witness for characterizing the non-Markovianity of quantum evolutions through the Hilbert-Schmidt speed (HSS), which is a special type of quantum statistical speed. This witness has the advantage of not requiring diagonalization of evolved density matrix. Its sensitivity is investigated by considering several paradigmatic instances of open quantum systems, such as one qubit subject to phase-covariant noise and Pauli channel, two independent qubits locally interacting with leaky cavities, V-type and  $\Lambda$ -type three-level atom (qutrit) in a dissipative cavity. We show that the proposed HSS-based non-Markovianity witness detects memory effects in agreement with the well-established trace distance-based witness, being sensitive to system-environment information backflows.

This chapter is related to Publication 4 in the List of Publications.

## 6.1 Introduction

The interaction of quantum systems with the surrounding environment leads to dissipating energy and losing quantum coherence [105]. Nevertheless, the process does not need to be monotonic and the quantum system may recover temporarily some of the lost energy or information due to memory effects during the evolution [133, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 128, 320]. This dynamical behavior, named non-Markovianity, can then act as a resource in various quantum information tasks such as teleportation with mixed states [101], improvement of capacity for long quantum channels [321], efficient entangling protocols [322, 323, 324], and work extraction from an Otto cycle [325].

Characterization and quantification of non-Markovianity has been a subject of intense study [308, 307, 326, 327]. One route is to investigate temporary increases of the entanglement shared by the open quantum system with an isolated ancilla, which amounts to measure the deviation from complete positivity (CP-divisibility) of the dynamical map describing the evolution of the system [328]. Another approach [329, 330] relies on measuring the distinguishability of two optimal initial states evolving through the same quantum channel and detecting any non-monotonicity (information backflows). Further witnesses of non-Markovianity have been proposed, based on different dynamical figures of merit, such as: negative time-dependent decoherence rates appearing in the canonical form of the master equation [331], channel capacities [321], quantum mutual information [332], local quantum uncertainty [333], quantum interferometric power [334, 335, 336, 337], coherence [338, 339], state fidelity [337, 340, 341], change of volume of the set of accessible states of the evolved system [342], Fisher information flow [343, 344], spectral analysis [345], entropy production rates [346, 347], correlation measures [348], Choi state [349] and quantum evolution speedup [350, 351, 352]. This variety of witnesses and approaches highlights the multifaceted nature of non-Markovian behavior which hence cannot be attributed to a unique feature of the system-environment interaction, preventing the characterization by means of a single tool for such a phenomenon.

CP-divisibility is the most common definition for Markovianity in open quantum systems [105, 308]. A dynamical map  $\{\mathcal{E}_t\}_{t \geq 0}$  is defined as a family of completely positive (CP) and trace-preserving (TP) maps acting on the system Hilbert space  $\mathcal{H}$ . Generally speaking, one calls a map  $k$ -positive if the composite map  $\mathcal{E}_t \otimes \mathbb{I}_k$  is positive, where  $k$ ,  $\mathbb{I}_k$  denote the dimensionality of the ancillary Hilbert space and its identity operator, respectively [353]. Provided that  $\mathcal{E}_t \otimes \mathbb{I}_k$  is positive for all  $k \geq 0$  and for all  $t$ , then the dynamical map is completely positive. One then says that the dynamical

map  $\mathcal{E}_t$  is CP-divisible (P-divisible) when the propagator  $V_{t,s}$ , defined by  $\mathcal{E}_t = V_{t,s} \circ \mathcal{E}_s$ , is completely positive (positive) for all  $t \geq s \geq 0$  [105]. According to the non-Markovianity measure introduced by Rivas-Huelga-Plenio (RHP) [328], the quantum evolution is considered Markovian if and only if the corresponding dynamical map  $\mathcal{E}_t$  is CP-divisible.

The non-Markovian character of the system dynamics can be identified through another well-known perspective proposed by Breuer-Laine-Piilo (BLP), namely the distinguishability of two evolving quantum states of the same system [329, 330]. This distinguishability is quantified by the trace distance, a commonly used distance measure for two arbitrary states  $\rho_1$  and  $\rho_2$ , defined as  $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}|\rho_1 - \rho_2|$ , where  $|A| = \sqrt{A^\dagger A}$  for some operator  $A$ . The trace distance  $D(\rho_1, \rho_2)$  is contractive under CPTP maps, i.e.  $D(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2)) \leq D(\rho_1, \rho_2)$ . Nevertheless, this does not mean generally that  $D(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2))$  is a monotonically decreasing function of time. In fact,  $\frac{d}{dt}D(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2)) > 0$  implies violation of P-divisibility and therefore of CP-divisibility [329, 354]. In other words, under any Markovian evolution of the quantum system, one gets  $dD(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2))/dt \leq 0$ , owing to the contraction property. Therefore, its non-monotonicity can be understood as a witness of non-Markovianity due to system-environment backflows of information.

Studies on the role of typical figures of merit for quantum metrology, based on quantum Fisher information metric, to witness non-Markovianity have been also reported [343, 134]. On the other hand, non-Markovian effects can speed up the quantum evolution of a system [350, 134, 142, 355, 356]. It is known that quantifiers of statistical speed in the system Hilbert space may be associated with measures adopted in quantum metrology to investigate the ultimate limit of precision in estimating a given physical quantity [357]. The sensitivity of an initial quantum state to changes of the parameter (e.g., an unknown phase shift) of a dynamical evolution can be then determined by measures of quantum statistical speed [358]. A higher sensitivity implies higher precision in the estimation of the parameter of interest [357, 359, 29]. These arguments naturally motivate one to inquire whether measures of quantum statistical speed can conveniently quantify the non-Markovian character of the system dynamics, a problem which has remained unexplored.

Here, we address this issue introducing a method for witnessing and measuring non-Markovianity by means of the Hilbert-Schmidt speed (HSS) [358], a type of quantum statistical speed which has the advantage of avoiding diagonalization of the evolved density matrix. We check the efficiency of the proposed HSS-based witness in several typical situations of open quantum systems made of qubits and qutrits. In particular, we consider: one qubit subject to phase-covariant noise [360], especially the so-called eternal non-Markovianity model [339, 361, 362, 331, 363]; a single qubit undergoing the Pauli channel [307, 361, 364]; two independent qubits locally interacting with leaky cavities; V-type and  $\Lambda$ -type three-level atom (qutrit) in a dissipative cavity. We find that the HSS-based non-Markovianity witness identifies memory effects in total agreement with the trace distance-based BLP witness, thus detecting system-environment information backflows.

The paper is organized as follows. In Sec. 6.2 we briefly review the definition of the Hilbert-Schmidt speed. In Sec. 6.3 we introduce the measure of quantum non-Markovianity via the HSS. Through various examples, the sensitivity of this measure in detecting memory effects is studied in Sec. 7.3.1. Finally, Sec. 6.5 summarizes the main results and prospects.

## 6.2 Hilbert-Schmidt speed (HSS)

We start by recalling the general framework leading to the definition of quantum statistical speed, whose the HSS is a particular case.

Let us consider the family of distance measures

$$[d_\alpha(p, q)]^\alpha = \frac{1}{2} \sum_x |p_x - q_x|^\alpha, \quad (6.1)$$

with  $\alpha \geq 1$  and where  $p = \{p_x\}_x$  and  $q = \{q_x\}_x$  are probability distributions. Here it is assumed that the random variable  $x$  takes only discrete values; in the case of a continuum of values, the sum is replaced by an integral. These distances satisfy the following basic properties: (i) non-negativity and normalization  $0 \leq d_\alpha(p, q) \leq 1$ , where  $d_\alpha(p, q) = 0 \Leftrightarrow p \equiv q$ ; (ii) triangle inequality  $d_\alpha(p_1, p_3) \leq d_\alpha(p_1, p_2) + d_\alpha(p_2, p_3)$ ; (iii) symmetry  $d_\alpha(p, q) = d_\alpha(q, p)$ .

Generally, in order to obtain the statistical speed from any statistical distance, one should quantify the distance between infinitesimally close distributions taken from a one-parameter family  $p_x(\varphi)$  with parameter  $\varphi$ . Then, the classical statistical speed is given by

$$s_\alpha[p(\varphi_0)] = \frac{d}{d\varphi} d_\alpha(p(\varphi_0 + \varphi), p(\varphi_0)). \quad (6.2)$$

Considering now a given pair of quantum states  $\rho$  and  $\sigma$ , one can extend these classical notions to the quantum case by taking  $p_x = \text{Tr}\{E_x \rho\}$  and  $q_x = \text{Tr}\{E_x \sigma\}$  as the measurement probabilities associated with the positive-operator-valued measure (POVM) defined by the set of  $\{E_x \geq 0\}$  satisfying  $\sum_x E_x = \mathbb{I}$ , where  $\mathbb{I}$  is the identity operator. Maximizing the classical distance over all possible choices of POVMs, one obtains the corresponding quantum distance

$$D_\alpha(\rho, \sigma) = \max_{\{E_x\}} d_\alpha(\rho, \sigma), \quad (6.3)$$

which leads to the expression [358]

$$[D_\alpha(\rho, \sigma)]^\alpha = \frac{1}{2} \text{Tr}|\rho - \sigma|^\alpha, \quad (6.4)$$



where  $|X|^\alpha$  can be computed using the spectral decomposition  $X \equiv \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ , i.e.,  $|X|^\alpha = \sum_i |\lambda_i|^\alpha |\lambda_i\rangle\langle\lambda_i|$ , so that  $\text{Tr}|X|^\alpha = \sum_i |\lambda_i|^\alpha$ . For  $\alpha = 1$ , the trace distance  $D(\rho_1, \rho_2) = \frac{1}{2}\text{Tr}|\rho_1 - \rho_2|$  is retrieved, while for  $\alpha = 2$  one gets the so-called Hilbert-Schmidt distance  $D_2(\rho, \sigma)$  allowing for a simple evaluation because it does not need diagonalization of the argument operator. This distance is of Riemann type and limited by the following inequality relation

$$0 \leq D_2(\rho, \sigma) \leq 2D(\rho, \sigma). \quad (6.5)$$

The Hilbert-Schmidt distance generally does not possess the contractivity property, although quantum systems such as qubits constitute useful exceptions. Necessary and sufficient conditions for contractivity of the Hilbert-Schmidt distance for the Lindblad operators have been discussed [365]. For a single qubit, it is straightforward to derive that trace and Hilbert-Schmidt distances are equivalent, namely

$$D_2(\rho, \sigma) = \sqrt{2}D(\rho, \sigma), \quad (6.6)$$

so that contractivity of trace distance implies contractivity of Hilbert-Schmidt distance. However, it worth to notice that this argument cannot be generalized to high-dimensional systems with Hilbert space dimension larger than two [365].

Extending Eq. (8.6) to the quantum case, one then obtains the quantum statistical speed as [358]

$$S_\alpha[\rho(\varphi)] = \max_{\{E_x\}} s_\alpha[p(\varphi)] = \left( \frac{1}{2} \text{Tr} \left| \frac{d\rho(\varphi)}{d\varphi} \right|^\alpha \right)^{1/\alpha}. \quad (6.7)$$

In the special case when  $\alpha = 2$ , the quantum statistical speed is given by the Hilbert-Schmidt speed (HSS) [358]

$$HSS(\rho_\varphi) = \sqrt{\frac{1}{2} \text{Tr} \left[ \left( \frac{d\rho_\varphi}{d\varphi} \right)^2 \right]}, \quad (6.8)$$

which, in analogy with the Hilbert-Schmidt distance, does not require the diagonalization of  $d\rho_\varphi/d\varphi$ . Notice that noncontractivity of the Hilbert-Schmidt distance does not consequently imply noncontractivity of the HSS. In fact, on the one hand, the Hilbert-Schmidt distance is computed by maximization over all the possible choices of POVMs  $\{E_x\}$  of the adopted distance measure  $d_2(\rho, \sigma)$  (see Eqs. (6.1) and (8.7)); on the other hand, the HSS is determined by maximization applied after the differentiation with respect to  $\varphi$ , starting from the adopted distance measure (see Eqs. (8.6) and (8.8)). Because of these computational subtleties, from the noncontractivity of the Hilbert-Schmidt distance, one cannot conclude that the HSS is also noncontractive. Indeed, we shall show in the following that the HSS can be regarded as a trustful, convenient non-Markovianity measure just because of its contractivity.

### 6.3 HSS-based non-Markovianity measure

It is known that non-Markovian effects may lead to faster quantum evolution from an initial state to a subsequent one [350, 142, 355, 356]. It thus seems natural that measures of quantum speed limits may play the role of proper quantifiers of memory effects occurring during a system dynamics. Some works along this direction based on quantum Fisher information metric have been reported [343, 134]. Here we aim at exploiting a convenient quantum statistical speed [358] as a figure of merit of the non-Markovian character of quantum evolutions, which avoids diagonalization of the system density matrix, with consequent practical advantages in the analysis. We stress that such a quantifier would be particularly useful, especially for detecting the memory effects of high-dimensional and multipartite open quantum systems. Looking at the various possible choices among the quantum statistical speeds of Eq. (8.8), the most natural candidate towards this aim is just that obtained for  $\alpha = 2$ , corresponding to the Hilbert-Schmidt speed (HSS) of Eq. (6.8). To assume the role of a faithful indicator of non-Markovianity, the HSS should not exhibit the problems of contractivity manifested by the Hilbert-Schmidt distance for dimensions larger than two [365]. We shall see that, interestingly, the HSS is indeed contractive at least for quantum systems having dimension  $n \leq 3$ .

In this regard, for a quantum system with  $n$ -dimensional Hilbert space  $\mathcal{H}$ , let us take an initial state defined as

$$|\psi_0\rangle = \frac{1}{\sqrt{n}} (e^{i\varphi} |\psi_1\rangle + \dots + |\psi_n\rangle), \quad (6.9)$$

where  $\varphi$  is an unknown phase shift and  $\{|\psi_i\rangle, i = 1, \dots, n\}$  constructs a complete and orthonormal set (basis) for  $\mathcal{H}$ . The form of  $|\psi_0\rangle$  is strategically chosen for phase-sensitive quantum statistical speed, being the standard initial state structure for quantum metrology phase estimation [359, 29]. With the idea that a nonmonotonic speed (positive acceleration) of the quantum dynamics is a signature of memory effects in the system dynamics, we then introduce the HSS-based witness of non-Markovianity as

$$\chi(t) := \frac{dHSS(\rho_\varphi(t))}{dt} > 0, \quad (6.10)$$

where  $\rho_\varphi(t)$  denotes the evolved state of the system and  $HSS(\rho_\varphi(t))$  is defined in Eq. (6.8). Given this witness, in analogy to what has been done for other measures [329, 330], a quantifier of the degree of non-Markovianity can be naturally defined as

$$\mathcal{N}_{\text{HSS}} := \max_{\varphi, \{|\psi_1\rangle, \dots, |\psi_n\rangle\}} \int_{\chi(t) > 0} \chi(t) dt, \quad (6.11)$$

where the maximization is taken over all the possible parametrizations of the single initial state of Eq. (6.9).

Notice that here we are interested in only detecting non-Markovian effects by the HSS-based witness, so that its actual value is not important and no optimization over the initial state parameters is required. The sanity check of  $\chi(t)$  as faithful witnesses of non-Markovianity is performed in the following section.

## 6.4 Qualitative analysis of non-Markovianity

In this section, we consider several typical examples of open quantum systems of both theoretical and experimental interest to qualitatively analyze the faithfulness of the HSS-based non-Markovianity witness defined above. Notice that, to this aim, it is sufficient to verify that the HSS is contractive for memoryless dynamics and sensitive to system-environment information backflows, occurring in correspondence of  $\chi(t) > 0$  (speedup of the quantum evolution, as identified by Eq. (6.10)). We shall study the time behavior of  $\chi(t)$ , verifying that whenever it is positive then the BLP (trace distance-based) witness  $\sigma(t) \equiv \frac{d}{dt} D(\rho_1(t), \rho_2(t))$  is also positive [329]. These properties provide evidence that the proposed HSS-based witness is a bona-fide identifier of non-Markovianity.

### 6.4.1 One-qubit systems

#### Phase-covariant noise

We start by considering a single qubit undergoing a so-called phase covariant noise. The general time-local master equation, in the interaction picture (in units of  $\hbar$ ), for the density matrix  $\rho$  for a single qubit subject to phase-covariant noise is written as [360, 366, 367]

$$\frac{d\rho}{dt} = -i\omega(t)[\sigma_z, \rho] + \sum_{i=1}^3 \frac{\gamma_i(t)}{2} L_i(\rho), \quad (6.12)$$

where  $\omega(t)$  represents a time-dependent frequency shift,  $\gamma_i(t)$  ( $i = 1, 2, 3$ ) denotes the time-dependent rate associated to each dissipator  $L_i(\rho)$ , whose expressions are [360]

$$\begin{aligned} L_1(\rho) &= \sigma_+ \rho \sigma_- - \frac{1}{2} \{\sigma_- \sigma_+, \rho\}, \\ L_2(\rho) &= \sigma_- \rho \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho\}, \\ L_3(\rho) &= \sigma_z \rho \sigma_z - \rho. \end{aligned}$$

In the above equations,  $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$  denote the inversion operators and  $\sigma_i$ 's ( $i = x, y, z$ ) are the Pauli operators. Moreover, the three dissipators  $L_1$ ,  $L_2$ , and  $L_3$  describe, respectively, the heating, dissipation, and dephasing. Special cases of master equations of the form of Eq. (6.12), describing the phase-covariant noise, are the amplitude damping model obtained for  $\gamma_1(t) = \gamma_3(t) = 0$  and the pure dephasing model achieved for  $\gamma_1(t) = \gamma_2(t) = 0$  [307, 368, 369].

Indicating with  $|0\rangle$  and  $|1\rangle$  the ground and excited states of the qubit, respectively, one can show that the solution of the master equation of Eq. (6.12) is given by [360]

$$\mathcal{E}_t(\rho(0)) = \rho(t) = \begin{pmatrix} P_1(t) & Q(t) \\ Q^*(t) & 1 - P_1(t) \end{pmatrix}, \quad (6.13)$$

where

$$P_1(t) = e^{-\Gamma(t)} [G(t) + P_1(0)], \quad Q(t) = \alpha(0) e^{i\Omega(t) - \Gamma(t)/2 - \Gamma(t)}, \quad (6.14)$$

with the time-dependent functions

$$\begin{aligned} \Gamma(t) &= \int_0^t dt' [\gamma_1(t') + \gamma_2(t')]/2, \quad \Gamma(t) = \int_0^{t'} dt' \gamma_3(t'), \\ \Omega(t) &= \int_0^{t'} dt' 2\omega(t'), \quad G(t) = \int_0^{t'} dt' e^{\Gamma(t')} \gamma_2(t')/2. \end{aligned} \quad (6.15)$$

The master equation of Eq. (6.12) leads to commutative dynamics, meaning  $\mathcal{E}_t \circ \mathcal{E}_s = \mathcal{E}_s \circ \mathcal{E}_t$  for any  $s, t \geq 0$ , iff  $\gamma_1(t) = \gamma(t)$  and  $\gamma_2(t) = \kappa\gamma(t)$ , in which  $0 \leq \kappa \leq 1$ . Moreover, the dynamics is unital, i.e. the corresponding channel  $\mathcal{E}_t$  satisfies  $\mathcal{E}_t(\mathbb{I}) = \mathbb{I}$  ( $\mathbb{I}$  denotes the identity operator), when it is commutative and  $\kappa = 1$ .

Preparing the qubit in the initial state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(e^{i\varphi}|+\rangle + |-\rangle), \quad (6.16)$$

where  $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ , the time derivative of the HSS, that is the quantity  $\chi(t)$  of Eq. (6.10), results to be

$$\begin{aligned} \chi(t) = & -\frac{1}{8}e^{-2\Gamma(t)} \frac{(\gamma_1(t) + \gamma_2(t) + 4\gamma_3(t)) \cos^2 \varphi}{\sqrt{e^{\Gamma(t)-2\Gamma(t)} \cos^2 \varphi + \sin^2 \varphi}} \\ & -\frac{1}{4}e^{-\Gamma(t)} \frac{(\gamma_1(t) + \gamma_2(t)) \sin^2 \varphi}{\sqrt{e^{\Gamma(t)-2\Gamma(t)} \cos^2 \varphi + \sin^2 \varphi}}. \end{aligned} \quad (6.17)$$

Accordingly, choosing  $\varphi = 0$ , the HSS-based witness  $\chi(t) > 0$  tells us that the process is non-Markovian when  $\gamma_1(t) + \gamma_2(t) + 4\gamma_3(t) < 0$ . On the other hand, choosing  $\varphi = \frac{\pi}{2}$ , the dynamics is non-Markovian by the HSS-based witness when  $\gamma_1(t) + \gamma_2(t) < 0$ . In other words, the dynamics is detected as non-Markovian if either of the conditions above holds. These conditions are a clear signature of dynamical information backflows, as can be deduced from the non-monotonicity of the off-diagonal terms of the evolved density matrix of Eq. (6.13). In fact, these conditions for  $\chi(t) > 0$  are exactly the ones that give  $\sigma(t) > 0$  (positive BLP witness) for the same dynamical instance [326]:  $\chi(t) > 0 \Leftrightarrow \sigma(t) > 0$ . Certainly, contractivity of the HSS is assured for Markovian conditions for which  $\chi(t) < 0$  for any  $t > 0$ . Notice that the sensitivity of the witness  $\chi(t)$  is investigated by considering general conditions for the phase-covariant noise, which encompass many of the most studied qubit dynamics such as pure dephasing, amplitude damping noise, depolarizing noise and the so-called eternal non-Markovianity [331]. As a general insight from this first example, we thus observe that the HSS-based witness performs in perfect agreement with the BLP measure. It is known that the BLP measure, for which breaking CP-divisibility is a consequence of breaking P-divisibility [329, 354], is tighter than other proposed non-Markovianity measures [326]. On the basis of the above results, the same property holds for the HSS-based witness.

### Pauli channel

In this section, we consider a qubit subject to a Pauli channel, whose corresponding master equation is [361, 370]

$$\frac{d\rho}{dt} = \sum_{i=1}^3 \gamma_i(t)(\sigma_i \rho \sigma_i - \rho), \quad (6.18)$$

where  $\gamma_i(t)$  ( $i = 1, 2, 3$ ) denote the decoherence rate associated to the  $i$ -th channel. The dynamics may be rewritten in the following equivalent form [361, 370]

$$\rho(t) = \mathcal{E}_t[\rho(0)] = \sum_{i=0}^3 p_i(t) \sigma_i \rho(0) \sigma_i, \quad t \geq 0 \quad (6.19)$$

where  $\sigma_0 = \mathbb{I}$  (identity operator),  $\sigma_i$ 's are the Pauli matrices, and  $p_i(t)$ 's denote the time-dependent probability distribution. Notice that  $p_0(0) = 1$  and  $p_i(0) = 0$  ( $i = 1, 2, 3$ ), guaranteeing that  $\mathcal{E}_0 = \mathbb{I}$  (identity channel). The explicit expressions of the time-dependent probabilities of the Pauli channel are

$$\begin{aligned} p_0(t) &= \frac{1}{4}[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)], \\ p_1(t) &= \frac{1}{4}[1 + \lambda_1(t) - \lambda_2(t) - \lambda_3(t)], \\ p_2(t) &= \frac{1}{4}[1 + \lambda_2(t) - \lambda_1(t) - \lambda_3(t)], \\ p_3(t) &= \frac{1}{4}[1 + \lambda_3(t) - \lambda_2(t) - \lambda_1(t)], \end{aligned} \quad (6.20)$$

where  $\lambda_1(t) = e^{-2(\Gamma_2(t) + \Gamma_3(t))}$ ,  $\lambda_2(t) = e^{-2(\Gamma_1(t) + \Gamma_3(t))}$ , and  $\lambda_3(t) = e^{-2(\Gamma_1(t) + \Gamma_2(t))}$ , with

$$\Gamma_i(t) = \int_0^t \gamma_i(\tau) d\tau. \quad (i = 1, 2, 3) \quad (6.21)$$

It is straightforward to show that this dynamics is unital ( $\mathcal{E}_t(\mathbb{I}) = \mathbb{I}$ ). When  $\gamma_1(t) = \gamma_2(t)$ , the unital case of the phase-covariant master equation and the Pauli channel with the same decay rates coincide with each other. It should be noted

that the general Pauli channel includes a larger set of dynamics than the unital phase-covariant noise, such as bit-flip and bit-phase-flip channels.

We now calculate the HSS-based witness  $\chi(t)$  introduced in Eq. (6.10), with the qubit initially prepared in a state parametrized as

$$|\psi_0^\pm(\varphi)\rangle = \frac{1}{\sqrt{2}}(e^{i\varphi}|0\rangle \pm |1\rangle). \quad (6.22)$$

For three different optimal initial parametrizations given by the set  $\{|\psi_0^+(0)\rangle, |\psi_0^+(\pi/2)\rangle, |\psi_0^-(\pi/2)\rangle\}$  one easily finds, respectively,

$$\begin{aligned} \chi(t) &= -(\gamma_1(t) + \gamma_3(t)) e^{-2\Gamma_1(t) - 2\Gamma_3(t)}, \\ \chi(t) &= -(\gamma_1(t) + \gamma_2(t)) e^{-2\Gamma_1(t) - 2\Gamma_2(t)}, \\ \chi(t) &= -(\gamma_2(t) + \gamma_3(t)) e^{-2\Gamma_2(t) - 2\Gamma_3(t)}. \end{aligned} \quad (6.23)$$

Therefore, according to the HSS-based criterion the dynamics is deemed Markovian if and only if  $\gamma_1(t) + \gamma_2(t) \geq 0$ ,  $\gamma_1(t) + \gamma_3(t) \geq 0$  and  $\gamma_2(t) + \gamma_3(t) \geq 0$  for all  $t \geq 0$ . Whenever at least one of the three conditions above is not satisfied, that is  $\gamma_i(t) + \gamma_j(t) < 0$  for some  $j \neq i$ , one gets  $\chi(t) > 0$  so that the qubit dynamics exhibits memory effects and is non-Markovian. The latter is exactly the same condition that makes  $\sigma(t) > 0$  (positive BLP witness) [361, 370], so once again:  $\chi(t) > 0 \Leftrightarrow \sigma(t) > 0$ . In fact, it is well known that the qubit dynamics for the Pauli channel is Markovian according to BLP non-Markovianity criterion if and only if the sum of all pairs of distinct decoherence rates remains positive, i.e.,  $\gamma_i(t) + \gamma_j(t) \geq 0$  for all  $j \neq i$ , for which contractivity of the HSS is verified ( $\chi(t) < 0$ ). Differently, for instance, according to the RHP non-Markovianity criterion, the dynamics is Markovian if and only if all of the decoherence rates remain positive for all  $t \geq 0$ , i.e.,  $\gamma_i(t) \geq 0$ , for all  $i = 1, 2, 3$ . Once again, the HSS-based witness is sensitive to system-environment information backflows in perfect agreement with the BLP measure.

## 6.4.2 Two-qubit system

We now investigate a composite quantum system consisting of two separated qubits, A and B, which independently interact with their own dissipative reservoir (leaky cavity). The general Hamiltonian is therefore written as  $H = H_A + H_B$ . The single qubit-reservoir Hamiltonian is ( $\hbar \equiv 1$ ) [105]

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger), \quad (6.24)$$

where  $\omega_0$  represents the transition frequency of the qubit,  $\sigma_\pm$  are the system raising and lowering operators,  $\omega_k$  is the frequency of the  $k$ -th field mode of the reservoir,  $b_k$  and  $b_k^\dagger$  denote, respectively, the  $k$ -mode creation and annihilation operators,  $B = \sum_k g_k b_k$  with  $g_k$  being the coupling constant with the  $k$ -th mode. At zero temperature and in the basis  $\{|1\rangle, |0\rangle\}$ , from the above Hamiltonian with a Lorentzian spectral density for the cavity modes, one finds that the dynamics of the qubit can be described by the evolved reduced density matrix [105, 371]

$$\rho_q(t) = \begin{pmatrix} \rho_{11}^S(0)P(t) & \rho_{10}^S(0)\sqrt{P(t)} \\ \rho_{01}^S(0)\sqrt{P(t)} & 1 - \rho_{00}^S(0)P(t) \end{pmatrix}, \quad (6.25)$$

where the coherence characteristic function  $P(t)$  is

$$P(t) = e^{-\lambda t} [\cos(\Gamma t/2) + (\lambda/\Gamma) \sin(\Gamma t/2)]^2, \quad (6.26)$$

with  $\Gamma = \sqrt{2\gamma_0\lambda - \lambda^2}$ . The rate  $\lambda$  denotes the spectral width for the qubit-reservoir coupling (photon decay rate) and is connected to the reservoir correlation time  $\tau_c$  by the relation  $\tau_c = 1/\lambda$ . The decay rate  $\gamma_0$  is instead related to the system (qubit) relaxation time scale  $\tau_r$  by  $\tau_r = 1/\gamma_0$ . In the strong coupling regime, occurring for  $\gamma_0 > \lambda/2$ , the non-Markovian effects become relevant [105].

The density matrix evolution of the two independent qubits can be then easily obtained knowing the evolved density matrix of a single qubit [371]. The elements of the two-qubit evolved density matrix  $\rho(t)$  are presented in Appendix A. Preparing the two-qubit system in the initial state

$$|\psi_0\rangle = \frac{1}{2}(e^{i\varphi}|11\rangle + |10\rangle + |01\rangle + |00\rangle), \quad (6.27)$$

we find that the HSS of Eq. (6.8) is given by

$$HSS(\rho_\varphi(t)) = \frac{1}{4} \sqrt{P(t)[P(t)(4P(t) - 3) + 2]}, \quad (6.28)$$

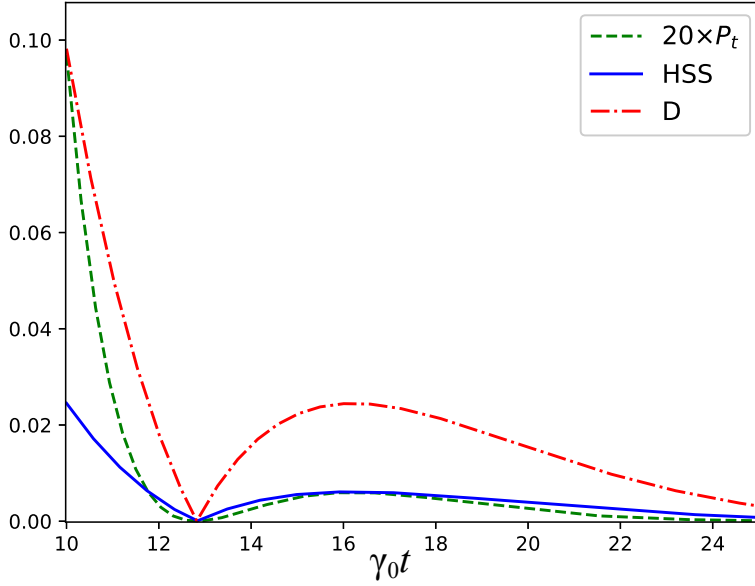


Figure 6.1: Dynamics of Hilbert-Schmidt speed  $HSS(\rho_\varphi(t))$  (blue solid line), trace distance  $D(\rho_1(t), \rho_2(t))$  (red dot-dashed line) and coherence characteristic function  $P(t)$  (amplified by 20 times for comparison, green dashed line) as a function of the dimensionless time  $\gamma_0 t$  for the two-qubit system in the strong coupling regime, with  $\lambda = 1.25\gamma_0$ .

which is independent of the phase  $\varphi$ . From this equation and from Eq. (6.10), one promptly gets that the two-qubit dynamics is non-Markovian whenever

$$\chi(t) = \frac{dHSS}{dP} \frac{dP}{dt} > 0, \quad (6.29)$$

where  $P = P(t) \in [0, 1]$  is the coherence characteristic function of Eq. (6.26). Since  $dHSS/dP$  is always positive, as easily seen from Eq. (6.28), we obtain that  $\chi(t) > 0 \Leftrightarrow dP/dt > 0$ , a clear signature of information backflows from the environment to the system. So, this is also expected to happen for  $\sigma(t) > 0$  (positive BLP witness). Using the definition  $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}|\rho_1 - \rho_2|$  and the optimal pair of two-qubit quantum states  $\rho_1(0) = |++\rangle\langle ++|$ ,  $\rho_2(0) = |--\rangle\langle --|$  with  $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ , the time-dependent trace distance is [372]

$$D(\rho_1(t), \rho_2(t)) = \sqrt{P(t)(2 - 2P(t) + P(t)^2)}. \quad (6.30)$$

As a consequence, from  $\sigma(t) = \frac{d}{dt} D(\rho_1(t), \rho_2(t))$ , we have  $\sigma(t) = \frac{dD}{dP} \frac{dP}{dt}$ . Seeing that  $dD/dP$  is always positive, one finds that  $\sigma(t) > 0$  whenever  $dP/dt > 0$ , as expected. We hence obtain:  $\chi(t) > 0 \Leftrightarrow \sigma(t) > 0$ . The computation immediately shows that, in the weak coupling regime ( $\lambda > 2\gamma_0$ ), the behavior of  $D(\rho_1(t), \rho_2(t))$ ,  $HSS(\rho_\varphi(t))$ , and  $P(t)$  is essentially a Markovian exponential decay controlled by  $\gamma_0$  (all of them are decreasing monotonic functions of time):  $\chi(t)$  and  $\sigma(t)$  are always negative, verifying contractivity of the HSS. Differently, in the strong coupling regime ( $\lambda < 2\gamma_0$ ), where memory effects arise,  $D(\rho_1(t), \rho_2(t))$ ,  $HSS(\rho_\varphi(t))$ , and  $P(t)$  simultaneously exhibit an oscillatory behavior such that their maximum and minimum points exactly coincide, as quantitatively shown in Fig. 6.1. This two-qubit dissipative model also leads to the conclusion that the HSS-based witness of non-Markovianity is equivalent to the trace distance-based measure.

### 6.4.3 One-qutrit systems

#### V-type three-level open quantum system

In this section, we investigate the non-Markovian dynamics of a V-type three level atom, playing the role of a qutrit, coupled to a dissipative environment [373, 374]. We recall that three-level quantum systems (qutrits) can be promising alternative candidates to be used in quantum processors instead of the standard two-level systems (qubits) [375, 376]. For a V-type qutrit interacting with a dissipative reservoir, the two upper levels, i.e.,  $|2\rangle$  and  $|1\rangle$  are coupled to the ground state  $|0\rangle$  with transition frequencies  $\omega_2$  and  $\omega_1$ , respectively. The Hamiltonian of the total system can be written as

$$H = H_0 + H_I, \quad (6.31)$$

where ( $\hbar \equiv 1$ )

$$H_0 = \sum_{j=1}^2 \omega_j \sigma_+^{(j)} \sigma_-^{(j)} + \sum_k \omega_k b_k^\dagger b_k, \quad (6.32)$$

represents the free Hamiltonian of the system plus the environment, while

$$H_I = \sum_{j=1}^2 \sum_k (g_{jk} \sigma_+^{(j)} b_k + g_{jk}^* \sigma_-^{(j)} b_k^\dagger), \quad (6.33)$$

is the interaction Hamiltonian in which  $\sigma_\pm^{(j)}$  ( $j = 1, 2$ ) are the standard raising and lowering operators between each of the two upper levels and the ground one. The index  $k$  denotes the different reservoir field modes with frequencies  $\omega_k$ , creation and annihilation operators  $b_k^\dagger, b_k$  and coupling constants  $g_{jk}$ .

We assume that the relaxation rates of the two upper levels are equal, the two upper atomic levels are degenerated, the atomic transitions are resonant with the central frequency of the reservoir and the photonic bath is initially with no excitation. Under these conditions and after applying the unitary transformation

$$\varrho(t) = U \rho_S(t) U^\dagger, \quad (6.34)$$

with

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.35)$$

on the evolved density matrix  $\rho(t)$  obtained in the interaction picture and written in the basis  $\{|2\rangle, |1\rangle, |0\rangle\}$ , one obtains the evolved state of the V-type atom by [377, 374]

$$\varrho(t) = \sum_{i=1}^3 \mathcal{K}_i \varrho(0) \mathcal{K}_i^\dagger. \quad (6.36)$$

In the above dynamical map, the Kraus operators are

$$\begin{aligned} \mathcal{K}_1 &= \begin{pmatrix} G_+(t) & 0 & 0 \\ 0 & G_-(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathcal{K}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{1 - |G_+(t)|^2} & 0 & 0 \end{pmatrix}, \\ \mathcal{K}_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{1 - |G_-(t)|^2} & 0 \end{pmatrix}, \end{aligned} \quad (6.37)$$

with

$$G_\pm(t) = e^{-\lambda t/2} \left[ \cosh\left(\frac{d_\pm t}{2}\right) + \frac{\lambda}{d_\pm} \sinh\left(\frac{d_\pm t}{2}\right) \right], \quad (6.38)$$

where  $d_\pm = \sqrt{\lambda^2 - 2\lambda\gamma(1 \pm |\theta|)}$ ,  $\lambda$  is the spectral width of the reservoir,  $\gamma$  is the relaxation rate of the two upper levels to the ground state, and  $\theta$  depends on the relative angle between two dipole moment elements associated with the transitions  $|2\rangle \rightarrow |0\rangle$  and  $|1\rangle \rightarrow |0\rangle$ . For example,  $\theta = 0$  means that the dipole moments of the two transitions are perpendicular to each other and corresponds to the case where there is no spontaneously generated interference (SGI) between the two decay channels; differently,  $\theta = \pm 1$  indicates that the two dipole moments are parallel or antiparallel, corresponding to the strongest SGI between the two decay channels. Moreover, the two coherence characteristic functions  $G_\pm(t)$  are associated, respectively, to the decay channels  $|\pm\rangle \rightarrow |0\rangle$ , where  $|\pm\rangle = (|2\rangle \pm |1\rangle)/\sqrt{2}$  [377, 374].

To assess the memory effects by the HSS-based measure, the qutrit is initially taken in the state

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(e^{i\varphi}|2\rangle + |1\rangle + |0\rangle), \quad (6.39)$$

where  $|i\rangle = U|i\rangle$  ( $i = 1, 2, 3$ ). The HSS of Eq. (6.8) is then easily obtained as

$$HSS(\varrho_\varphi(t)) = \frac{1}{3}|G_+(t)|\sqrt{|G_-(t)|^2 + 1}, \quad (6.40)$$

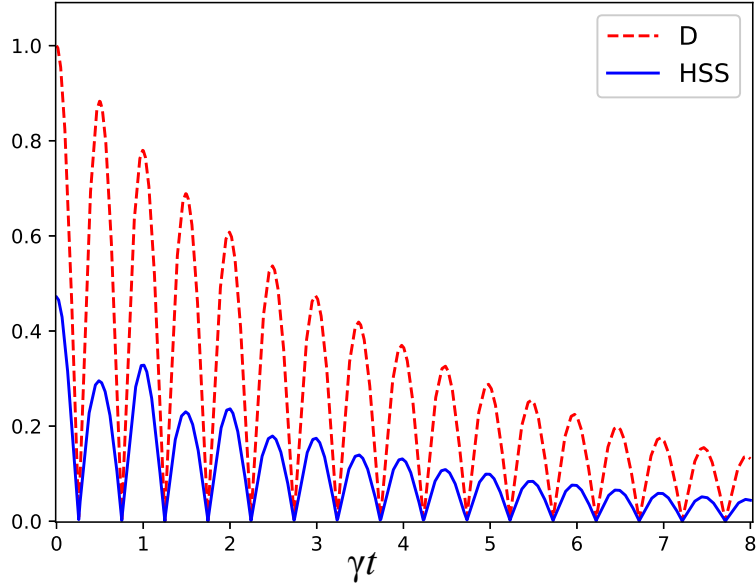


Figure 6.2: Dynamics of Hilbert-Schmidt speed  $HSS(\rho_\varphi(t))$  (blue solid line) and trace distance  $D(\rho_1(t), \rho_2(t))$  (red dashed line) as a function of the dimensionless time  $\gamma t$  for the V-type three-level atom, with  $\lambda = 5 \times 10^{-3}\gamma$  and  $\theta = 0.6$ .

being independent of the initial phase  $\varphi$ . Firstly we notice that, as physically expected, the HSS above depends on both  $G_\pm(t)$  so taking into account the interplay (interference effects) of the two decay channels. Also, under Markovian (memoryless) evolution of the qutrit, occurring for  $\lambda > 4\gamma$  (weak-coupling regime),  $HSS(\rho_\varphi(t))$  is monotonically decreasing and thus contractive. Memory effects are therefore detected when  $\chi(t) > 0$ , that is when the combination of the two channel contributions provides a net information backflow from the environment to the system (quantum speedup). On the other hand, the trace distance-based measure, obtained by choosing a pair of initial orthogonal pure states  $\rho_1(0) = |\psi_+\rangle\langle\psi_+|$  and  $\rho_2(0) = |\psi_-\rangle\langle\psi_-|$ , where  $|\psi_\pm\rangle = (|+\rangle \pm |0\rangle)/\sqrt{2}$ , is given by [374]

$$D(\rho_1(t), \rho_2(t)) = |G_+(t)|. \quad (6.41)$$

It is worth to notice that this expression does not encompass the contribution due to  $|G_-(t)|$  governing the decay channel  $|-\rangle \rightarrow |0\rangle$ , which makes us doubt whether the pair of initial states  $\rho_1(0), \rho_2(0)$  above is really the optimal one or not. Indeed, it is known that maximizing the trace distance for systems with dimension larger than 2 may be a challenging task in general. However, for  $\theta = 0, \pm 1$ , from Eqs. (6.38), (6.40) and (6.41), one immediately finds that the qualitative dynamics of  $HSS(\rho_\varphi(t))$  and  $D(\rho_1(t), \rho_2(t))$  perfectly agree, giving:  $\chi(t) > 0 \Leftrightarrow \sigma(t) > 0$ . For intermediate values of the parameter  $\theta$ ,  $HSS(\rho_\varphi(t))$  and  $D(\rho_1(t), \rho_2(t))$  maintain the general property of having the same zeros (in the oscillatory strong-coupling regime,  $\lambda < 4\gamma$ ), but their maximum points do not exactly coincide (we recall that this may be due to a nonoptimal choice of the initial states for maximizing the trace distance). The more intensely the strong coupling regime is satisfied ( $\lambda \ll \gamma$ , that means stronger memory effects), the tighter the accordance between their maximum points appears. The time behaviors of the two non-Markovianity witnesses are plotted in Fig. 6.2 for  $\theta = 0.6$  and  $\lambda = 5 \times 10^{-3}\gamma$ . One can appreciate that the trace distance and the HSS exhibit an excellent qualitative agreement, with very close maximum points. Overall, the HSS-based measure results to be a valid non-Markovianity identifier for this open V-type qutrit dynamics.

### **$\Lambda$ -type three-level open quantum system**

The last system considered in our case study analysis is the so-called  $\Lambda$  model, consisting of a three-level atom (qutrit) with excited state  $|a\rangle$  and two ground states  $|b\rangle$  and  $|c\rangle$  which interacts off-resonantly with a cavity field [373]. The cavity modes are assumed to have a Lorentzian spectral density

$$J(\omega) = \frac{\gamma_0}{2\pi} \frac{\lambda^2}{(\omega_{\text{cav}} - \omega)^2 + \lambda^2}, \quad (6.42)$$

where  $\lambda$  is the cavity spectral width,  $\omega_{\text{cav}}$  represents the resonance frequency of the cavity, and the rate  $\gamma_0$  quantifies the strength of the system-environment coupling. Moreover,  $\Delta_i = \omega_i - \omega_{\text{cav}}$  denotes the detuning of the  $i$ -th transition frequency of the atom from the cavity resonance frequency, being  $\omega_1 \equiv \omega_{ab}$  and  $\omega_2 \equiv \omega_{ac}$ . The master equation describing

the reduced dynamics of the  $\Lambda$ -type atom and its analytical solution are reported, for convenience, in Appendix B. This is characterized by two Lindblad operators  $|b\rangle\langle a|$  and  $|c\rangle\langle a|$  corresponding to the time-dependent decay rates, respectively,  $\gamma_1(t)$  and  $\gamma_2(t)$ .

To find the conditions for dynamical memory effects by means of the HSS-based measure, we prepare the  $\Lambda$ -type atom in the initial state

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(e^{i\varphi}|a\rangle + |b\rangle + |c\rangle), \quad (6.43)$$

which gives, from Eq. (6.8),  $HSS(\rho_\varphi(t)) = \frac{\sqrt{2}}{3}e^{-[\Gamma_1(t)+\Gamma_2(t)]/2}$ , where  $\Gamma_i(t) = \int_0^t ds\gamma_i(s)$ . Therefore, the non-Markovianity witness  $\chi(t)$  of Eq. (6.10) is

$$\chi(t) = \frac{-(\gamma_1(t) + \gamma_2(t))}{3\sqrt{2}}e^{-[\Gamma_1(t)+\Gamma_2(t)]/2}. \quad (6.44)$$

This equation reveals that the non-Markovian character of the system dynamics is identified by the sum of the time-dependent decay rates  $\gamma_1(t) + \gamma_2(t)$ , which takes into account the competing processes of the two decay channels associated to  $\gamma_1(t)$  and  $\gamma_2(t)$ , respectively. This is physically expected, also on the basis of previous analysis of such a  $\Lambda$ -type system in terms of non-Markovian quantum jumps [378].

Let us qualitatively discuss some particular conditions. As promptly seen from the canonical master equation given in Appendix B, if both the decay rates  $\gamma_1(t)$ ,  $\gamma_2(t)$  are nonnegative during the evolution, the open dynamics is Markovian (memoryless) [331], giving  $\chi(t) \leq 0$  and so verifying contractivity of the HSS: in this case, the rate of information flow may change but the direction of the flow remains constant, namely from the system to the environment. On the other hand, it is known that, when the detunings  $\Delta_i$  are large enough, the decay rates  $\gamma_i$  assume temporary negative values which produce information backflows from the cavity to the system [378, 330]: hence, memory effects occur ( $\chi(t) > 0$ ) when  $\gamma_1(t) + \gamma_2(t) < 0$  with an overall backflow of information. For  $\Delta_1 = \Delta_2$  the decay rates are simultaneously negative in the same time regions, while for  $\Delta_1 \neq \Delta_2$  the decay rates can have opposite signs [330]. In the latter situation, the cooperative action of the two channels become relevant. When the channel corresponding to the decay rate  $\gamma_i(t)$  ( $i = 1, 2$ ) produces more information flow from environment to system than the other channel associated to  $\gamma_j(t)$  ( $j \neq i$ ), then  $|\gamma_i(t)| > |\gamma_j(t)|$ . This means that  $\gamma_j(t) < -\gamma_i(t)$  during the time intervals when  $\gamma_i(t)$  is negative and  $\gamma_j(t)$  is positive: it is thus sufficient that only  $\gamma_i(t)$  is negative to assure non-Markovianity ( $\chi(t) > 0$ ). These results are fully consistent with the previous findings obtained by the BLP (trace distance-based) witness and  $\sigma(t)$  [330]. This open  $\Lambda$ -type qutrit system thus gives:  $\chi(t) > 0 \Leftrightarrow \sigma(t) > 0$ , confirming the faithfulness of the HSS-based measure to detect memory effects in open quantum systems of dimension three.

## 6.5 Conclusions

We have established a relation between the non-Markovian dynamics of open quantum systems and the positive changing rate of the Hilbert-Schmidt speed (HSS), which is a special case of quantum statistical speed. The idea underlying this definition is grounded on the fact that the nonmonotonic speed (positive acceleration) of quantum evolutions is a signature of memory effects in the dynamics of the system interacting with the surrounding environment. By the introduced HSS-based witness, one can then define a quantitative measure of dynamical memory effects.

We have shown, in an extensive case study analysis, that the proposed witness is as efficient as the well-known trace distance-based (BLP) witness in detecting the non-Markovianity. The models considered for our study encompass many of the most paradigmatic open quantum systems (single qubits, two qubits and single qutrits undergoing dissipative and nondissipative dynamics), and supply evidence for the sensitivity of our HSS-based witness to system-environment information backflows. Besides its conceptual interest, we remark that the HSS-based witness does not require diagonalization of the reduced system density matrix, with consequent practical advantages in the analysis. In fact, a valid quantifier with this characteristic would be highly desired, especially for assessing memory effects of high-dimensional and multipartite open quantum systems.

The HSS is related to the Hilbert-Schmidt metric. However, despite the noncontractivity of the Hilbert-Schmidt distance for quantum systems of dimension  $n > 2$ , we have shown that the HSS-based witness is a faithful non-Markovianity measure (satisfying contractivity) for all the systems studied, including qutrits ( $n = 3$ ). As a prospect, these results stimulate the investigation for systems of higher dimension to assess the extent of validity.

Our study supplies an alternative useful tool to detect non-Markovianity based on the concept of quantum statistical speed detecting system-environment backflows of information. It thus motivates further analyses on the role of memory effects in composite open quantum systems and their relation to quantum speedup.



## Appendix

### A Two-qubit evolved density matrix

Following the procedure described in Ref. [371] to construct the reduced density matrix  $\rho(t)$  for the two-qubit system discussed in Sec. 6.4.2, one finds that the diagonal and nondiagonal elements of  $\rho(t)$  in the computational basis  $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$  are given by

$$\begin{aligned}\rho_{11}(t) &= \rho_{11}(0)P(t)^2, \\ \rho_{22}(t) &= \rho_{22}(0)P(t) + \rho_{11}(0)P(t)(1 - P(t)), \\ \rho_{33}(t) &= \rho_{33}(0)P(t) + \rho_{11}(0)P(t)(1 - P(t)), \\ \rho_{44}(t) &= 1 - [\rho_{11}(t) + \rho_{22}(t) + \rho_{33}(t)],\end{aligned}\tag{6.45}$$

and

$$\begin{aligned}\rho_{12}(t) &= \rho_{12}(0)P(t)^{3/2}, \quad \rho_{13}(t) = \rho_{13}(0)P(t)^{3/2}, \\ \rho_{14}(t) &= \rho_{12}(0)P(t), \quad \rho_{23}(t) = \rho_{23}(0)P(t), \\ \rho_{24}(t) &= \sqrt{P(t)}[\rho_{24}(0) + \rho_{13}(0)(1 - P(t))], \\ \rho_{34}(t) &= \sqrt{P(t)}[\rho_{34}(0) + \rho_{12}(0)(1 - P(t))],\end{aligned}\tag{6.46}$$

with  $\rho_{ji}(t) = \rho_{ij}^*(t)$ .

### B Solutions for $\Lambda$ -type three-level system

This appendix presents the formal analytical solutions for the  $\Lambda$ -type three-level systems [378, 330]. The weak-coupling master equation for this model is written as follows

$$\begin{aligned}\frac{d}{dt}\rho(t) &= -i\lambda_1(t)[|a\rangle\langle a|, \rho(t)] - i\lambda_2(t)[|a\rangle\langle a|, \rho(t)] + \gamma_1(t)\left[|b\rangle\langle a|\rho(t)|a\rangle\langle b| - \frac{1}{2}\{\rho(t), |a\rangle\langle a|\}\right] \\ &+ \gamma_2(t)\left[|c\rangle\langle a|\rho(t)|a\rangle\langle c| - \frac{1}{2}\{\rho(t), |a\rangle\langle a|\}\right],\end{aligned}\tag{6.47}$$

where

$$\begin{aligned}\lambda_i(t) &= \int_0^t ds \int_0^\infty ds J(\omega) \sin[(\omega - \omega_i)s], \\ \gamma_i(t) &= \int_0^t ds \int_0^\infty ds J(\omega) \cos[(\omega - \omega_i)s].\end{aligned}\tag{6.48}$$

Introducing the short-hand notation

$$D_i(t) = \int_0^t ds \gamma_i(s), \quad L_i(t) = \int_0^t ds \lambda_i(s),\tag{6.49}$$

one finds that the solution of the master equation is given by [378, 330]

$$\begin{aligned}\rho_{aa}(t) &= \rho_{aa}(0)e^{-[D_1(t)+D_2(t)]}, \\ \rho_{bb}(t) &= \rho_{aa}(0) \int_0^t ds \gamma_1(s) e^{-[D_1(s)+D_2(s)]} + \rho_{bb}(0), \\ \rho_{cc}(t) &= \rho_{aa}(0) \int_0^t ds \gamma_2(s) e^{-[D_1(s)+D_2(s)]} + \rho_{cc}(0), \\ \rho_{ab}(t) &= \rho_{ab}(0)e^{-[D_1(t)+D_2(t)]/2} e^{-i[L_1(t)+L_2(t)]}, \\ \rho_{ac}(t) &= \rho_{ac}(0)e^{-[D_1(t)+D_2(t)]/2} e^{-i[L_1(t)+L_2(t)]}, \\ \rho_{bc}(t) &= \rho_{bc}(0).\end{aligned}\tag{6.50}$$

# Memory effects in high-dimensional systems faithfully identified by Hilbert-Schmidt speed-based witness

## Abstract

A witness of non-Markovianity based on the Hilbert-Schmidt speed (HSS), a special type of quantum statistical speed, has been recently introduced for low-dimensional quantum systems. Such a non-Markovianity witness is particularly useful, being easily computable since no diagonalization of the system density matrix is required. We investigate the sensitivity of this HSS-based witness to detect non-Markovianity in various high-dimensional and multipartite open quantum systems with finite Hilbert spaces. We find that the time behaviors of the HSS-based witness are always in agreement with those of quantum negativity or quantum correlation measure. These results show that the HSS-based witness is a faithful identifier of the memory effects appearing in the quantum evolution of a high-dimensional system with a finite Hilbert space. This chapter is related to Publication 5 in the List of Publications.

## 7.1 Introduction

The unavoidable interaction of quantum systems with their environments induces decoherence and dissipation of energy. Recently, because of important developments in both theoretical and experimental branches of quantum information theory, studies of memory effects (non-Markovianity) during the evolution of quantum systems have attracted much attention (see Refs. [308, 379, 380] for some reviews). Some approaches used for a quantitative description of non-Markovian processes are either related to the presence of information backflows [120] or to the indivisibility of the dynamical map [117]. However, while well-defined for classical evolution, the notion of non-Markovianity appears to still lack a unique definition in the quantum scenario [381].

Non-Markovian processes, exhibiting quantum memory effects, have been characterized and observed in various realistic systems such as quantum optical systems [90, 382, 128, 314, 383, 317], superconducting qubits [384, 385], photonic crystals [95, 93, 386], light-harvesting complexes [94], chemical compounds [96, 97]. Moreover, it is known that non-Markovianity can be a resource for quantum information tasks [98, 99, 100, 101, 102]. Accordingly, various witnesses have been proposed to identify non-Markovianity based on, for example, distinguishability between evolved quantum states of the system [120], fidelity [387, 388, 337], quantum relative entropies [121, 389], quantum Fisher information [390], capacity measure [391, 392, 393] and Bloch volume measure [394, 395, 396].

It has been shown that the nonmonotonic behavior of quantum resources such as entanglement [115], quantum coherence [397, 398, 399, 400] and quantum mutual information [401] can be interpreted as a witness of quantum non-Markovianity. Using entanglement to witness non-Markovianity was first proposed in Ref. [117]. This proposal has been theoretically investigated for qubits coupled to bosonic environments [123, 402, 403], for a damped harmonic oscillator [404], and for random unitary dynamics and classical noise models [405, 313, 406]. It is also shown that entanglement cannot capture all the quantumness of correlations because there are some separable mixed states with vanishing entanglement, while they can have nonzero quantum correlations [407]. Therefore, in this sense, quantum correlations are more robust than entanglement [408, 409, 410, 411], while entanglement may suffer sudden death [412, 413]. Consequently, many methods to quantify quantum correlations have been provided, among which quantum discord [414, 415] and measurement-induced disturbance [416] are proper for any bipartite state.

Recently, Hilbert-Schmidt speed (HSS) [358], a measure of quantum statistical speed which has the advantage of avoiding diagonalization of the evolved density matrix, has been proposed and employed as a faithful witness of non-

Markovianity in Hermitian systems [144, 417, 418, 419] and an efficient tool in quantum metrology [420, 421]. These studies are so far especially limited to low-dimensional systems, while high-dimensional ones have not been investigated in detail. We know that high-dimensional systems play a crucial role in increasing the security in quantum cryptography [422, 423], as well as in enhancing quantum logic gates, fault-tolerant quantum computation and quantum error correction [424]. This motivates us to check the sensitivity of HSS-based witness to detect non-Markovianity in high-dimensional and multipartite open quantum systems.

In this work, we analyze the validity of our HSS-based witness in various examples of high-dimensional open quantum systems with finite Hilbert spaces, such as qudits and hybrid qubit-qutrit systems. In particular, we consider a single qudit (spin-S systems) subject to a squeezed vacuum reservoir [425], and a hybrid qubit-qutrit system coupled to quantum as well as classical noises [426]. We observe that the HSS-based witness is consistent with established non-Markovianity quantifiers based on the dynamical breakdown of monotonicity for the quantum information resources.

The paper is organized as follows. In Sec. 7.2 we briefly review the definition of quantifiers. In Sec. 7.3 the sensitivity of HSS-based witness in high-dimensional and multipartite open quantum systems with finite Hilbert spaces through various examples is studied. Finally, Sec. 7.4 summarizes the main results and prospects.

## 7.2 Preliminaries

In this section, we briefly review the relevant quantifiers and concepts employed in this paper.

### 7.2.1 Non-Markovianity definition

A classical *Markov process* is described by a family of random variables  $\{X(t), t \in I \subset \mathbb{R}\}$ , for which the probability that  $X$  takes a value  $x_n$  at any arbitrary time  $t_n \in I$ , provided that it took value  $x_{n-1}$  at some previous time  $t_{n-1} < t_n$ , can be determined uniquely and may not be influenced by the possible values of  $X$  at previous times to  $t_{n-1}$ . It can be formulated in terms of conditional probabilities as follows:  $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0) = \mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1})$  for all  $\{t_n \geq t_{n-1} \geq \dots \geq t_0\} \subset I$ . Roughly speaking, its concept is connected with the memorylessness of the process and informally encapsulated by the statement that "a Markov process has no memory of the history of past values of  $X$ , i.e., the future of the process is independent of its history". To achieve a similar formulation in the quantum scenario, we should find a way to define

$\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$  for quantum systems. In the classical realm, we may sample a stochastic variable without affecting its posterior statistics. However, 'sampling' a quantum system requires measuring process and hence disturbs the state of the system, affecting the subsequent outcomes. Therefore,  $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$  depends on not only the dynamics but also the measurement process. Since in such a case the Markovian character of a quantum dynamical system is dependent on the measurement scheme, chosen to obtain  $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$ , a definition of quantum Markovianity in terms of which is a challenging task. In fact, a reliable definition of quantum Markovianity should be independent of what is required to verify it.

The aforesaid problem may be solved by adopting a different approach focusing on studying one-time probabilities  $\mathbb{P}(x, t)$ . For these, in *linear* quantum evolutions, the definition of Markovianity reduces to the concept of *divisibility* defined without any explicit reference to measurement processes in the quantum scenario [308]. To introduce the divisibility concept, let us assume that the inverse of a quantum dynamical map  $\mathcal{E}_t$  exists for all times  $t \geq 0$ . Then it is possible to define a two-parameter family of maps by means of  $\mathcal{E}_{t,s} = \mathcal{E}_t \mathcal{E}_s^{-1}$  ( $t \geq s \geq 0$ ) such that  $\mathcal{E}_{t,0} = \mathcal{E}_t$  and  $\mathcal{E}_{t,0} = \mathcal{E}_{t,s} \mathcal{E}_{s,0}$ . It should be noted that the existence of the inverse for all positive times guarantees the possibility of introducing the notion of divisibility. While  $\mathcal{E}_{t,0}$  and  $\mathcal{E}_{s,0}$  are required to be completely positive by construction, the map  $\mathcal{E}_{t,s}$  need not be completely positive and not even positive. It stems from the fact that the inverse  $\mathcal{E}_s^{-1}$  of a completely positive map  $\mathcal{E}_s$  need not be positive. The family of dynamical maps is called (C)P divisible when  $\mathcal{E}_{t,s}$  is (completely) positive for all  $t \geq s \geq 0$ .

The trace norm given by  $\|\rho\|_1 = \text{Tr} \sqrt{\rho^\dagger \rho} = \sum_k \sqrt{a_k}$ , in which  $a_k$ 's represent the eigenvalues of  $\rho^\dagger \rho$ , leads to an important measure, called *trace distance*  $D(\rho^1, \rho^2) = \frac{1}{2} \|\rho^1 - \rho^2\|_1$ , for the distance between two quantum states  $\rho^1$  and  $\rho^2$ . The trace distance  $D(\rho^1, \rho^2)$  is interpreted as the *distinguishability* between states  $\rho^1$  and  $\rho^2$ . Moreover, it is *contractive* for any completely positive and trace preserving (CPTP) map  $\mathcal{E}$  affecting two arbitrary quantum states  $\rho^{1,2}$ , i.e.,  $D(\mathcal{E}(\rho^1), \mathcal{E}(\rho^2)) \leq D(\rho^1, \rho^2)$  [380]. Because the dynamics of an open quantum system is described by a CPTP map  $\mathcal{E}_t$ , the trace distance between the initial states is always larger than the trace distance between the time-evolved quantum states. Nevertheless, this fact does *not* mean that  $D(\rho^1(t), \rho^2(t))$ , in which  $\rho^{1,2}(t) \equiv \mathcal{E}_t(\rho^{1,2}(0))$ , exhibits a monotonically decreasing function versus time [427].

There are various ways to define and detect non-Markovianity or memory effects in quantum mechanics (see [308] for a review). In [121, 120], Breuer-Laine-Piilo (BLP) proposed one of the most well-known approaches, based on variation of distinguishability of quantum states, to characterize the non-Markovian feature of the system dynamics. This is the definition of non-Markovianity that we mention in our paper. According to BLP measure, for a Markovian process, the distinguishability between any two initial states of the open system, continuously diminishes over time. In other words, a quantum evolution, mathematically described by a quantum dynamical map  $\mathcal{E}_t$ , is called Markovian if,

for any arbitrary pair of initial quantum states  $\rho^1(0)$  and  $\rho^2(0)$ , the evolved trace distance  $D(\rho^1(t), \rho^2(t))$  monotonically decreases with time. Hence, quantum Markovian dynamics exhibits a continuous loss of information from the open system to the environment. Consequently, a non-Markovian evolution is defined as a process in which, for certain time intervals,  $dD(\rho^1(t), \rho^2(t))/dt > 0$ , usually interpreted as the information flows back into the system temporarily. Provided that  $\mathcal{E}_t$  is invertible, one can show that the quantum process is BLP Markovian iff  $\mathcal{E}_t$  is P-divisible [428, 380].

### 7.2.2 HSS-based witness of non-Markovianity

Considering the distance measure [358]

$$[d(p, q)]^2 = \frac{1}{2} \sum_x |p_x - q_x|^2, \quad (7.1)$$

where  $p = \{p_x\}_x$  and  $q = \{q_x\}_x$  denote the probability distributions, one can quantify the distance between infinitesimally close distributions taken from a one-parameter family  $p_x(\phi)$  and then define the classical statistical speed as

$$s[p(\phi_0)] = \frac{d}{d\phi} d(p(\phi_0 + \phi), p(\phi_0)). \quad (7.2)$$

These classical notions can be generalized to the quantum case by assuming a pair of quantum states  $\rho$  and  $\sigma$ , and writing  $p_x = \text{Tr}\{E_x \rho\}$  and  $q_x = \text{Tr}\{E_x \sigma\}$  which represent the measurement probabilities corresponding to the positive-operator-valued measure (POVM) defined by the  $\{E_x \geq 0\}$  satisfying  $\sum_x E_x = \mathbb{I}$ . The associated quantum distance which called Hilbert-Schmidt distance [429] can be achieved by maximizing the classical distance over all possible choices of POVMs [430]

$$D(\rho, \sigma) \equiv \max_{\{E_x\}} d(p, q) = \sqrt{\frac{1}{2} \text{Tr}[(\rho - \sigma)^2]}. \quad (7.3)$$

Consequently, the HSS, the corresponding quantum statistical speed is defined as follows

$$HSS(\rho_\phi) \equiv HSS_\phi \equiv \max_{\{E_x\}} s[p(\phi)] = \sqrt{\frac{1}{2} \text{Tr} \left[ \left( \frac{d\rho_\phi}{d\phi} \right)^2 \right]}, \quad (7.4)$$

which can be easily computed without diagonalization of  $\frac{d\rho_\phi}{d\phi}$ .

Now the recently proposed protocol, completely consistent with the BLP witness, to detect non-Markovianity based on the HSS is briefly recalled [431]. We consider an  $n$ -dimensional quantum system whose initial state is given by

$$|\psi_0\rangle = \frac{1}{\sqrt{n}} (e^{i\phi} |\psi_1\rangle + \dots + |\psi_n\rangle), \quad (7.5)$$

where  $\phi$  is an unknown phase shift and  $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$  denotes a complete and orthonormal set (basis) for the corresponding Hilbert space  $\mathcal{H}$ . Given this initial state, the HSS-based witness of non-Markovianity is defined by

$$\text{Non-Markovianity Witness : } \chi(t) \equiv \frac{dHSS(\rho_\phi(t))}{dt} > 0, \quad (7.6)$$

in which  $\rho_\phi(t)$  is the evolved state of the system.

### 7.2.3 Quantum entanglement measure

Quantum entanglement is a kind of quantum correlation which, from an operational point of view, can be defined as those correlations between different subsystems which cannot be generated by local operations and classical communication (LOCC) procedures. We use negativity [432] to quantify the quantum entanglement of the state, which is a reliable measure of entanglement in the case of qubit-qubit and qubit-qutrit systems [433].

For any bipartite state  $\rho_{AB}$  the negativity is defined as

$$\mathcal{N}(\rho_{AB}) = \sum_i |\lambda_i|, \quad (7.7)$$

where  $\lambda_i$  is the negative eigenvalue of  $\rho^{T_k}$ , with  $\rho^{T_k}$  denoting the partial transpose of the density matrix  $\rho_{AB}$  with respect to subsystem  $k = A, B$ . The negativity can also be computed by the following formula [434]:

$$\mathcal{N}(\rho_{AB}) = \frac{1}{2} (\|\rho^{T_k}\| - 1), \quad (7.8)$$

in which the trace norm of  $\rho^{T_k}$  is equal to the sum of the absolute values of its eigenvalues [435]:

$$\|\rho^{T_k}\| = \sum_i |\mu_i|, \quad (7.9)$$

where the spectral decomposition of  $\rho^{T_k}$  is given by  $\sum_i \mu_i |i\rangle \langle i|$ .

### 7.2.4 Quantum correlation quantifier: Measurement-induced disturbance

We use measurement-induced disturbance MID [436] as an alternative nonclassicality indicator for quantifying the quantum correlations of the bipartite quantum systems. It is defined as the minimum disturbance caused by local projective measurements leaving the reduced states invariant.

Considering the spectral resolutions of the reduced density states  $\rho_A = \sum_i p_i^A \Pi_i^A$  and  $\rho_B = \sum_j p_j^B \Pi_j^B$ , one can compute the MID as follows

$$\mathcal{M}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{I}(\Pi(\rho_{AB})), \quad (7.10)$$

where  $\mathcal{I}$  is the mutual quantum information given by

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (7.11)$$

in which  $S(\rho) = -\text{tr} \rho \log(\rho)$  denotes the von Neumann entropy and

$$\Pi(\rho_{AB}) = \sum_{i,j} (\Pi_i^A \otimes \Pi_j^B) \rho_{AB} (\Pi_i^A \otimes \Pi_j^B). \quad (7.12)$$

## 7.3 Analyzing the efficiency of the HSS witness in high-dimensional systems with finite Hilbert spaces

In this section we check the sanity of HSS-based witness through several paradigmatic high dimensional quantum systems with finite Hilbert spaces. The analyses are based on the fact that for systems in which the corresponding subsystems are coupled to independent environments, the oscillations of quantum correlations with time are associated with the non-Markovian evolution of the system [405, 317, 437], resulting in the transfer of correlations back and forth among the various parts of the total system. Moreover, by comparing the results presented in references [438, 144, 314, 309], we can demonstrate that the BLP measure of non-Markovianity can be used as a valid definition of non-Markovianity, when we intend to detect non-Markovianity by revivals of quantum correlations.

In particular, we consider a single qudit subject to a quantum environment and a hybrid qubit-qudit system coupled to independent as well as common quantum and classical noises. We show that the oscillation of HSS-based witness is in qualitative agreement with nonmonotonic variations of the quantum resources, and hence it can be introduced as a faithful identifier of non-Markovianity in such high dimensional systems with finite Hilbert spaces.

It should be noted that the efficiency of the HSS-based witness in detecting the non-Markovian nature of the dynamics directly depends on adopting the correct parametrization of the initial state (8.9), as discussed in [144]. However, often choosing the computational basis as the complete orthonormal set  $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$  is enough to capture the non-Markovianity, as shown in this paper. In all examples discussed below, the HSS is computed for the pure initial states while the quantum correlations may be calculated for mixed ones to illustrate the general efficiency of the HSS-based witness.

### 7.3.1 Single-qudit interacting with a quantum environment

#### Coupling to a thermal reservoir

Let's consider the spin-S systems interacting with a thermal reservoir modeled by an infinite chain of quantum harmonic oscillators with  $\omega_k$ ,  $b_k$ , and  $b_k^\dagger$  being, respectively, the frequency, annihilation, and creation operators for the  $k$ -th oscillator. The total Hamiltonian of the system is given by

$$H = \omega_0 S_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k S_z (g_k b_k^\dagger + g_k^* b_k), \quad (7.13)$$

in which  $\omega_0$  denote the transition frequency between any neighboring energy states of the spin, and  $S_z$ , the  $z$  component of spin operator, can be represented by a diagonal matrix  $S_z = \text{diag}[s, s-1, \dots, -s]$  in the eigen-basis  $\{|i\rangle, i = s, \dots, -s\}$ . In the interaction picture Eq. (7.13) into is expressed as

$$H_I = \sum_k S_z (g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t}), \quad (7.14)$$

where  $g_k$  denotes the coupling strength between the spin and the environment through the dephasing interaction. Up to an overall phase factor, the corresponding unitary propagator is obtained as

$$V(t) = \exp \left[ \frac{1}{2} S_z \sum_k \left( \alpha_k b_k^\dagger - \alpha_k^* b_k \right) \right], \quad (7.15)$$

where  $\alpha_k = 2g_k (1 - e^{i\omega_k t}) / \omega_k$ .

It is assumed that the initial state of the spin-bath system is in a product state  $\rho_T(0) = \rho(0) \otimes \rho_B$  in which  $\rho(0)$  denotes the initial state of spin, and

$$\rho_B = \frac{1}{Z_B} e^{-\beta \sum_k \omega_k b_k^\dagger b_k} \quad (7.16)$$

represents the thermal equilibrium state of the bath with partition function  $Z_B$  and inverse temperature  $\beta = \frac{1}{k_B T}$ . The evolved state of the system can be calculated by [439]

$$\rho_{nm}(t) = \rho_{nm}(0) \exp[-(n-m)^2 \Gamma(t)], \quad (7.17)$$

where  $n, m = -s, -s+1, \dots, 0, \dots, s-1, s$  and, in the continuum-mode limit, the decoherence function is given by

$$\Gamma(t) = \int_0^\infty J(\omega) \coth\left(\frac{\omega}{2k_B T}\right) \frac{1 - \cos(\omega t)}{\omega^2} d\omega, \quad (7.18)$$

with spectral density  $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$ .

The  $\Gamma(t)$  behavior closely depends on the characteristics of the environment. Here we consider the Ohmic-like reservoirs with spectral density

$$J(\omega) = \alpha \frac{\omega^s}{\omega_c^{s-1}} \exp\left(-\frac{\omega}{\omega_c}\right), \quad (7.19)$$

where  $\alpha$  represents a dimensionless coupling strength, and  $\omega_c$  denotes the cutoff frequency of the bath. Changing the Ohmic parameter  $s$ , one can obtain sub-Ohmic ( $0 < s < 1$ ), Ohmic ( $s = 1$ ) and super-Ohmic ( $s > 1$ ) reservoirs.

### Coupling to a squeezed vacuum reservoir

In the case that the spin system is coupled to a squeezed vacuum reservoir, the reduced density-matrix elements are similar to ones presented in Eq. (7.17) when the decoherence function  $\Gamma(t)$  is replaced by

$$\gamma(t) = \int_0^\infty J(\omega) \frac{(1 - \cos(\omega t))}{\omega^2} [\cosh(2r) - \sinh(2r) \cos(\omega t - \theta)] d\omega, \quad (7.20)$$

where  $r$  is the squeezed amplitude parameter, and  $\theta$  denotes the squeezed angle.

Because the structures of the density matrices are the same in both scenarios (coupling to thermal and squeezed vacuum reservoirs), we only focus on the interaction of the system with the squeezed vacuum reservoir, noting that the general results also holds for the thermal reservoir.

We take the qutrit in the pure initial state

$$|\psi\rangle = \frac{1}{\sqrt{2s+1}} (e^{i\phi}|s\rangle + |s-1\rangle + |s-2\rangle + \dots + |-s\rangle), \quad (7.21)$$

which leads to the evolved state  $\rho(t)$  given by

$$\rho(t) = \frac{1}{2s+1} \begin{pmatrix} 1 & e^{-\gamma(t)} e^{i\phi} & \dots & e^{-(2s)^2 \gamma(t)} e^{i\phi} \\ e^{-\gamma(t)} e^{-i\phi} & 1 & \dots & e^{-(2s-1)^2 \gamma(t)} \\ e^{-4\gamma(t)} e^{-i\phi} & e^{-\gamma(t)} & \dots & e^{-(2s-2)^2 \gamma(t)} \\ \vdots & 1 & \ddots & \\ e^{-(2s)^2 \gamma(t)} e^{-i\phi} & e^{-(2s-1)^2 \gamma(t)} & \dots & 1 \end{pmatrix}. \quad (7.22)$$

Therefore, the time derivative of the HSS-based witness is obtained as

$$\chi(t) = -\frac{1}{2s+1} \frac{\partial \gamma(t)}{\partial t} \frac{\sum_{k=1}^{2s} k^2 e^{-2k^2 \gamma(t)}}{\sum_{k=1}^{2s} e^{-2k^2 \gamma(t)}}. \quad (7.23)$$

The HSS-based witness  $\chi(t) > 0$  tells us that the process is non-Markovian whenever  $\frac{\partial \gamma(t)}{\partial t} < 0$ , which corresponds to time intervals in which the decoherence function decreases, leading to the re-coherence phenomenon. As known, in this system the non-Markovian effects, originating from the non-divisible maps, appear when the decoherence function temporarily decays with time [336]. Therefore, our witness correctly predicts the intervals at which the memory effects arise in this single-qutrit system. Moreover, when  $\gamma(t)$  is a monotonous increasing function of time, the dynamics is Markovian because the coherence decays monotonously with time.

### 7.3.2 Hybrid qubit-qutrit system interacting with various quantum and classical environments

The composite hybrid qubit( $A$ )-qutrit( $B$ ) system consists of a spin- $\frac{1}{2}$  subsystem (qubit  $A$ ) and a spin-1 subsystem (qutrit  $B$ ). In the following we study the interaction of this composite system with local non-Markovian environments  $A$  and  $B$  or with common environment  $C$  modeling quantum or classical noises. The theoretical schematic of this system is depicted in Fig. 7.1.

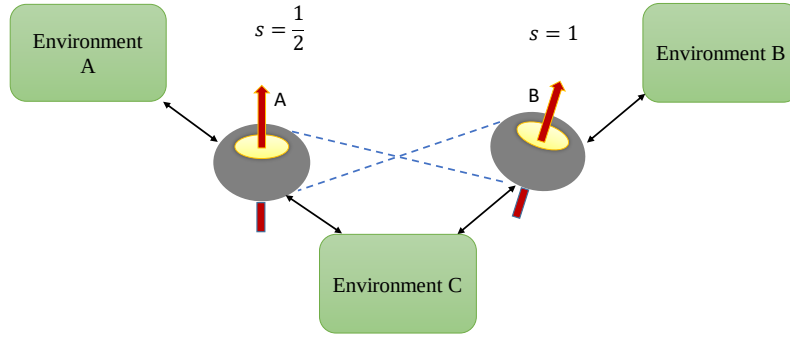


Figure 7.1: Illustration of the composite qubit(A)-qutrit(B) system; Blue dashed lines represent entanglement between the subsystems. The bipartite system can interact either with independent local environments  $E_A$ ,  $E_B$  or with a common environment  $E_C$ .

### Coupling to independent squeezed vacuum reservoirs

Now we investigate the scenario in which each of the subsystems, i.e., the qubit  $A$  ( $s_A = \frac{1}{2}$ ) and qutrit  $B$  ( $s_B = 1$ ), interacts independently with its local squeezed vacuum reservoir. For simplicity we assume that the characteristics of the reservoirs are similar. Equation (7.17) with the decoherence factor introduced in Eq. (8.17) gives the reduced density matrices of the subsystems. Computing them and applying the method presented in [438], one can obtain the elements of the evolved density matrix of the composite system as follows [440]

$$\rho_{ABnm}(t) = \rho_{ABnm}(0) \exp [-(n_A - m_A)^2 - (n_B - m_B)^2] \gamma(t), \quad (7.24)$$

where  $n_A, m_A = -s_A, \dots, s_A$  and  $n_B, m_B = -s_B, \dots, s_B$ .

### Pure initial state

We take the hybrid qubit-qutrit system initially in the pure state [144]

$$|\psi\rangle = \frac{1}{\sqrt{6}}(e^{i\phi}|00\rangle + |01\rangle + |02\rangle + |10\rangle + |11\rangle + |12\rangle), \quad (7.25)$$

which leads to a dynamics of the system described by the evolved reduced density matrix  $\rho(t)$  whose elements are presented in Appendix A. Then, the HSS is obtained as

$$HSS = \frac{1}{6} \sqrt{2e^{-2\gamma(t)} + e^{-4\gamma(t)} + e^{-8\gamma(t)} + e^{-10\gamma(t)}}. \quad (7.26)$$

The dynamics of negativity, MID and HSS computed by the evolved state of the system are plotted in Fig. 7.2. We find that each of the measures initially decreases with time, then starts to increase, and finally remains approximately constant over time, a behavior known as the freezing phenomenon [441, 442, 443, 444, 445, 446, 447]. As discussed, the revival of the quantum correlation measures can be attributed to the non-Markovian evolution of the system [405]. We see that the behaviors of the HSS, negativity and quantum correlation exhibit an excellent qualitative agreement. Consequently, the HSS-based witness can precisely capture the non-Markovian dynamics of the composite system.

### Mixed initial state

The non-Markovianity of the system, as faithfully individuated by quantum correlation measures, may in general depend on the initial state. It is thus important to investigate whether the HSS witness, obtained from the initial pure state of Eq. (7.25) by definition, is capable to identify the non-Markovian character of the system dynamics also when the system starts from a mixed state. We shall study this aspect here and in all the other environmental conditions considered hereafter (see sections below devoted to a mixed initial state).

We consider the one-parameter mixed entangled state as the initial state of the hybrid qubit-qutrit system [448]

$$\rho_0(p) = \frac{p}{2}(|01\rangle\langle 01| + |11\rangle\langle 11|) + p|\psi^+\rangle\langle\psi^+| + (1 - 2p)|\psi^-\rangle\langle\psi^-|, \quad (7.27)$$

where

$$\begin{aligned} |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |12\rangle), \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|02\rangle + |10\rangle), \end{aligned} \quad (7.28)$$

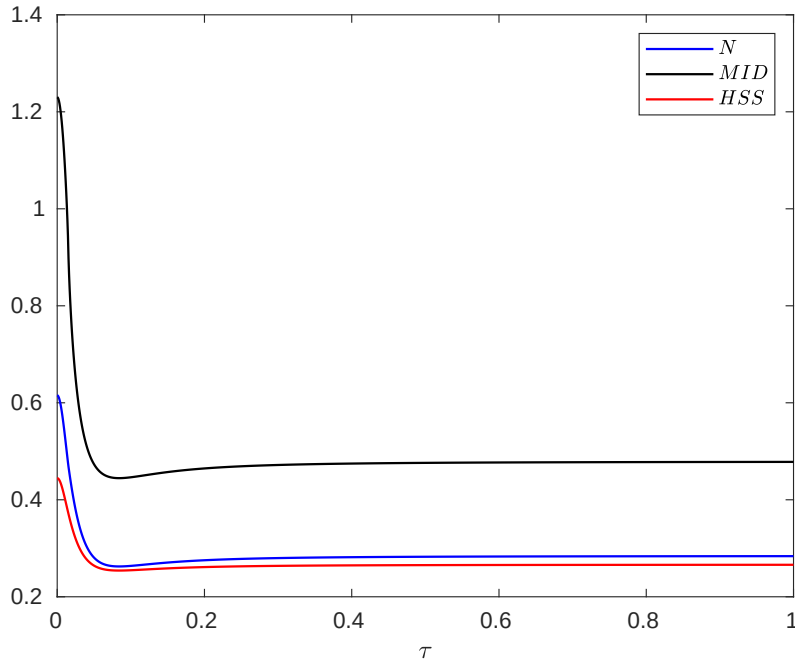


Figure 7.2: Evolution of the negativity, MID and HSS as a function of dimensionless time  $\tau = \omega_0 t$  when each subsystem of the hybrid qubit-qutrit system, starting from the initial pure state, is independently subject to a squeezed vacuum reservoir. The values of the other parameters are  $\alpha = 0.1$ ,  $\omega_c = 20\omega_0$ ,  $r = 0.3$ ,  $\phi = \pi$  and  $s = 3$ .

in which the entanglement parameter  $p$  varies from 0 to 1 such that  $\rho(p)$  is entangled except for  $p = \frac{1}{3}$ . We point out that such a state is taken as the initial state of the system for the dynamics of the quantum correlation quantifiers, namely negativity and MID. We find that Eq. (7.27) leads to the evolved state of the system

$$\rho(t) = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{p}{2}\mathcal{F} \\ 0 & \frac{p}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2}\mathcal{F} & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2}\mathcal{F} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{2} & 0 \\ \frac{p}{2}\mathcal{F} & 0 & 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}, \quad (7.29)$$

where  $\mathcal{F} = e^{-5\gamma(t)}$ . Then, the negativity is given by [426]

$$\mathcal{N} = \frac{(p-1)}{2} + \frac{1}{4}|p + (1-p)\mathcal{F}| + \frac{1}{4}|p - (1-p)\mathcal{F}| + \frac{1}{4}|p - (1-2p)\mathcal{F}| + \frac{1}{4}|p + (1-2p)\mathcal{F}|. \quad (7.30)$$

Moreover, using Eq. (7.10) we can compute the MID as

$$\mathcal{M} = \frac{(1-p)}{2} [(1+\mathcal{F}) \log(1+\mathcal{F}) + (1-\mathcal{F}) \log(1-\mathcal{F})]. \quad (7.31)$$

In Fig. 7.3 we compare the evolution of HSS, obtained from the initial pure state of Eq. (7.25), with the dynamics of negativity and MID, computed for the mixed initial state of Eq. (7.15), for different values of  $p$ . The dynamics of the HSS is again in perfect agreement with that observed for the entanglement and quantum correlations as quantified by the negativity and MID, respectively. Therefore, the HSS-based witness, computed versus the phase parameter encoded into an initial pure state of the system, can efficiently detect the non-Markovian dynamics even in the case when the initial state of our high-dimensional system is not pure. It should be noted that in the presence of sudden death of entanglement, which occurs for some values of the entanglement parameter (for example, for  $p = 0.4$ ), only the HSS and MID show the same dynamics. Hence, the negativity cannot be used as a faithful witness of non-Markovianity when it exhibits the sudden death phenomenon.

In the case of initially entangled noninteracting qubits in independent non-Markovian quantum environments, entanglement or quantum correlation revivals can be explained in terms of transfer of correlations back and forth from the composite system to the various parts of the total system. This is due to the back-action via the environment on the system, which creates correlations between qubits and environments and between the environments themselves. Accordingly, in this case the non-Markovianity is defined as backflow of information from the environment(s) to the system(s).



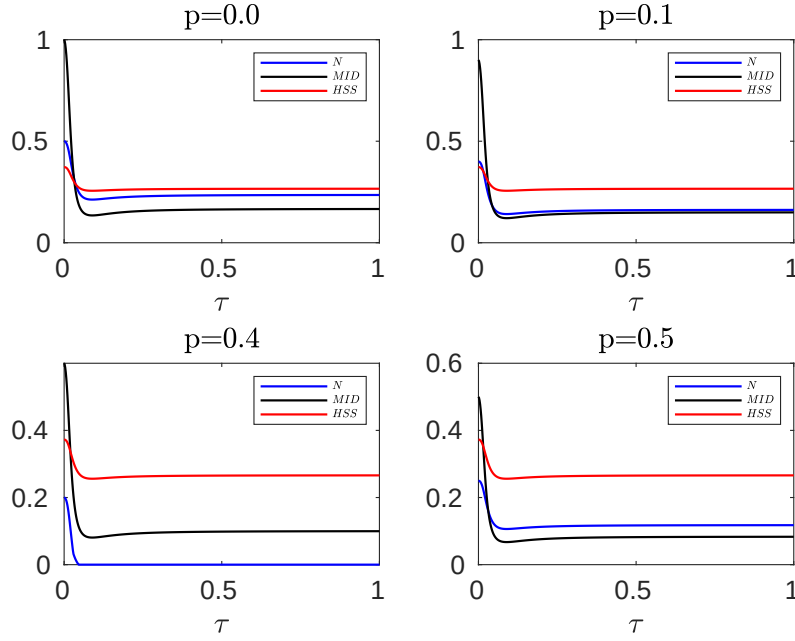


Figure 7.3: Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when each subsystem is independently coupled to a squeezed vacuum reservoir, with HSS (obtained from the initial pure state) for different values of the entanglement parameter  $p$ . In all plots the remaining parameters are  $\alpha = 0.1$ ,  $s = 3$ ,  $\omega_c = 20\omega_0$ ,  $r = 0.3$ .

### Coupling to classical environments

Here we assume that the hybrid qubit-qutrit system is affected by a classical environment implemented by the random telegraph noise (RTN) with a Lorentzian spectrum. It is a famous class of non-Gaussian noises used to generate the low-frequency  $\frac{1}{f^\alpha}$  noise both theoretically and experimentally. It is also responsible for coherent dynamics in quantum solid-state nanodevices [449, 450, 451]. Physically, the RTN may result from one of the following scenarios: (i) charges flipping between two locations in space (charge noise); (ii) electrons trapping in shallow subgap formed at the boundary between a superconductor and an insulator (noise of critical current); and (iii) spin diffusion on a superconductor surface generated by the exchange mediated by the conduction electrons (flux noise) [452, 453]. The Hamiltonian of the qubit-qutrit system under the RTN is given by

$$\mathcal{H}(t) = \mathcal{H}_0 + \mathcal{H}_I$$

$$\mathcal{H}_0 = \sum_{k=A,B} \epsilon_k S_k^Z, \mathcal{H}_I = \sum_{k=A,B} [J_k L_k(t) + J_c C(t)] S_z^k, \quad (7.32)$$

where  $\epsilon_k$  denote the energy of an isolated qubit (qutrit),  $S_z^A = \sigma_z$  and  $S_z^B$  represent the spin operators of, respectively, the qubit and the qutrit in the  $z$ -direction. Moreover,  $J_k$  and  $J_c$  represent the coupling strengths of each marginal system to the local and non-local RTN, such that we consider two types of the system-environment interactions, namely

- 1) local or independent environments (*ie*):  $J_k = \nu \neq 0$  and  $J_c = 0$ ;
- 2) non-local or common environments (*ce*):  $J_k = 0$  and  $J_c = \nu \neq 0$ .

Furthermore,  $L_k(t)$  and  $C(t)$  denote the random variables used to introduce the stochastic processes. They are used to describe the different conditions under which the subsystems undergo the decoherence due to the environment. Here, they represent classical random fluctuating fields such as bistable fluctuators flipping between two fixed values  $\pm m$  at rates  $\gamma_k$  and  $\gamma$ , respectively. For simplicity, we assume that  $\gamma_k = \gamma$ . For the *autocorrelation function* of the random variable  $\eta(t) = \{L_k(t); C(t)\}$  we have  $\langle \delta\eta(t) \delta\eta(t') \rangle = \exp[-2\gamma|t - t'|]$  with a Lorentzian power spectrum  $S(\omega) = \frac{4\gamma}{\omega^2 + \gamma^2}$ . Defining the parameter  $q = \frac{\gamma}{\nu}$ , we can identify two regimes for the dynamics of quantum correlations: the Markovian regime ( $q \gg 1$ : fast RTN), and the non-Markovian regime ( $q \ll 1$ : slow RTN). The time-evolving state of the system under the influence of the RTN is given by

$$\rho(\{\eta\}, t) = U(\{\eta\}, t) \rho(0) U^\dagger(\{\eta\}, t). \quad (7.33)$$

in which the time-evolution operator  $U(\{\eta\}, t)$  called the stochastic unitary operator in the interaction picture is given by

$$U(\{\eta\}, t) = \exp \left[ -i \int_0^t \mathcal{H}_I(t') dt' \right]. \quad (7.34)$$

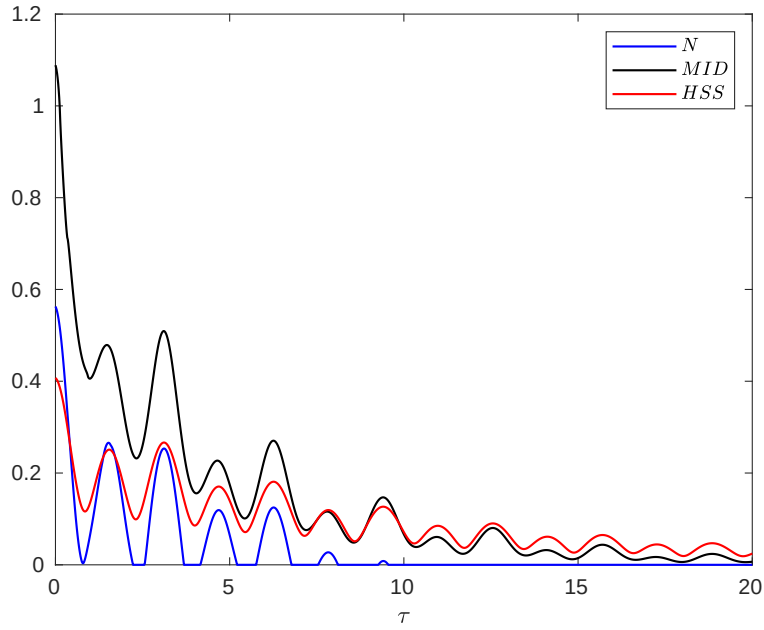


Figure 7.4: Evolution of negativity, MID and HSS as a function of dimensionless time  $\tau = \nu t$  when each subsystem of the hybrid qubit-qutrit system, starting from the initial pure state, is independently subject to a random telegraph noise in non-Markovian regime  $q = 0.1$ .

where  $\eta(t) = \{L_k(t); C(t)\}$  stands for the different realizations of the stochastic process. Because  $U(\{\eta\}, t)$  depends on the noise, we should perform the ensemble average over the noise fields to obtain the reduced density matrix of the open system, i.e.,

$$\rho_{ie(ce)} = \langle \rho(\{\eta\}, t) \rangle_{\eta(t)}. \quad (7.35)$$

The evolved state of the system in the presence of independent environments (ie) and collective environments (ce) is obtained as

$$\begin{aligned} \rho_{ie}(t) &= \langle \langle \rho(\theta_A(t), \theta_B(t), t) \rangle_{\theta_A} \rangle_{\theta_B} \\ \rho_{ce}(t) &= \langle \rho(\theta(t), t) \rangle_{\theta}, \end{aligned} \quad (7.36)$$

where  $\theta_k(t) = \nu \int_0^t L_k(t') dt'$  ( $k = A, B$ ) and  $\theta(t) = \nu \int_0^t C(t') dt'$ . Calculation of the above terms requires the computation of averaged terms of the type  $\langle e^{\pm i n \theta} \rangle$  ( $n \in N$ ) given by [454]

$$\begin{aligned} \langle e^{i n \theta} \rangle &= D_n(\tau) = \langle \cos(n\theta) \rangle \pm i \langle \sin(n\theta) \rangle, \\ \langle \sin(n\theta) \rangle &= 0, \end{aligned} \quad (7.37)$$

$$\langle \cos(n\theta) \rangle = \begin{cases} e^{-q\tau} \left[ \cosh(\xi_{qn}\tau) + \frac{q}{\xi_{qn}} \sinh(\xi_{qn}\tau) \right], & q > n \\ e^{-q\tau} \left[ \cos(\xi_{nq}\tau) + \frac{q}{\xi_{nq}} \sin(\xi_{nq}\tau) \right], & q < n \end{cases}$$

where  $\xi_{ab} = \sqrt{a^2 - b^2}$  ( $(a, b) = (n, q)$ ), and  $\tau = \nu t$  denotes the scaled (dimensionless) time [455].

### Pure initial state in the presence of independent classical environments

Here we assume that each of the qubit and the qutrit locally interacts with its local RTN while the composite system starts with the pure initial state of Eq. (7.25). For this case, the elements of the evolved density matrix are given in Appendix A. Then the HSS is obtained as

$$HSS = \frac{1}{6} \sqrt{D_1^2(\tau) + 2D_2^2(\tau) + D_2^2(\tau)D_1^2(\tau) + D_2^4(\tau)}. \quad (7.38)$$

In Fig. 7.4 we illustrate the time behaviors of the negativity, MID, and HSS in the non-Markovian regime as a function of the dimensionless time. It is clear that when the entanglement sudden death occurs, the HSS and MID synchronously oscillate with time as they are suppressed to the minimum value and then rise. Moreover, at the first revival of the measures, the minimum point of the HSS exactly coincides with that of the negativity. After that moment we see that the maximum (minimum) points of the HSS are in complete coincidence with the maximum (minimum) points of the negativity as well as the MID. This perfect qualitative agreement between HSS and entanglement or quantum correlations is evidence that the HSS-based witness can precisely detect non-Markovianity in the presence of classical noises.

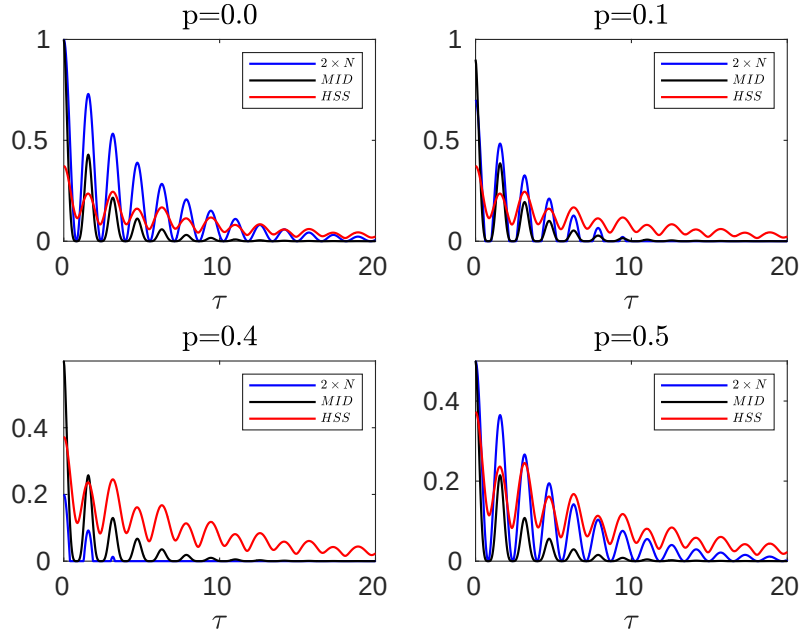


Figure 7.5: Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when each subsystem is independently coupled to a random telegraph noise, with HSS (obtained from the initial pure state) for different values of the entanglement parameter  $p$  in the non-Markovian regime:  $q = 0.1$ .

#### Mixed initial state in the presence of independent classical environments

Now we compare the dynamics of the HSS, obtained from the initial pure state of Eq. (7.25) by definition, with the evolution of the negativity and quantum correlation computed for the initially mixed state of Eq. (7.27). The evolved density matrix, the corresponding negativity, and quantum correlation are obtained from, respectively, Eqs. (7.29-7.31) replacing  $\mathcal{F}$  with  $D_2(\tau)^2$ .

Figure 7.5 exhibits this comparison for different values of the entanglement parameter  $p$ . Not considering the periods when the sudden death of the entanglement occurs, we observe that the maximum and minimum points of the measures are very close to each other and small deviations originate from the fact that the initial state, used for computation of the HSS-based measure, should be optimized over all possible parametrizations. Therefore, the HSS-based measure remains a valid non-Markovianity identifier in the presence of classical noises.

#### Mixed initial state in the presence of a common classical environment

Let us now compare the dynamics of the HSS, obtained as usual from the initial pure state of (7.25) by definition, with the evolution of the negativity and quantum correlation computed for the initial mixed state of Eq. (7.27), when both the qubit and the qutrit are embedded into a common RTN source in the non-Markovian regime. The elements of the evolved dynamical density matrix are given in Appendix A. Then, one can easily determine the HSS as

$$HSS = \frac{1}{6} \sqrt{D_1(\tau)^2 + 2D_2(\tau)^2 + D_3(\tau)^2 + D_4(\tau)^2}. \quad (7.39)$$

Moreover, the evolved density matrix of the hybrid qubit-qutrit system for the initially mixed state of Eq. (7.27) is obtained as

$$\rho(t) = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{p}{2}\mathcal{F}e^{i\phi} \\ 0 & \frac{p}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{2} & 0 \\ \frac{p}{2}\mathcal{F}e^{-i\phi} & 0 & 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}, \quad (7.40)$$

where  $\mathcal{F} = D_4(\tau)$ . As a consequence, we find that the negativity and MID are, respectively,

$$\mathcal{N} = \frac{1}{4} [(p-1) + |3p-1| + |(1-2p) - p\mathcal{F}| + |(1-2p) + p\mathcal{F}|], \quad (7.41)$$

$$\mathcal{M} = (1-2p) + \frac{p}{2} (1+\mathcal{F}) \log(1+\mathcal{F}) + \frac{p}{2} (1-\mathcal{F}) \log(1-\mathcal{F}). \quad (7.42)$$

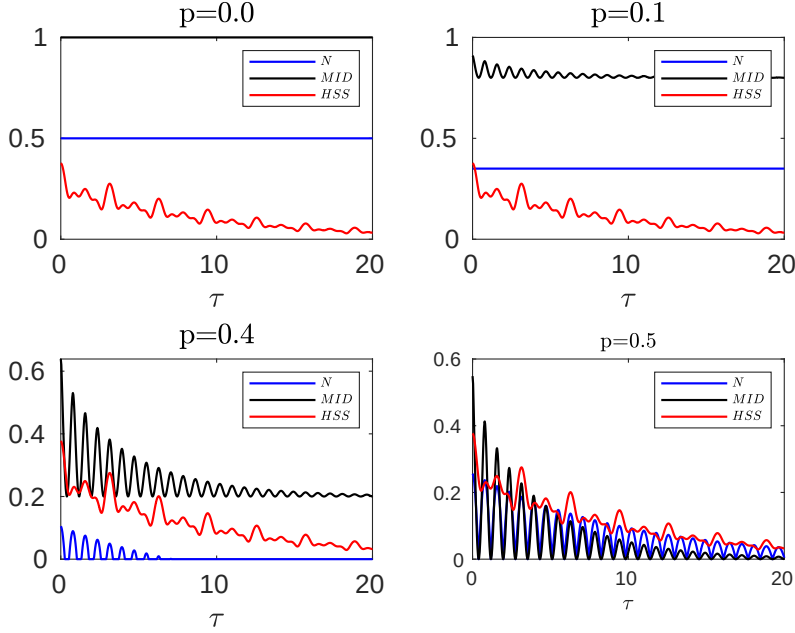


Figure 7.6: Comparing the evolution of negativity and MID computed for the initial mixed state of the hybrid qubit-qutrit system, when its subsystems are subject to a common RTN source, with HSS (obtained from the initial pure state) for different values of the entanglement parameter  $p$  in the non-Markovian regime:  $q = 0.1$ .

For common environments, we know that mutual interaction between subsystems, induced by the common environment, may lead to the preservation of correlations or even result in the creation of quantum correlations between the subsystems [456, 457, 458, 459]. Therefore, revivals of the quantum correlations cannot be necessarily linked to pure non-Markovianity effects and hence we do not expect complete consistency between the HSS and quantum correlation behaviours (see Fig. 7.6 demonstrating this feature of common environments causing the MID to fail in detecting the non-Markovianity). Except for these situations, we see that the maximum (minimum) points of the HSS computed for the initial pure state are very close to those of the MID calculated for the initial mixed state.

It should be noted that the classical environments cannot store any quantum correlations on their own, and hence they do not become entangled with their respective quantum systems. Accordingly, the common interpretation of non-Markovianity in accordance with the inflow (outflow) of information to (from) the system may be problematic in the presence of the RTN and other similar classical noises [405, 460]. In other words, it is somewhat misleading to talk about information flow from the system(s) to the environment(s) or information backflow from the environment(s) to the system(s). The better interpretation is to say that the quantum system has a recording memory of the events affecting its dynamics. When the quantum memory starts remembering, the information about the past events becomes accessible, leading to revival of the quantum correlations and hence to the appearance of quantum non-Markovianity [347].

### Composite classical-quantum environments

Here we investigate a hybrid system formed by a qubit subjected to a random telegraph noise and a qutrit independently subjected to a squeezed vacuum reservoir. The Hamiltonian of such a system can be written as

$$\mathcal{H} = \mathcal{H}_{qb}(t) \otimes \mathcal{I}_{qt} + \mathcal{I}_{qb} \otimes \mathcal{H}_{qt}(t). \quad (7.43)$$

where  $\mathcal{I}_{qb(qt)}$  denotes the identity operator acting on the subspace of the qubit (qutrit). Moreover, the Hamiltonians of the local interaction of the qubit and qutrit,  $\mathcal{H}_{qb}(t)$  and  $\mathcal{H}_{qt}(t)$ , as well as their corresponding evolution operators,  $\mathcal{U}_{qb}(\theta, t)$  and  $\mathcal{U}_{qt}(\theta, t)$  can be extracted from Secs. 7.3.2 and 7.3.1. In addition, one can consider the unitary evolution operator of the system as  $\mathcal{U} = \mathcal{U}_{qb}(\theta, t) \otimes \mathcal{U}_{qt}(t)$ .

Then, the evolved density matrix of this system can then be obtained by averaging the unitary evolved density matrix over the stochastic process induced by the RTN.

### Pure initial state

The elements of the evolved density matrix when starting from the pure state of Eq. (7.25) are given in Appendix A, leading to the following expression for the HSS:

$$HSS = \frac{1}{6} \sqrt{(e^{-2\gamma(t)} + e^{-8\gamma(t)}) \left(1 + D_2(\tau)^2\right) + D_2(\tau)^2}. \quad (7.44)$$

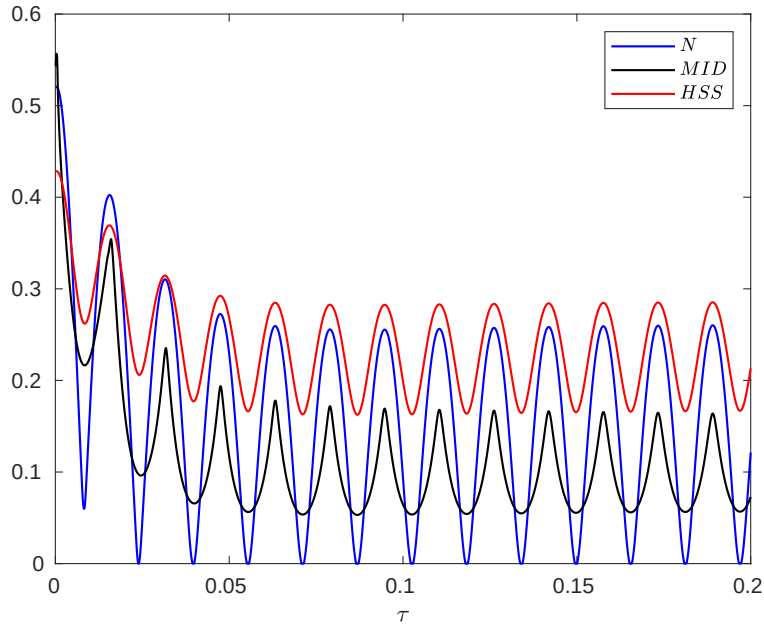


Figure 7.7: Evolution of negativity, MID and HSS as a function of dimensionless time  $\tau$  when the subsystems of the hybrid qubit-qutrit system, starting from the initial pure state, are independently subject to composite classical-quantum environments. The values of the other parameters are given by  $\alpha = 0.1$ ,  $\omega_c = 20\omega_0$ ,  $r = 0.3$ , and  $\nu = 100$ .

The time behaviors of negativity, MID and HSS are shown in Fig. 7.7 illustrating that all measures exhibit simultaneous oscillations with time such that their maximum and minimum points exactly coincide. This excellent agreement confirms the faithfulness of the HSS-based measure to detect memory effects.

#### Mixed initial state

Using Eq. (7.27) as the initial state and computing the evolved state of the system (See Appendix B), we find that the negativity and MID, respectively, are in the form of Eqs. (7.30) and (7.31) with  $\mathcal{F} = D_2(\tau)e^{-4\gamma(t)}$ . In Fig. 7.8 the dynamics of negativity and MID, obtained for the initial mixed state, has been compared with that of the HSS (computed for the initial pure state) in the non-Markovian regime.

The analyses are similar to ones in other discussed scenarios, showing that the HSS-based witness may be a proper non-Markovianity identifier even if the initial state of high dimensional systems is not pure.

## 7.4 Conclusions

Recently, the HSS-based witness, a quantifier of quantum statistical speed which has the advantage of avoiding the diagonalization of the evolved density matrix, has been introduced as a trustful witness of non-Markovianity in low-dimensional systems [144]. In this work we have generalized this result showing that the proposed witness is a bona-fide identifier of non-Markovianity for high-dimensional and multipartite open quantum systems with finite Hilbert spaces. This result stems from the observation that the HSS-based witness is in perfect agreement with established non-Markovianity identifiers based on the dynamical breakdown of monotonicity for quantum information resources, such as negativity and measurement-induced disturbance. We have found that, despite the common interpretation of non-Markovianity in terms of backflow of information from the environment to the system may be problematic [381], the HSS-based witness is capable to detect memory effects of the evolved quantum system.

In order to construct a non-Markovianity measure on the basis of a geometric distance between two quantum states, one of desirable properties is that the distance is contractive, i.e., nonincreasing under any completely positive trace preserving (CPTP) map. It has been shown that the HSS is contractive under CPTP maps in low-dimensional Hermitian systems [144]. Checking all of the dynamical cases presented here, we have found that the contractivity of the HSS holds not only in low dimensional systems but also in finite high-dimensional ones. Recently, an HSS-like measure has been used to analyze the quantum speed limit for continuous-variable systems following Gaussian preserving dynamics [461]. Therefore, our results also motivate further studies about HSS applications in detecting non-Markovianity in continuous variable systems.

Recently K. Goswami *et al.* [462] have reported a quantum-optics experimental setup to implement a non-Markovian process specifically, a process with initial classical correlations between system and environment. It should be noted that in all systems investigated in this paper we have adopted the usual assumption that the system and its environment are

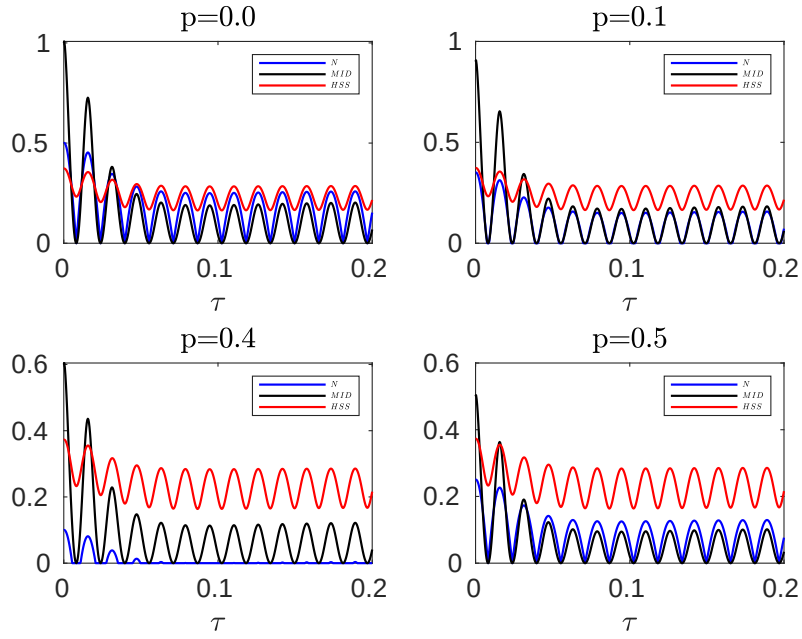


Figure 7.8: Comparing the evolution of the negativity and MID, computed for the initial mixed state of the hybrid qubit-qutrit system, when the subsystems are independently subject to composite classical-quantum environments, with the HSS obtained from the initial pure state for different values of the entanglement parameter  $p$  in the non-Markovian regime:  $q = 0.1$ . The values of the other parameters are given by  $\alpha = 0.1$ ,  $s = 3$ ,  $\omega_c = 20\omega_0$ ,  $p = 0$  and  $v = 100$ .

initially uncorrelated. It would be interesting to generalize the application of the HSS-based non-Markovianity witness to scenarios in which initial correlations between the system and environment rise. This will be studied in detail in our future works.

## Appendix

### A Pure hybrid qubit-qutrit evolved density matrix

This appendix presents the elements of the evolved density matrix of hybrid qubit-qutrit system, starting from the initial pure state of Eq. (7.25), in the presence of quantum and classical noises. This evolved state is required for the assessment of non-Markovianity via the HSS-based witness.

### Squeezed vacuum reservoirs

The elements of the evolved density matrix, when each subsystem of the hybrid qubit-qutrit system is independently subject to a squeezed vacuum reservoir, in the computational basis  $|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle$  are given by

$$\begin{aligned}
\rho_{11}(t) &= \rho_{22}(t) = \rho_{33}(t) = \rho_{44}(t) = \rho_{55}(t) = \rho_{66}(t) = \frac{1}{6} \\
\rho_{12}(t) &= \rho_{14}(t) = \rho_{21}^*(t) = \rho_{41}^*(t) = \frac{1}{6}e^{i\phi}e^{-\gamma(t)} \\
\rho_{13}(t) &= \rho_{31}^*(t) = \frac{1}{6}e^{i\phi}e^{-4\gamma(t)} \\
\rho_{15}(t) &= \rho_{51}^*(t) = \frac{1}{6}e^{-2\gamma(t)} \\
\rho_{16}(t) &= \rho_{61}^*(t) = e^{i\phi}e^{-5\gamma(t)} \\
\rho_{23}(t) &= \rho_{25}(t) = \rho_{32}(t) = \rho_{36}(t) = \rho_{45}(t) = \rho_{52}(t) = \rho_{54}(t) = \rho_{56}(t) \\
&= \rho_{63}(t) = \rho_{65}(t) = \frac{1}{6}e^{-\gamma(t)} \\
\rho_{46}(t) &= \rho_{64}(t) = \frac{1}{6}e^{-4\gamma(t)} \\
\rho_{24}(t) &= \rho_{26}(t) = \rho_{35}(t) = \rho_{42}(t) = \rho_{53}(t) = \rho_{62}(t) = \frac{1}{6}e^{-2\gamma(t)} \\
\rho_{34}(t) &= \rho_{43}(t) = \frac{1}{6}e^{-5\gamma(t)}.
\end{aligned} \tag{7.45}$$

### Independent random telegraph noise

The elements of the evolved density matrix, when each subsystem of the hybrid qubit-qutrit system is independently subject to the classical random telegraph noise, can be obtained as

$$\begin{aligned}
\rho_{11}(t) &= \rho_{22}(t) = \rho_{33}(t) = \rho_{44}(t) = \rho_{55}(t) = \rho_{66}(t) = \frac{1}{6} \\
\rho_{12}(t) &= \rho_{21}^*(t) = \frac{1}{6}e^{i\phi}D_1(\tau) \\
\rho_{23}(t) &= \rho_{32}(t) = \rho_{45}(t) = \rho_{54}(t) = \rho_{56}(t) = \rho_{65}(t) = \frac{1}{6}D_1(\tau) \\
\rho_{13}(t) &= \rho_{14}(t) = \rho_{31}^*(t) = \rho_{41}^*(t) = \frac{1}{6}e^{i\phi}D_2(\tau) \\
\rho_{25}(t) &= \rho_{36}(t) = \rho_{46}(t) = \rho_{52}(t) = \rho_{63}(t) = \rho_{64}(t) = \frac{1}{6}D_2(\tau) \\
\rho_{15}(t) &= \rho_{51}^*(t) = \frac{1}{6}e^{i\phi}D_2(\tau)D_1(\tau) \\
\rho_{24}(t) &= \rho_{26}(t) = \rho_{35}(t) = \rho_{42}(t) = \rho_{53}(t) = \rho_{62}(t) = \frac{1}{6}D_2(\tau)D_1(\tau) \\
\rho_{16}(t) &= \rho_{61}^*(t) = \frac{1}{6}e^{i\phi}D_2^2(\tau) \\
\rho_{34}(t) &= \rho_{43}(t) = \frac{1}{6}D_2^2(\tau).
\end{aligned} \tag{7.46}$$

### Common random telegraph noise

The elements of the evolved density matrix, when the qubit and qutrit are subject to a common RTN source, are given by

$$\begin{aligned}
\rho_{11}(t) &= \rho_{22}(t) = \rho_{33}(t) = \rho_{44}(t) = \rho_{55}(t) = \rho_{66}(t) = \frac{1}{6} \\
\rho_{12}(t) &= \rho_{21}^*(t) = \frac{1}{6} e^{i\phi} D_1(\tau) \\
\rho_{23}(t) &= \rho_{32}(t) = \rho_{24}(t) = \rho_{42}(t) = \rho_{35}(t) = \rho_{53}(t) = \\
\rho_{45}(t) &= \rho_{54}(t) = \rho_{56}(t) = \rho_{65}(t) = \frac{1}{6} D_1(\tau) \\
\rho_{13}(t) &= \rho_{14}(t) = \rho_{31}^*(t) = \rho_{41}^*(t) = \frac{1}{6} e^{i\phi} D_2(\tau) \\
\rho_{25}(t) &= \rho_{36}(t) = \rho_{46}(t) = \rho_{52}(t) = \rho_{63}(t) = \rho_{64}(t) = \frac{1}{6} D_2(\tau) \\
\rho_{15}(t) &= \rho_{51}^*(t) = \frac{1}{6} e^{i\phi} D_3(\tau) \\
\rho_{24}(t) &= \rho_{26}(t) = \rho_{35}(t) = \rho_{42}(t) = \rho_{53}(t) = \rho_{62}(t) = \frac{1}{6} D_2(\tau) D_1(\tau) \\
\rho_{16}(t) &= \rho_{61}^*(t) = \frac{1}{6} e^{i\phi} D_4(\tau) \\
\rho_{34}(t) &= \rho_{43}(t) = \frac{1}{6}.
\end{aligned} \tag{7.47}$$

### Composite classical-quantum environments

The elements of the evolved density matrix, when the qubit and qutrit are independently subject to, respectively, random telegraph noise channel and squeezed vacuum reservoirs, can be obtained as

$$\begin{aligned}
\rho_{11}(t) &= \rho_{22}(t) = \rho_{33}(t) = \rho_{44}(t) = \rho_{55}(t) = \rho_{66}(t) = \frac{1}{6} \\
\rho_{12} &= \rho_{21}^* = \frac{1}{6} e^{i\phi} e^{-\gamma(t)} \\
\rho_{23}(t) &= \rho_{32}(t) = \rho_{45}(t) = \rho_{54}(t) = \rho_{56}(t) = \rho_{65}(t) = \frac{1}{6} e^{-\gamma(t)} \\
\rho_{13}(t) &= \rho_{31}^*(t) = \frac{1}{6} e^{i\phi} e^{-4\gamma(t)} \\
\rho_{14}(t) &= \rho_{41}^*(t) = \frac{1}{6} e^{i\phi} D_2(\tau) \\
\rho_{15}(t) &= \rho_{51}^*(t) = \frac{1}{6} e^{i\phi} D_2(\tau) e^{-\gamma(t)} \\
\rho_{16}(t) &= \rho_{61}^*(t) = \frac{1}{6} e^{i\phi} D_2(\tau) e^{-4\gamma(t)} \\
\rho_{25}(t) &= \rho_{36}(t) = \rho_{52}(t) = \rho_{63}(t) = \frac{1}{6} D_2(\tau) \\
\rho_{24}(t) &= \rho_{26}(t) = \rho_{35}(t) = \rho_{42}(t) = \rho_{53}(t) = \rho_{62}(t) = \frac{1}{6} D_2(\tau) e^{-\gamma(t)} \\
\rho_{34}(t) &= \rho_{43}(t) = \frac{1}{6} D_2(\tau) e^{-4\gamma(t)} \\
\rho_{46}(t) &= \rho_{64}(t) = \frac{1}{6} e^{-4\gamma(t)}.
\end{aligned} \tag{7.48}$$

## B Mixed hybrid qubit-qutrit evolved density matrix

This appendix presents the elements of the evolved density matrix of hybrid qubit-qutrit system, starting from the initial mixed state of Eq. (7.27), in the presence of quantum and classical noises.



### Squeezed Vacuum reservoirs

The elements of the evolved density matrix, when each subsystem of the hybrid qubit-qutrit system is independently subject to a squeezed vacuum reservoir, are given by

$$\rho(t) = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{p}{2}\mathcal{F} \\ 0 & \frac{p}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2}\mathcal{F} & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2}\mathcal{F} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{2} & 0 \\ \frac{p}{2}\mathcal{F} & 0 & 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}, \quad (7.49)$$

and the partial transpose with respect to the subsystem  $A$  is

$$\left(\rho(t)^{AB}\right)^{T_A} = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{1-2p}{2}\mathcal{F} \\ 0 & \frac{p}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{p}{2}\mathcal{F} & 0 & 0 \\ 0 & 0 & \frac{p}{2}\mathcal{F} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{2} & 0 \\ \frac{1-2p}{2}\mathcal{F} & 0 & 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}, \quad (7.50)$$

where the  $\mathcal{F} = e^{-5\gamma(t)}$ .

### Independent random telegraph noise

The elements of the evolved density matrix, when each subsystem of the hybrid qubit-qutrit system is independently subject to the classical random telegraph noise, are given by Eq. (7.49) with  $\mathcal{F} = D_2(\tau)^2$ .

### Common random telegraph noise

The evolved density matrix, when the qubit and qutrit are subject to a common RTN source, is given by

$$\rho(t) = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 & 0 & \frac{p}{2}\mathcal{F} \\ 0 & \frac{p}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & \frac{1-2p}{2} & \frac{1-2p}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{p}{2} & 0 \\ \frac{p}{2}\mathcal{F} & 0 & 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}, \quad (7.51)$$

where  $\mathcal{F} = D_4(\tau)$ .

### Composite classical-quantum environments

The elements of the evolved density matrix, when the qubit and qutrit are independently subject to, respectively, random telegraph noise channel and squeezed vacuum reservoirs, are given by Eq. (7.49) with  $\mathcal{F} = D_2(\tau)e^{-4\gamma(t)}$ .

# Witnessing global memory effects of multiqubit correlated noisy channels by Hilbert-Schmidt speed

## Abstract

In correlated noisy channels, the global memory effects on the dynamics of a quantum system depend on both non-Markovianity of the single noisy channel (intrinsic memory) and classical correlations between multiple uses of the channel itself (correlation-based memory). We show that the Hilbert-Schmidt speed (HSS), a measure of non-Markovianity, serves as a reliable figure of merit for evaluating the role of this correlation-based memory on the global memory effects, for both unital and non-unital channels. The intensity of the correlation-based memory is ruled by a classical correlation strength between consecutive applications of the channel. We demonstrate that, for unital noisy channels, increasing the number of qubits of the system significantly weakens the sensitivity of the HSS to this classical correlation strength. Such a pattern indicates that the state evolution of large quantum systems may be less prone to be affected by classical correlations between noisy channels. Moreover, assuming the qubits are affected by independent or classically correlated local non-Markovian unital channels, we observe that, as the number of qubits increases, the collective behavior of the multiqubit system inhibits the non-Markovian features of the overall system dynamics.

This chapter is related to Publication 6 in the List of Publications.

## 8.1 Introduction

The inevitable interaction of quantum systems with their environment creates a noisy channel which induces decoherence, leading to the loss of quantum properties of the system [105, 308]. This phenomenon is well described by the theory of open quantum systems. These system-environment interactions can be either memoryless, known as Markovian regime, or possess memory and involve information backflow, referred to as non-Markovian regime. Memory effects occurring within the non-Markovian regime can mitigate the detrimental effects of environment-induced decoherence. This notion of memory stems from the intrinsic structure of the noisy environment, such as its spectral density, and defines the non-Markovianity of the system dynamics. Non-Markovianity has been identified as a valuable resource for quantum information and communication protocols [98, 99, 100, 309, 101, 102, 405, 463, 464, 465, 314, 466, 467, 468, 469, 418].

On the other hand, in quantum information theory memory effects of quantum channels are identified through multiple uses of the channel on a sequence of quantum systems. Channels with memory show classical correlations between multiple uses, while memoryless channels do not exhibit such correlations. This different notion of memory, due to correlations between consecutive applications of a channel, plays a role in protecting and enhancing quantum resources. Specifically, these effects can suppress decoherence [470] and enhance quantum coherence [471], quantum correlations [472], entanglement [473, 474], teleportation fidelity [475], quantum Fisher information [476], and measurement uncertainty [477].

In correlated non-Markovian channels, the global memory effects encompass both concepts of memory defined above: (i) the intrinsic memory dictated by the channel's non-Markovianity, and (ii) the memory stemming from classical correlations between consecutive uses of the quantum channel. The study presented in Ref. [145] explores this interplay under a dephasing scenario, utilizing two measures of non-Markovianity: trace distance [120], which assesses the backflow of information from the environment to the system, and entanglement-based measures [117]. The latter measures reveal that classical correlations between multiple uses of non-Markovian quantum channels amplify the overall dynamical

non-Markovianity, while the trace distance measure is not affected by these classical correlations. Importantly, for both measures, correlation effects do not modify the time intervals during which non-Markovianity appears.

Additionally, the impact of channel correlations has been studied using the entanglement-based non-Markovianity indicator and a non-Markovianity witness based on the variation of the volume of accessible states, as detailed in Ref. [147]. These models, however, are limited to scenarios involving modified Ornstein-Uhlenbeck noise, correlated random telegraph noise [478], and correlated non-Markovian amplitude damping channels [479]. None of these approaches comprehensively address the critical aspect of classical channel correlation on information backflow, as discussed in Ref. [120].

The Hilbert-Schmidt speed (HSS) [144, 420, 480, 417, 481], a specialized quantum statistical measure, offers a computationally efficient method to detect non-Markovianity without requiring the diagonalization of the system's density matrix [156]. Recognizing the significance of correlated non-Markovian channels in enhancing the quantum speed limit [146], increasing channel capacity, and mitigating noise in quantum error correction [147], we investigate the global dynamic memory effect in correlated noisy channels. Our focus is on how classical correlation influences non-Markovianity, specifically through the backflow of information, utilizing HSS.

This study generalizes the findings of Ref. [145] and examines both unital and non-unital channels with varying noisy spectral densities, including Pauli and depolarizing channels for unital cases and amplitude-damping channels for non-unital cases. Our results indicate that HSS reliably measures these correlations and that classical memory significantly impacts non-Markovianity. Furthermore, we demonstrate that in unital noisy channels with a large number of qubits, the HSS sensitivity to the classical correlation strength between consecutive channel uses is considerably reduced. Initially, we analyze the effect of correlated noisy channel uses on two qubits and subsequently extend our analysis to multiqubit noisy channels, capitalizing on the computational simplicity of HSS. By examining the behavior of these multi-qubit systems, we illustrate how the collective non-Markovian behavior of the entire system diminishes as the number of qubits, each locally interacting with non-Markovian environments, increases.

The paper is organized as follows: in Sec. 8.2, we present the general representation of the quantum channel. The quantifier of the non-Markovian memory effect is discussed in Sec. 8.3. In Sec. 8.4, we examine the two-qubit correlated unital noisy quantum channels under various noisy environments. Further, in Sec. 8.5, we investigate the non-unital channel. In Sec. 8.6, we extend our discussion to multiqubit correlated unital noisy channels. Finally, Sec. 8.7 summarizes our results.

## 8.2 Correlated quantum channels

Quantum channels are fundamental constructs in quantum information theory, representing the physical processes that convert an initial quantum state into a final one. These channels can be classified into two primary types: memory channels and memoryless channels. Memory channels exhibit correlations between successive uses, while memoryless channels treat each use independently. This memory effect is distinct from non-Markovian memory effects, which arise from temporal correlations in the system's dynamics.

Consider a quantum channel  $\Phi$  acting on a system of  $n$  qubits, each independently influenced by a noise described by Kraus operator  $K_{i_n}$ , and hence given by [473]

$$\Phi(\rho) = \sum_{i_1 \dots i_n} (K_{i_n} \otimes \dots \otimes K_{i_1}) \rho (K_{i_n}^\dagger \otimes \dots \otimes K_{i_1}^\dagger), \quad (8.1)$$

A channel is termed unital if it maps the identity operator  $\mathbb{I}$  to itself  $\Phi(\mathbb{I}) = \mathbb{I}$ . Non-unital channels, on the other hand, do not preserve the identity operator  $\Phi(\mathbb{I}) \neq \mathbb{I}$ .

For unital channels, such as Pauli channels, a generalized framework can be represented by Kraus operators in the following form [473]

$$K_{i_1 \dots i_n} = \sqrt{p_{i_1 \dots i_n}} \sigma_{i_1} \dots \sigma_{i_n}, \quad (8.2)$$

where  $\sum_{i_1 \dots i_n} p_{i_1 \dots i_n} = 1$ ,  $\sigma_0$  denotes the  $2 \times 2$  identity matrix, and  $\sigma_i$  are the Pauli operators in the  $x$ ,  $y$ , and  $z$  directions. For memoryless channels, the probabilities factorize as  $p_{i_1 \dots i_n} = p_{i_1} p_{i_2} \dots p_{i_n}$ . An interesting extension involves Markov chains, defined as

$$p_{i_1 \dots i_n} = p_{i_1} p_{i_2|i_1} \dots p_{i_n|i_{n-1}}, \quad (8.3)$$

where  $p_{i_n|i_{n-1}}$  represents the conditional probability that the channel affects the  $n$ th qubit, given it was applied to the  $(n-1)$ th qubit. In channels with partial memory, where consecutive uses are correlated, one assumes [473]

$$p_{i_n|i_{n-1}} = (1 - \mu) p_{i_n} + \mu \delta_{i_n, i_{n-1}}. \quad (8.4)$$

Here,  $\mu$  is the classical correlation factor (or strength) between consecutive channels, indicating that with probability  $\mu$  the same operation is applied to both qubits, while with probability  $1 - \mu$  the operations are uncorrelated, as depicted in Fig. 8.1.

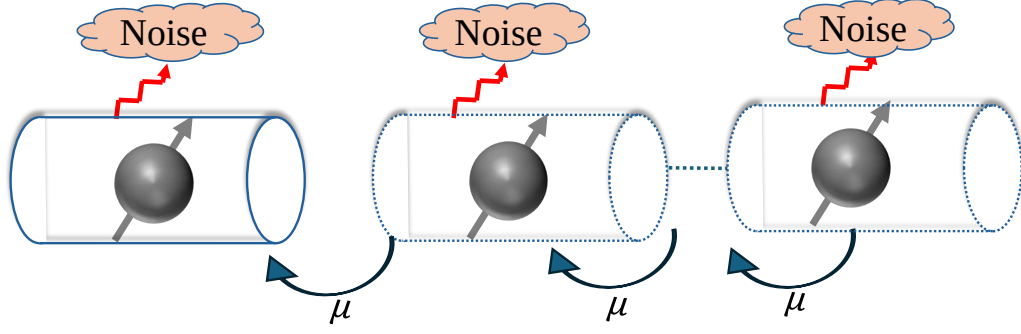


Figure 8.1: Illustration of the action of correlated noisy channels. The parameter  $\mu$  is the classical correlation factor between consecutive uses of a quantum channel on the sequence of quantum systems.

### 8.3 Characterizing Non-Markovianity with Hilbert-Schmidt speed

In this section, we briefly review the definition of the Hilbert-Schmidt speed, previously introduced as a quantifier of the non-Markovian memory effect in open quantum systems.

We begin by introducing the distance measure [358]

$$[d(p, q)]^2 = \frac{1}{2} \sum_x |p_x - q_x|^2, \quad (8.5)$$

where  $p = \{p_x\}_x$  and  $q = \{q_x\}_x$  are the probability distributions. The classical statistical speed can then be defined by quantifying the distance between infinitesimally close distributions taken from a one-parameter family ( $p_x(\phi)$ ) as follows:

$$s[p(\phi_0)] = \frac{d}{d\phi} d(p(\phi_0 + \phi), p(\phi_0)). \quad (8.6)$$

These classical notions can be extended to the quantum case by considering a pair of quantum states  $\rho$  and  $\sigma$ , and writing  $p_x = \text{Tr}\{E_x \rho\}$  and  $q_x = \text{Tr}\{E_x \sigma\}$ , which represent the measurement probabilities corresponding to the positive-operator-valued measure (POVM) defined by the  $\{E_x \geq 0\}$  satisfying  $\sum_x E_x = \mathbb{I}$ .

The corresponding quantum distance, known as the Hilbert-Schmidt distance [429], can be obtained by maximizing the classical distance over all possible choices of POVMs [430]

$$D(\rho, \sigma) \equiv \max_{\{E_x\}} d(p, q) = \sqrt{\frac{1}{2} \text{Tr}[(\rho - \sigma)^2]}. \quad (8.7)$$

Consequently, the Hilbert-Schmidt speed (HSS), the corresponding quantum statistical speed, is introduced as

$$\text{HSS}(\rho_\phi) \equiv \text{HSS}\phi \equiv \max_{\{E_x\}} s[p(\phi)] = \sqrt{\frac{1}{2} \text{Tr} \left[ \left( \frac{d\rho_\phi}{d\phi} \right)^2 \right]}, \quad (8.8)$$

which can be easily computed without the need for diagonalizing the matrix  $d\rho_\phi/d\phi$ .

It is now useful to recall the proposed non-Markovianity witness based on the HSS [144]. We consider an  $n$ -dimensional quantum system whose initial state is given by

$$|\psi_0\rangle = \frac{1}{\sqrt{n}} (e^{i\phi} |\psi_1\rangle + \dots + |\psi_n\rangle), \quad (8.9)$$

where  $\phi$  is an unknown phase shift, and  $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$  indicates a complete and orthonormal set (basis) for Hilbert space  $\mathcal{H}$ . To do so, the HSS-based witness of non-Markovianity is introduced as

$$\chi(t) \equiv \frac{d\text{HSS}(\rho_\phi(t))}{dt} > 0, \quad (8.10)$$

and, consequently, the degree of non-Markovianity is defined as

$$\mathcal{N}_{\text{HSS}} = \max_{\varphi, \{|\psi_1\rangle, \dots, |\psi_n\rangle\}} \int_{\chi(t) > 0} \chi(t) dt, \quad (8.11)$$

where  $\rho_\phi(t)$  denotes the evolved state of the system.

## 8.4 Two-qubit Correlated unital noisy quantum channels

In this section, we discuss Pauli and depolarizing channels as examples of unital channels under various noisy environments.

### 8.4.1 Model I: Two-qubit Pauli channels under two different noisy environments

#### Two-qubit Pauli Channels under Colored Pure Dephasing Reservoir

We investigate the dynamics of two consecutive Pauli channels influenced by colored pure dephasing noise with partial memory on two qubits, forming two-qubit correlated noisy channels. Simplifying Eq. (8.1), the two consecutive channel uses with partial memory can be described as follows:

$$\rho \rightarrow \Phi(\rho) = \sum_{i,j} p_{i,j} (\sigma_i \otimes \sigma_j) \rho (\sigma_i \otimes \sigma_j), \quad (8.12)$$

where  $p_{i,j} = (1-\mu)p_i p_j + \mu p_i \delta_{i,j}$ . To analyze the relationship between multiple uses of correlated quantum channels and non-Markovian memory effects, the coefficient  $p_{i,j}$  is time-dependent. The Kraus operators governing these dynamics are given by [482]

$$K_i = \sqrt{p_i} \sigma_i, \quad (i = 0, 1, 2, 3), \quad (8.13)$$

where  $p_0 = \frac{1}{2}(1 + \eta(\tau))$ ,  $p_1 = 0$ ,  $p_2 = 0$ ,  $p_3 = \frac{1}{2}(1 - \eta(\tau))$ , and  $\eta(\tau) = e^{-\tau} \left( \frac{\sin(\tau u)}{u} + \cos(\tau u) \right)$ , with  $u = \sqrt{16\nu^2 - 1}$  and  $\tau = \frac{t}{2\nu}$ . The parameter  $\nu$  controls the non-Markovianity degree of the dephasing process that induces dynamical memory effects. Accordingly, the correlated quantum channel in Eq. (8.12), describing the dynamical evolution of the open quantum system, is expressed as

$$\begin{aligned} \Phi(\rho) = & p_{0,3} (\sigma_0 \otimes \sigma_3) \rho (\sigma_0 \otimes \sigma_3) + \\ & p_{3,0} (\sigma_3 \otimes \sigma_0) \rho (\sigma_3 \otimes \sigma_0) + \\ & p_{0,0} (\sigma_0 \otimes \sigma_0) \rho (\sigma_0 \otimes \sigma_0) + \\ & p_{3,3} (\sigma_3 \otimes \sigma_3) \rho (\sigma_3 \otimes \sigma_3). \end{aligned} \quad (8.14)$$

The time evolution of the density matrix for two-qubit systems can be easily computed [145]. When parametrizing the initial state in the standard computational basis (see Appendix A), the Hilbert Schmidt speed (HSS) is given by

$$\text{HSS} = \frac{1}{4} \sqrt{((1-\mu)\eta(\tau)^2 + \mu)^2 + 2\eta(\tau)^2}. \quad (8.15)$$

HSS depends on the effect of classical correlation in multiple uses of the quantum channel, irrespective of the initial basis (refer to Appendix B for the expressions of the HSS in other relevant bases, whose definition is given in Appendix A).

The dynamics of HSS as a function of the correlation coefficient  $\mu$  is shown in Fig. 8.2a. The curves demonstrate that HSS, as a bona-fide witness, can evaluate the global memory effect of correlated non-Markovian channels. It serves as both a witness of non-Markovianity and an indicator for monitoring classical memory. An increase in classical correlation between multiple uses of the channel (thus increasing the memory) corresponds to an amplification of the HSS.

Notably, increasing the memory of the quantum channel  $\mu$  is also associated with an attenuation of the time fluctuations of the HSS. To examine the effect of classical correlation between multiple uses of the channel on the backflow of information, we consider two factors: the time intervals of non-Markovianity and the degree of non-Markovianity.

From the dynamics of HSS, it is apparent that the time intervals for the appearance of non-Markovianity do not change with increasing classical correlation between sequences of noisy Pauli channels. Additionally, we compute the HSS-based measure of non-Markovianity  $\mathcal{N}_{\text{HSS}}$  through numerical optimization over numerous initial states. As displayed in Fig. 8.2b, this measure, which quantifies the strength of the backflow of information from the environment to the system, does not increase with the degree of classical correlation between consecutive uses of the channel. This finding aligns with the results presented in Ref. [145], which investigates the same scenario using a trace-distance based (BLP) measure of non-Markovianity. This consistency corroborates the complete similarity between the HSS-based and BLP measures of non-Markovianity, even in correlated channels with memory.

#### Two-Qubit Pauli Channels under Squeezed Vacuum Reservoir

We examine a scenario where two spin- $s$  systems, specifically two-qubit Pauli channels ( $s = \frac{1}{2}$ ), are subjected to a squeezed vacuum reservoir. These two-qubit Pauli channels undergo dephasing and experience decoherence without dissipation. The Kraus operators for such system are given by Eq. (8.13), with

$$p_0 = \frac{1}{2} (e^{-\gamma(t)} + 1), p_1 = p_2 = 0, p_3 = \frac{1}{2} (1 - e^{-\gamma(t)}), \quad (8.16)$$

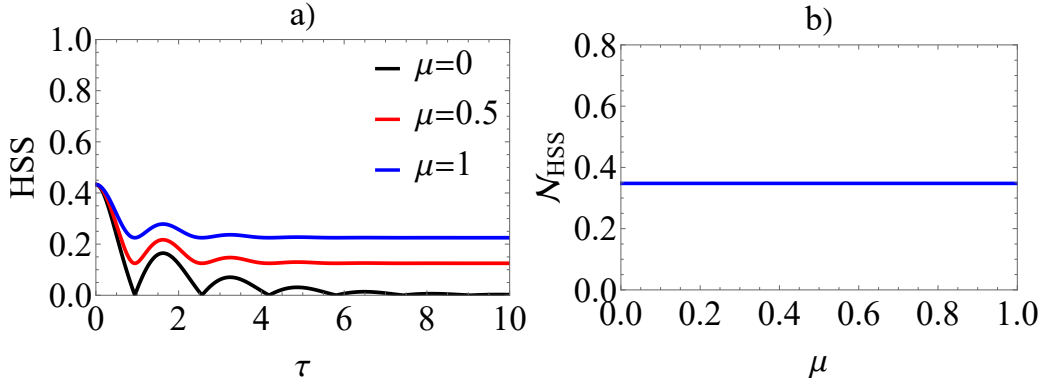


Figure 8.2: a) Dynamics of HSS as a function of dimensionless time  $\tau = \frac{t}{2\nu}$  for two-qubit Pauli channels with memory under the colored pure dephasing evolution in the standard basis with  $\nu = 1$ . Three different values of the correlation factor  $\mu$  are considered. b) HSS-based measure  $\mathcal{N}_{\text{HSS}}$  as a function of the correlation factor  $\mu$ .

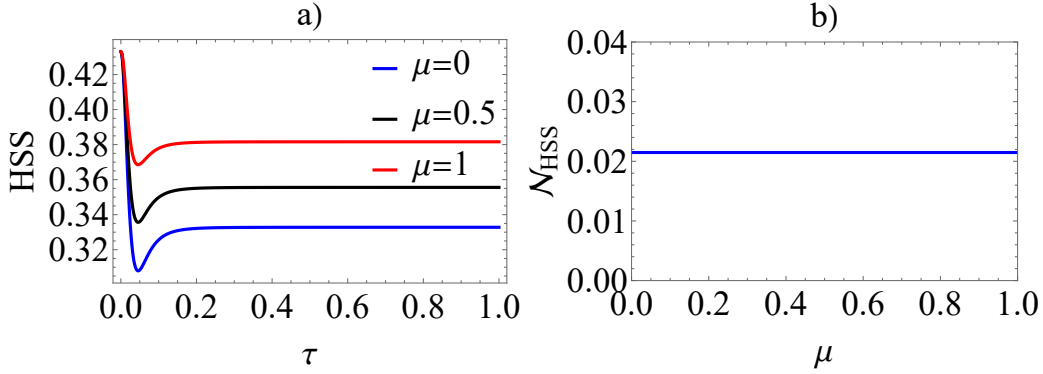


Figure 8.3: a) The dynamic of HSS as a function of dimensionless time  $\tau = \omega t$  for a two-qubit Pauli channel with memory under squeezed vacuum reservoir with  $\alpha = 0.5$ ,  $s = 4$ ,  $r = 0.5$  and  $\theta = \frac{3\pi}{2}$ . b) HSS-based measure  $\mathcal{N}_{\text{HSS}}$  as a function of the correlation factor  $\mu$ .

where  $\gamma(t)$  denotes the dephasing function. For a squeezed vacuum reservoir characterized by the squeezing parameter  $r$  and the squeezing angle  $\theta$ , the dephasing function is defined as follows [425]:

$$\gamma(t) = \int_0^\infty J(\omega) \frac{1 - \cos(\omega t)}{\omega^2} \times [\cosh(2r) - \sinh(2r) \cos(\omega t - \theta)] d\omega, \quad (8.17)$$

where  $J(\omega)$  is the spectral density of the reservoir. For Ohmic-like reservoirs, the spectral density is given by

$$J(\omega) = \alpha \frac{\omega^s}{\omega_c^{s-1}} \exp\left(-\frac{\omega}{\omega_c}\right), \quad (8.18)$$

in which  $\alpha$  represents a dimensionless coupling strength,  $\omega_c$  denotes the cutoff frequency of the bath, and the parameter  $s$  is positive and characterizes the environmental properties. By varying the Ohmic parameter  $s$ , one can distinguish between sub-Ohmic ( $0 < s < 1$ ), Ohmic ( $s = 1$ ), and super-Ohmic ( $s > 1$ ) reservoirs. Using algebraic manipulations and assuming  $\omega_c = 20\omega$ , we obtain the dephasing factor  $\gamma(\tau)$  in terms of the dimensionless time  $\tau = \omega t$  as follows: [483]

$$\begin{aligned} \gamma(\tau) = & \frac{1}{2} \alpha \left( \cosh(2r) [-(1 - 20i\tau)^{1-s} - (1 + 20i\tau)^{1-s} + 2] + \right. \\ & \left. \sinh(2r) \cos(\theta) [-2(1 - 20i\tau)^{1-s} + (1 - 40i\tau)^{1-s} + 1] \right) \\ & \times \Gamma(s - 1), \end{aligned} \quad (8.19)$$

where  $\Gamma(\cdot)$  is the Euler Gamma function.

As a consequence, for the two-qubit Pauli channel with classical correlation under a squeezed vacuum reservoir, the amount of Hilbert-Schmidt speed on the standard basis is calculated as

$$\text{HSS} = \frac{1}{4} \sqrt{(\mu - 1)^2 e^{-4\gamma(\tau)} + 2(1 - \mu^2 + \mu) e^{-2\gamma(\tau)} + \mu^2}, \quad (8.20)$$

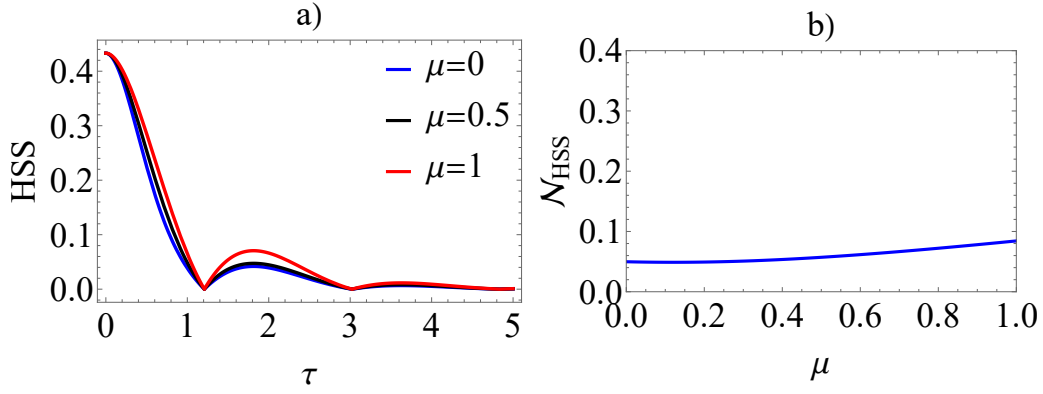


Figure 8.4: a) Dynamics of HSS as a function of dimensionless time  $\tau = \frac{t}{2\nu}$  for a two-qubit depolarizing channel with memory under the evolution of colored noise, for  $\theta = 0.5$  and  $\phi = \pi$ . b) HSS-based measure  $\mathcal{N}_{\text{HSS}}$  as a function of the correlation factor  $\mu$ .

We find that the HSS is sensitive to the presence of classical correlation, irrespective of the basis (see Appendix B for other bases), and hence it is capable of detecting global memory effects. The dynamics of the HSS are represented in Fig. 8.3a). Similar to the previous example, as the memory coefficient  $\mu$  increases, the HSS enhances. The presence of classical correlation does not alter the time intervals during which temporary backflow of information occurs.

Additionally, we display the degree of non-Markovianity,  $\mathcal{N}_{\text{HSS}}$ , in Fig. 8.3b). The degree of non-Markovianity remains constant and is not affected by the correlation factor of Pauli channels under a squeezed vacuum reservoir. This indicates that while classical correlation enhances the HSS, it does not impact the overall structure of non-Markovianity in the system.

#### 8.4.2 Model II: Two-Qubit Depolarizing Channels under Colored Noise Reservoir

Here, we study an example of a completely positive, trace-preserving map resulting from a system-environment coupling that is fundamentally non-Markovian and defines a depolarizing channel with colored noise [106]. The depolarizing channel, an important model for describing noise in quantum systems, can be represented by the Kraus operators given by  $K_i = \sqrt{p_i} \sigma_i$ , where  $p_i$  are the non-negative linear combinations defined as [482]

$$\begin{aligned} p_0 &= \frac{1}{4}(\Lambda_1 + \Lambda_2 + \Lambda_3 + 1), \\ p_1 &= \frac{1}{4}(\Lambda_1 - \Lambda_2 - \Lambda_3 + 1), \\ p_2 &= \frac{1}{4}(-\Lambda_1 + \Lambda_2 - \Lambda_3 + 1), \\ p_3 &= \frac{1}{4}(-\Lambda_1 - \Lambda_2 + \Lambda_3 + 1). \end{aligned}$$

In terms of the dimensionless time  $\tau = \frac{t}{2\nu}$ , the functions  $\Lambda_i = e^{-\tau} \left( \frac{\sin(\tau\Omega_i)}{\Omega_i} + \cos(\tau\Omega_i) \right)$  are damped harmonic oscillators with frequencies  $\Omega_i = \sqrt{(4\theta_i)^2 - 1}$ . We choose  $\theta_1 = \theta_2 = \theta_3 = \theta$ , under the condition that  $\theta$  is in the range  $[0, \frac{1}{4}\sqrt{1 + (\frac{\pi}{\log 3})^2}]$ , which results in  $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda$ .

The corresponding Kraus operators are given by  $K_i = \sqrt{\frac{1-\Lambda}{2}} \sigma_i$  for  $i = 1, 2, 3$ , and  $K_0 = \sqrt{\frac{1+3\Lambda}{4}} \sigma_0$ . Thus, the amount of HSS for a two-qubit depolarizing channel with classical correlation under the evolution of colored noise in the standard basis for  $\phi = \pi$  is obtained as

$$\text{HSS} = \frac{1}{4} \sqrt{\Lambda^2 (2(\Lambda(-\mu) + \Lambda + \mu)^2 + 1)}. \quad (8.21)$$

It is worth mentioning that, similar to previous examples, HSS serves as a faithful witness of the global memory effect in this context. We demonstrate the dynamics of the HSS in Fig. 8.4a), exhibiting behavior analogous to the previous example. The impact of classical correlation is evident in the amplification of HSS. Importantly, it does not affect the number of oscillations or the time intervals in which the non-Markovian signature appears.

Furthermore, we calculate the HSS-based measure of non-Markovianity through numerical optimization over various initial states and their parameterizations, as shown in Fig. 8.4b). The degree of non-Markovianity is observed to be partially enhanced by increasing the memory coefficient between successive uses of quantum channels. This enhancement

originates from the increased time fluctuations of the HSS with a rise in the classical correlation between multiple channel uses.

The findings underscore the robustness of HSS in detecting temporal backflow of information and global memory effects in quantum systems. By providing a reliable measure of non-Markovianity, HSS offers valuable insights into the dynamics of quantum information processes under the influence of colored noise and classical correlations.

## 8.5 Two-Qubit Correlated Non-Unitary Noisy Quantum Channel

In this section, we examine the amplitude damping channel as an example of a non-unital channel.

The amplitude damping channel describes the dynamics of a qubit interacting with a dissipative environment, associated with spontaneous emission. The time-dependent Kraus operators for the single-qubit amplitude damping channel under a Lorentzian reservoir are given by [469]

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & G(\tau) \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{1-G(\tau)^2} \\ 0 & 0 \end{pmatrix}, \quad (8.22)$$

where  $G(\tau)^2$  represents the decay of the excited population and is given by

$$G(\tau) = e^{-\tau/2} \left( \frac{\sinh\left(\sqrt{\frac{1}{4} - \frac{a}{2}}\tau\right)}{\sqrt{1-2a}} + \cosh\left(\sqrt{\frac{1}{4} - \frac{a}{2}}\tau\right) \right). \quad (8.23)$$

Here,  $a = \frac{\gamma_0}{\lambda}$  and  $\tau = \lambda t$ , with  $\gamma_0$  and  $\lambda$  describing the coupling strength and the spectral width, respectively. In the weak coupling regime ( $\gamma_0 < \frac{\lambda}{2}$ ),  $|G(\tau)|$  decreases monotonically. In the strong coupling regime ( $\gamma_0 > \frac{\lambda}{2}$ ,  $a > \frac{1}{2}$ ),  $|G(\tau)|$  oscillates, showing nonmonotonic behavior.

The evolution of two sequences of amplitude-damping channels with classical memory is expressed as [484]

$$\rho \rightarrow \Phi(\rho) = (1-\mu) \sum_{i,j=0}^1 K_{i,j} \rho K_{i,j}^\dagger + \mu \sum_{l=0}^1 F_l \rho F_l^\dagger, \quad (8.24)$$

in which  $K_{i,j} = K_i \otimes K_j$ , and the Kraus operators for the correlated part, obtained by solving the correlated Lindblad equation, are given by [485]

$$F_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & G(\tau) \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{1-G^2(\tau)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8.25)$$

For the two-qubit amplitude damping channel with classical correlation under a Lorentzian reservoir, the Hilbert-Schmidt speed (HSS) on the standard basis is obtained as

$$\text{HSS} = \frac{1}{4} \sqrt{(G(\tau)^2 + 2)(G(\tau) + \mu - \mu G(\tau))^2}. \quad (8.26)$$

Similar to the behavior observed in non-dissipative (unital) channels, the HSS is shown to be an effective tool for detecting global memory effects even in dissipative (non-unital) channels. As illustrated in Fig. 8.5a), the dynamics of HSS for a correlated amplitude damping channel, a typical non-unital channel, exhibit distinct characteristics when compared to unital channels, particularly as the memory parameter of the channel increases. Consequently, the time interval during which non-Markovian effects re-emerge shifts, highlighting a dependency on the memory parameter. In more detail, when starting from a special initial state, the HSS fluctuations over time are notably weakened as classical correlations increase. To further investigate the impact of these classical correlations on the strength of non-Markovianity, we computed the HSS-based measure of non-Markovianity through optimization over initial bases. The results, presented in Fig. 8.5b), demonstrate that as the correlation factor increases, the strength of non-Markovianity initially decreases, followed by an increase.

Accordingly, the HSS provides a nuanced quantification of quantum state dynamics, revealing intricate details about the temporal evolution of quantum systems. In non-unital channels, where dissipative effects play a crucial role, HSS acts as a precise indicator of memory retention and system coherence. This capability is essential in distinguishing between different types of quantum noise and their impacts on system dynamics.

The weakened HSS fluctuations with increasing classical correlations, starting from a specific initial state, imply that these correlations play a stabilizing role, effectively shielding the system from environmental disturbances. This stabilization can be attributed to the decoherence mitigation provided by stronger correlations, which enhance the resilience of the quantum state against external noise.



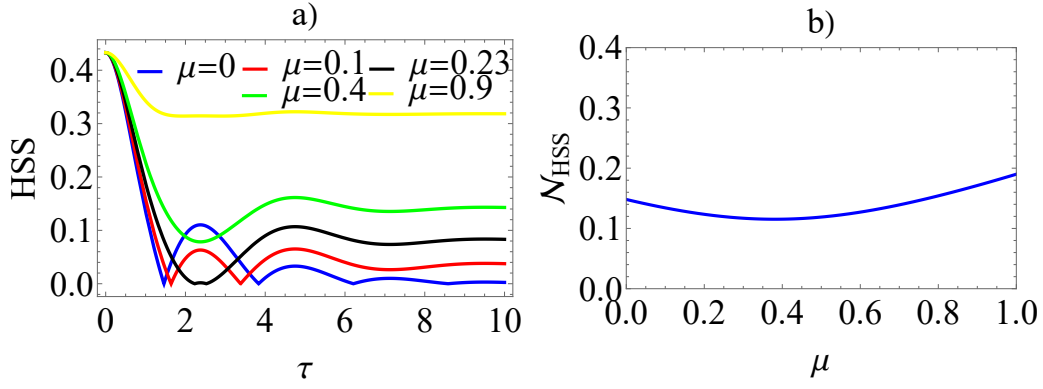


Figure 8.5: a) Dynamics of HSS as a function of dimensionless time  $\tau = \lambda t$  for two-qubit amplitude damping channels with memory in the standard basis under the evolution of a Lorentzian reservoir for  $a = 4$  and  $\phi = \pi$ . b) HSS-based measure  $N_{\text{HSS}}$  as a function of the correlation factor  $\mu$ .

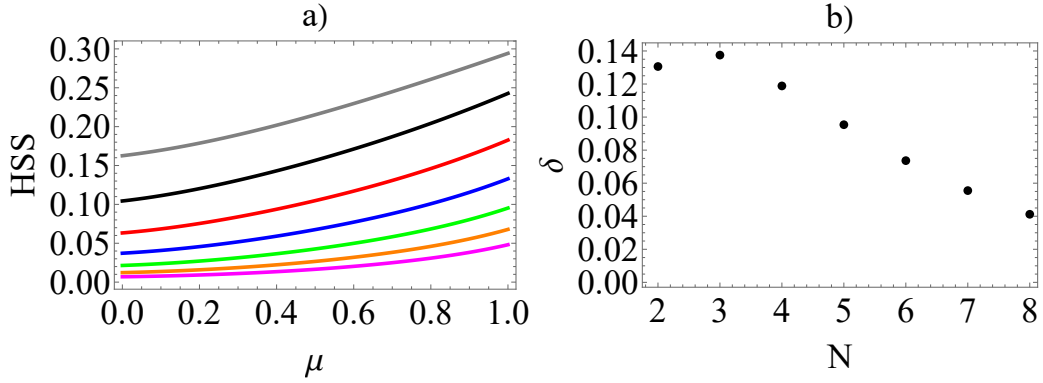


Figure 8.6: a) Dynamics of HSS for multiqubit correlated Pauli channel under colored pure dephasing reservoirs with constant reservoir parameters  $\nu = 1$  at dimensionless time  $\tau = 1.62$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and 8 qubits. b) Range of variation  $\delta$  versus the number of qubits.

Furthermore, the non-monotonic relationship between the correlation factor and the strength of non-Markovianity, as derived from the HSS-based measure, suggests a complex interplay between memory effects and system-environment interactions. Initially, the increasing classical correlations reduce the extent of non-Markovian behavior by enhancing the systems coherence. However, as the correlation factor continues to rise, the system starts exhibiting increased memory retention, leading to a resurgence of non-Markovian dynamics. This resurgence implies that beyond a certain threshold, classical correlations not only preserve the quantum state for a specific evolution but also enhance its ability to retain information about its past interactions.

## 8.6 Multiqubit correlated unital noisy quantum channel

In this study, we extend our analysis beyond the classical correlations observed in consecutive applications of noisy channels on two-qubit systems. We explore scenarios where noisy channels operate multiple times on sequences of multiqubit systems, with a particular focus on unital channels, such as multiqubit correlated Pauli and depolarizing noisy channels. To investigate the memory effects of noisy channels on high-qubit systems, we calculate the HSS for systems with up to eight qubits, utilizing the unital noisy channel on a standard basis. Given the complexity of the analytical expressions for the HSS in high-dimensional systems, our analysis prioritizes the interpretation of the figures rather than presenting the explicit formulas.

The HSS dynamics under constant reservoir parameters at given times are illustrated for the Pauli channel under a colored pure dephasing reservoir, the Pauli channel under a squeezed vacuum reservoir, and the depolarizing channel under a colored noise reservoir, as shown in Fig. 8.6a), Fig. 8.7a), and Fig. 8.8a), respectively.

Across all scenarios, regardless of the number of qubits, increasing the classical correlation typically amplifies the dynamics of the HSS. However, as demonstrated, this trend does not hold for a high number of qubits. In these instances, increasing the classical correlation does not significantly amplify or enhance the dynamics of the HSS. In other words, the effect of the presence of classical correlation  $\mu$  between multiple uses of quantum channels becomes almost negligible on the global memory effect. Detecting the presence of classical correlation  $\mu$  between multiple uses of quantum channels via the HSS tool becomes more challenging as the number of active qubits increases.

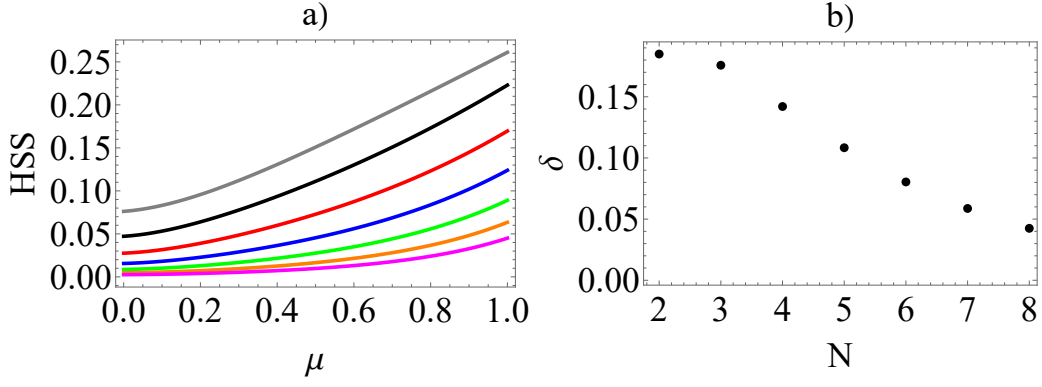


Figure 8.7: a) Dynamics of HSS for multiqubit correlated Pauli channel under squeezed vacuum reservoirs:  $\alpha = 0.5$ ,  $s = 4$ ,  $r = 0.5$ , and  $\theta = \frac{3\pi}{2}$  at dimensionless time  $\tau = 0.2$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and  $8$  qubits. b) Range of variation  $\delta$  versus the number of qubits.

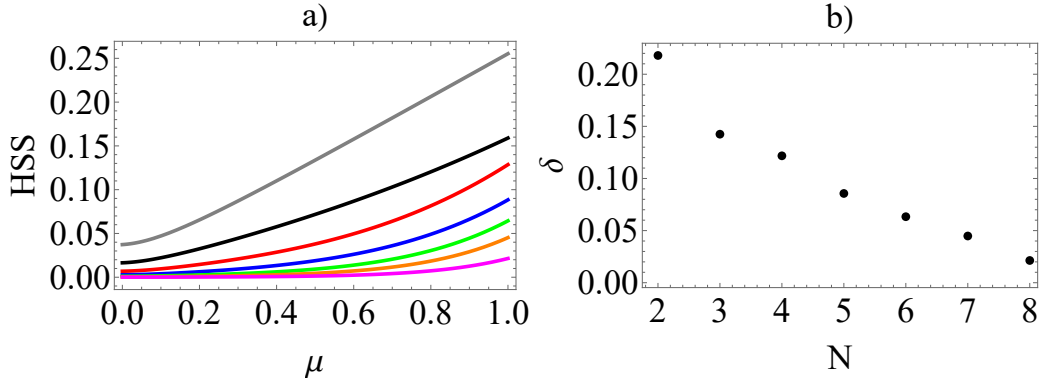


Figure 8.8: a) Dynamics of HSS for multiqubit correlated depolarizing channel under colored noise reservoirs:  $\eta = 0.5$ ,  $\Phi = \frac{\pi}{2}$  at dimensionless time  $\tau = 1.6$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and  $8$  qubits. b) Range of variation  $\delta$  versus the number of qubits.

To further characterize the system behavior, we study the difference between the maximum and minimum values of HSS at a given dimensionless time  $\tau$  and other fixed parameters. This difference  $\delta$ , characterizing the range of variation, is defined as

$$\delta = \max(\text{HSS}(\mu = 1, \tau, \dots)) - \min(\text{HSS}(\mu = 0, \tau, \dots)), \quad (8.27)$$

where " $\dots$ " denotes the other fixed parameters. To do so, we demonstrate  $\delta$  based on the number of qubits in Figs. 8.6b, 8.7b, and 8.8b, respectively, for Pauli channels under colored pure dephasing, squeezed vacuum reservoir, and depolarizing under colored noise reservoir. In which  $\delta$  decreases with an increasing number of qubits. It is anticipated that a large number of qubits, specifically beyond 10, will approach zero. This behavior indicates that the effects of the correlation strength between consecutive noise channels on the system are inhibited while increasing the number of qubits.

Further, for the multiqubit Pauli channel under a colored pure dephasing reservoir, we calculate the HSS for systems ranging from three to eight qubits. The dynamics of the HSS are depicted in Fig. 8.9. Increasing the number of active qubits, we find that the analysis given for the two-qubit model is also valid here. We also see that an increase in the number of qubits does not alter the time intervals in which non-Markovianity appears.

For the multi-qubit Pauli channel under a squeezed vacuum reservoir, the results align with those observed in the colored pure dephasing reservoir and the corresponding two-qubit model (see Fig. 8.10). Again, starting from a specific initial state, the time intervals at which non-Markovianity occurs remain unchanged despite an increase in the number of qubits. Similar observations can be made for multi-qubit depolarizing channels under a colored noise reservoir, as shown in Fig. 8.11.

*One of our most significant findings is that, assuming the qubits of a multipartite system are affected by independent or classically correlated local non-Markovian unital channels, the Hilbert-Schmidt speed (HSS) of the global system tends to zero as the number of qubits increases. This holds true regardless of the classical correlation strength between the noise channels. Our explicit results up to 8 qubits strongly suggest this general behavior for larger systems. Therefore, our findings indicate that the collective behavior of the multiqubit system inhibits the non-Markovian features of the overall system dynamics, as shown in Figs. 8.6-8.11.*

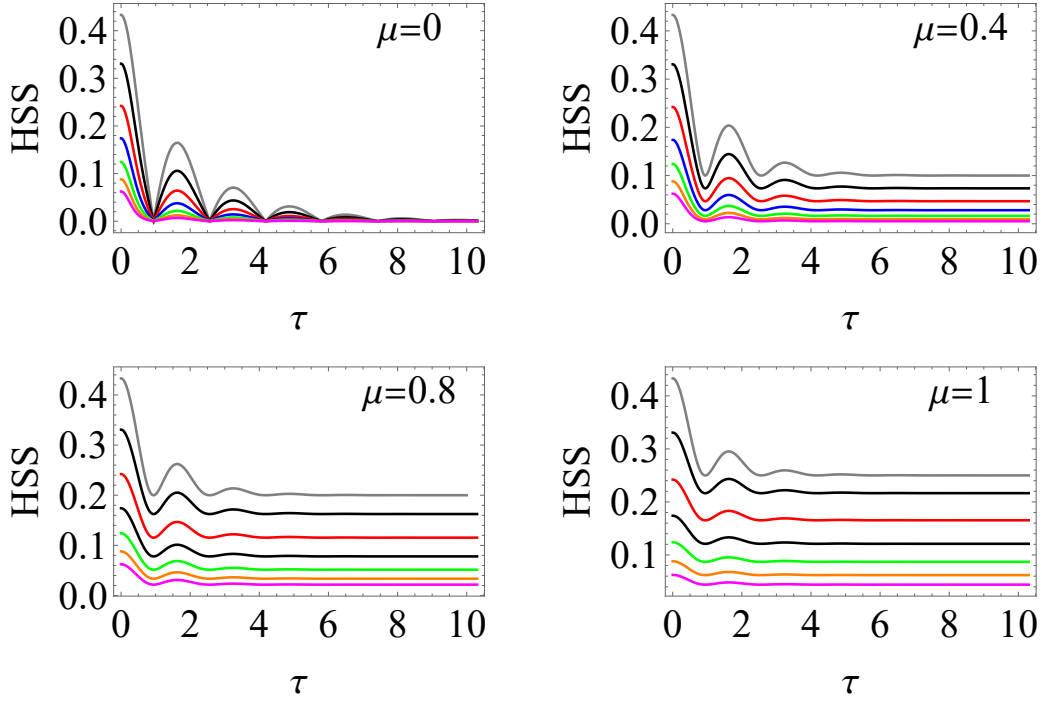


Figure 8.9: Dynamics of HSS as a function of dimensionless time  $\tau = \frac{t}{2\nu}$  for correlated multi-qubit Pauli channels under colored pure dephasing evolution in the standard basis with  $\nu = 1$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and 8 qubits.

## 8.7 Conclusion

In correlated noisy channels, the global memory effect arises from both the temporal correlation created during evolution and the classical correlation between multiple uses of the quantum channel. Our study has shown that Hilbert Schmidt speed (HSS), a genuine witness, effectively captures the global memory effect in both unital and non-unital noisy channels. HSS is consistently sensitive to classical correlations, irrespective of the basis employed. While fluctuations in HSS dynamics can be attributed to the quantum non-Markovian evolution of the system, the amplification of these dynamics signals the presence of classical correlation between multiple uses of the quantum unital channel.

Furthermore, we investigated the influence of classical correlation on the non-Markovian memory effect of correlated noisy channels. For unital channels, classical correlation does not alter the duration of temporary revivals. However, in non-unital channels, increasing classical correlation varies the time intervals at which the non-Markovian effects appear. *This makes Hilbert-Schmidt speed an effective tool to distinguish between unital and non-unital channels.*

Considering that the HSS is easily computable and does not require diagonalization, we extended our analysis to multiple qubits, extending up to eight qubits for unital noisy channels. The observed behavior in two-qubit noisy channels is replicated here: classical correlation amplifies the HSS dynamics, irrespective of the number of qubits. However, the impact of classical correlation between multiple applications of noisy channels on high-qubit systems is not dominant. As the number of qubits surpasses eight, HSS becomes less sensitive to classical correlations and primarily reflects the non-Markovian memory effect.

This study delves deep into the behavior of multiqubit systems under various noisy channels, particularly unital ones. *As the number of qubits increases, the influence of classical correlations diminishes, suggesting a saturation point in the system capacity to encode additional classical information.*

A key result of our study is that when qubits in a multipartite system are influenced by either independent or classically correlated local non-Markovian unital channels, the Hilbert-Schmidt speed of the global system tends to zero as the number of qubits increases. This trend holds true irrespective of the classical correlation strength between the noise channels. Our comprehensive results, extending to systems with up to 8 qubits, strongly suggest that this behavior persists in larger systems. *These findings underscore that the collective dynamics of multiqubit systems effectively suppress the non-Markovian features of the overall system evolution.* In our scenario where qubits are only entangled without any other direct interactions, the suppression of non-Markovianity can be understood through the following points:

a) The entanglement between qubits ensures that the system responds in a coordinated manner to local noises. Entanglement causes the qubits to respond collectively to local noises, resulting in a smoothing out of fluctuations through collective averaging and increased symmetry in the system dynamics. This coordinated response aligns more closely with Markovian behavior, and as the number of qubits increases, these uniform interactions become more pronounced, further diminishing the influence of memory effects.

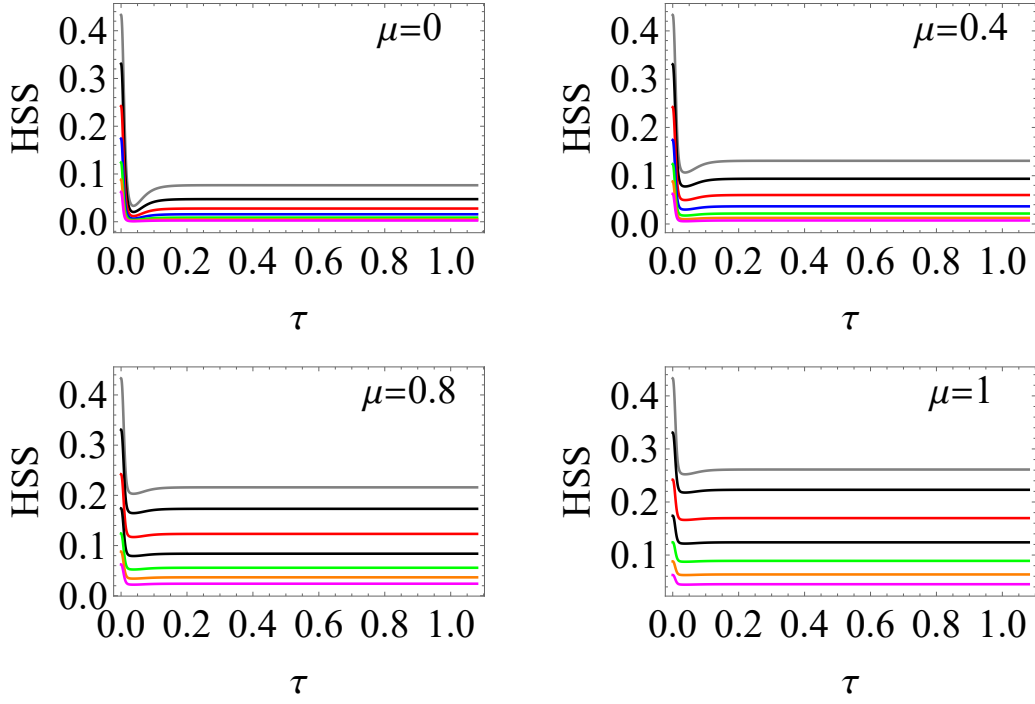


Figure 8.10: Dynamics of HSS as a function of dimensionless time  $\tau = \omega t$  for correlated multi-qubit Pauli channels under squeezed vacuum evolution in the standard basis with  $\alpha = 0.5$ ,  $s = 4$ ,  $r = 0.5$ , and  $\theta = \frac{3\pi}{2}$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and  $8$  qubits.

b) Larger systems are more susceptible to pronounced decoherence, which accelerates the loss of coherence and dominates over non-Markovian characteristics. This strong decoherence obscures the signatures of non-Markovianity, such as information backflow, making them harder to detect and effectively suppressing their impact. Moreover, the scaling effects of decoherence, amplified by the interconnectedness of entangled systems, further diminish the observable non-Markovian behavior, ultimately leading to its suppression as the system size increases.

Our results are crucial for advancing scalable quantum technologies, indicating that larger quantum systems may inherently resist classical noise correlations and mitigate collective non-Markovian effects. These findings pave the way for future research into the fundamental limits of quantum stability in complex systems. By leveraging the Hilbert-Schmidt speed as a diagnostic tool, researchers can further explore the resilience of quantum systems to environmental noise and develop strategies to enhance stability in practical quantum computing and communication applications. This study opens new avenues for understanding and optimizing the performance of large-scale quantum networks and devices.

## Appendix

### A All Bases

In this appendix, we provide an overview of various bases, including the standard, Bell, and Hadamard bases for two-qubit systems.

The standard computational basis is

$$\begin{aligned}
 |\psi_{00}\rangle &= |0\rangle \otimes |0\rangle, \\
 |\psi_{01}\rangle &= |0\rangle \otimes |1\rangle, \\
 |\psi_{10}\rangle &= |1\rangle \otimes |0\rangle, \\
 |\psi_{11}\rangle &= |1\rangle \otimes |1\rangle, \\
 |\psi_S\rangle &= \frac{1}{2} (e^{i\varphi} |\psi_{00}\rangle + |\psi_{01}\rangle + |\psi_{10}\rangle + |\psi_{11}\rangle).
 \end{aligned}$$

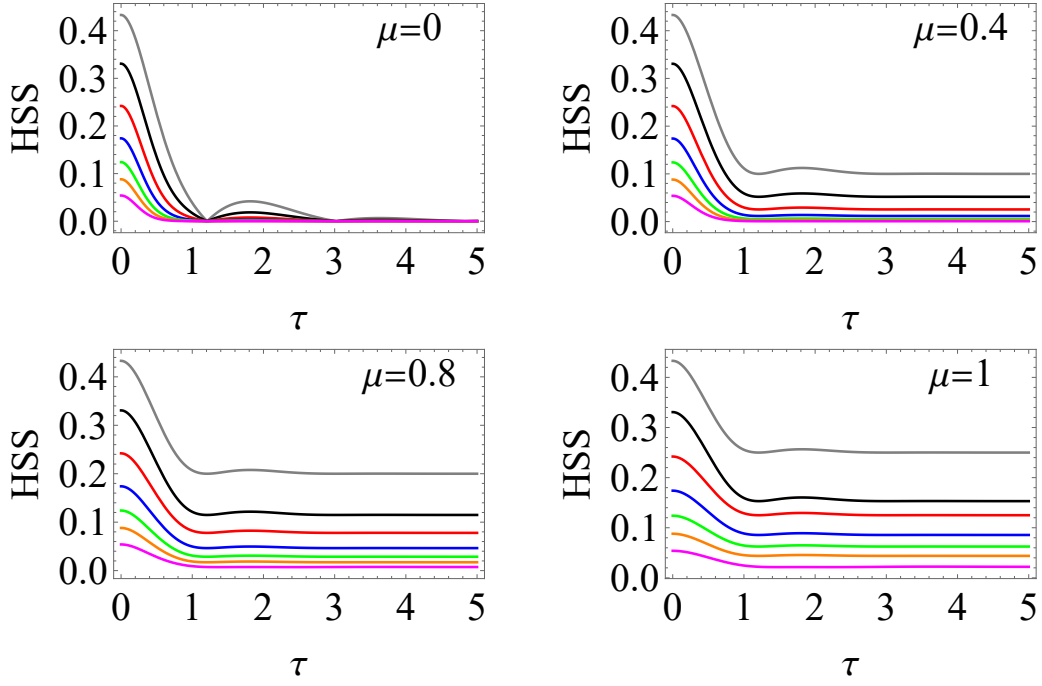


Figure 8.11: Dynamics of HSS as a function of dimensionless time  $\tau = \frac{t}{2\nu}$  for correlated multi-qubit depolarizing channels under colored noise evolution in the standard basis with  $\theta = 0.5$  and  $\phi = \frac{\pi}{2}$ . The solid gray, black, red, blue, green, orange, and magenta lines correspond to  $N = 2, 3, 4, 6, 7$ , and 8 qubits.

The Bell basis is

$$\begin{aligned}
 |\psi_+\rangle &= \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\
 |\psi_-\rangle &= \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\
 |\phi_+\rangle &= \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \\
 |\phi_-\rangle &= \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle), \\
 |\psi_B\rangle &= \frac{1}{2} (e^{i\varphi} |\psi_+\rangle + |\psi_-\rangle + |\phi_+\rangle + |\phi_-\rangle).
 \end{aligned}$$

The Hadamard basis is

$$\begin{aligned}
 |\psi_{+0}\rangle &= \frac{1}{\sqrt{2}} (|\psi_{00}\rangle + |\psi_{10}\rangle), \\
 |\psi_{+1}\rangle &= \frac{1}{\sqrt{2}} (|\psi_{01}\rangle + |\psi_{11}\rangle), \\
 |\psi_{-0}\rangle &= \frac{1}{\sqrt{2}} (|\psi_{00}\rangle - |\psi_{10}\rangle), \\
 |\psi_{-1}\rangle &= \frac{1}{\sqrt{2}} (|\psi_{01}\rangle - |\psi_{11}\rangle), \\
 |\psi_H\rangle &= \frac{1}{2} (e^{i\varphi} |\psi_{+0}\rangle + |\psi_{+1}\rangle + |\psi_{-0}\rangle + |\psi_{-1}\rangle).
 \end{aligned}$$

## B Two-qubit correlated unital noisy quantum channels

In this appendix, we provide the explicit expressions of the HSS, in the Bell and Hadamard bases, for three different unital noisy channels considered in the manuscript. We denote the HSS in the Bell and Hadamard bases as  $\text{HSS}_B$  and  $\text{HSS}_H$ , respectively.

### Two-qubit Pauli channel under a colored pure dephasing reservoir.

For a two-qubit Pauli channel under a colored pure dephasing reservoir, the two relevant HSS expressions in the Bell and Hadamard bases are given by, respectively,

$$\begin{aligned} \text{HSS}_B &= \frac{1}{4} \sqrt{((1-\mu)\eta(\tau)^2 + \mu)^2 \cos^2(\phi) + \sin^2(\phi) + 2\eta(\tau)^2}, \\ \text{HSS}_H &= \frac{1}{4} [((1-\mu)\eta(\tau)^2 + \mu)^2 + \eta(\tau)^2(1 + \cos^2(\Phi)) \\ &\quad + \sin^2(\phi)]^{1/2}. \end{aligned} \quad (8.28)$$

This indicates that the HSS is sensitive to  $\mu$  regardless of the basis used.

### Two-qubit Pauli channel under a squeezed vacuum reservoir

For a two-qubit Pauli channel under a squeezed vacuum reservoir, the HSS expressions in the Bell and Hadamard bases are obtained, respectively, as

$$\begin{aligned} \text{HSS}_B &= \frac{e^{-2\gamma(\tau)}}{4} \left[ \cos^2(\Phi) \left( \mu \left( e^{2\gamma(\tau)} - 1 \right) + 1 \right)^2 + e^{4\gamma(\tau)} \sin^2(\Phi) + 2e^{2\gamma(\tau)} \right]^{1/2}, \\ \text{HSS}_H &= \frac{1}{4\sqrt{2}(\cos(\Phi) + 2)^2} \left( 4(\mu - 1)^2 e^{-4\gamma(\tau)} (8 \cos(\Phi) + 3 \cos(2\Phi) + 7) \right. \\ &\quad + 4(8\mu^2 \cos(\Phi) + 3(\mu^2 - 1) \cos(2\Phi) + 7\mu^2 + 3) \\ &\quad + e^{-2\gamma(\tau)} ((444 - 64(\mu - 1)\mu) \cos(\Phi) - 56(\mu - 1)\mu + 327) \\ &\quad \left. + e^{-2\gamma(\tau)} (-24(\mu - 3)(\mu + 2) \cos(2\Phi) + 20 \cos(3\Phi) + \cos(4\Phi)) \right)^{1/2}. \end{aligned} \quad (8.29)$$

### Two-qubit depolarizing channel under a colored noise reservoir

The HSS functions for a two-qubit depolarizing channel under a colored noise reservoir in the Bell and Hadamard bases are given, respectively, by

$$\begin{aligned} \text{HSS}_B &= \frac{1}{4\sqrt{2}} \left( (\Lambda - 1)\Lambda^2(\mu - 1)(\Lambda(\mu - 1) - \mu - 1) \cos(2\Phi) + 3\Lambda^2 ((\Lambda(-\mu) + \Lambda + \mu)^2 + 1) \right)^{1/2}, \\ \text{HSS}_H &= \frac{1}{2\sqrt{2}(\cos(\Phi) + 2)^2} \left( 8\Lambda^2 (7(\Lambda(-\mu) + \Lambda + \mu)^2 + 6) \cos(\Phi) \right. \\ &\quad + 28\Lambda^4(\mu - 1)^2 - 50\Lambda^3(\mu - 1)\mu + \Lambda^2(\mu(19\mu + 6) + 28) + 3\mu^2 \\ &\quad + \Lambda^2(\mu(49\mu - 6) + 30) \cos^2(\Phi) \\ &\quad + \Lambda^2 ((\Lambda(-\mu) + \Lambda + \mu)^2 + 1) \cos^3(\Phi)(\cos(\Phi) + 10) \\ &\quad \left. + (40\Lambda^4(\mu - 1)^2 - 86\Lambda^3(\mu - 1)\mu - 3\mu^2) \cos^2(\Phi) \right)^{1/2}. \end{aligned} \quad (8.30)$$

## C Two-qubit correlated non-unital noisy quantum channels

For the amplitude damping channel, the HSS functions for the two Bell and Hadamard bases, respectively, are

$$\begin{aligned} \text{HSS}_B &= \frac{1}{4\sqrt{2}} \left( (-2G^3 - 4G)(\mu - 1)\mu + 2\mu^2 + 2G^8(\mu - 1)^2 + G^4(6 - 5\mu)\mu + G^2(\mu(7\mu - 8) + 4) \right. \\ &\quad \left. + G^2(4G^4(\mu - 1)^2 - 2G^6(\mu - 1)^2) \cos(2\Phi) + G^2(G^2(-((\mu - 2)\mu + 2)) - 2G(\mu - 1)\mu + \mu^2) \cos(2\Phi) \right)^{1/2}, \\ \text{HSS}_H &= \frac{1}{4\sqrt{2}(\cos(\Phi) + 2)^2} \left( (40G^4 - 6G^4)(\mu - 1)^2 + G^2(\mu - 1)(247\mu - 327) + 367\mu^2 + \right. \\ &\quad 4(8G^4(\mu - 1)^2 - 222G(\mu - 1)\mu) \cos(\Phi) + 4(G^2(\mu - 1)(95\mu - 111) + 8119\mu^2) \cos(\Phi) + \\ &\quad \left. (G(-\mu) + G + \mu)^2(144 \cos(2\Phi) + 20 \cos(3\Phi) + \cos(4\Phi)) \right)^{1/2}. \end{aligned} \quad (8.31)$$

# Conclusive remarks

This doctoral dissertation presents research centered on the theoretical and experimental investigation of quantum correlations and memory effects in composite quantum systems. Part I explores the use of the indistinguishability of identical quantum systems as a genuine resource for quantum information processing as well as the study of quantum features based on indistinguishability within a dynamic framework. Part II introduces a new witness of non-Markovianity and evaluates its validity and efficiency in various systems.

Part I comprises three chapters that investigate various aspects of quantum information processing leveraging the indistinguishability of identical particles and examine the behavior of quantum properties within a dynamic framework. **Chapter 2** presents a controllable and comprehensive scheme capable of generating, through three steps preparation, deformation, and post-selection a broad class of multipartite entangled states, including W, Dicke, GHZ, and cluster states, for both bosonic and fermionic statistics. In **Chapter 3**, I discuss quantum-enhanced phase estimation using time-bin entangled states and the use of discrete-time quantum walk as a generalized measurement device to realize optimal POVMs in coupled fiber-loop systems. **Chapter 4** proposes experimentally controllable inhomogeneous quantum walk dynamics as a platform for exploring the influence of coherent disorder on quantum correlations between indistinguishable photons, offering valuable insights into dynamic quantum systems.

The indistinguishability of identical particles serves as a valuable resource for generating both bipartite [12, 13, 14] and multipartite [148] entangled states. Through sLOCC measurements, this resource becomes applicable to various quantum information processing tasks [12, 13, 149, 150, 151, 152], as identical particles are rendered distinguishable and individually addressable via their spatial regions. Using spatial wave functions, experimentally feasible entanglement witness operators can be constructed to test the generated states [153]. Our algorithmic approach leverages the indistinguishability of identical particles to generate a broad class of multipartite entangled states, including W, Dicke, GHZ, and cluster states. By employing a graph-based representation, we translate and map the scheme into colored, complex, and weighted directed graphs (digraphs), each corresponding to an experimental setup. This approach facilitates the analysis of the output state, the assessment of its genuine entanglement, and the evaluation of resources in terms of postselection probability. Furthermore, it enables the optimization of resources, considering constraints and particle statistics, to explore arbitrary forms of multipartite state generation. Consequently, this framework serves as an effective interface for optimizing the generation of multipartite entangled states with specific success probabilities and maximum fidelity.

The flow of light in the synthetic photonic lattice (SPL) implemented via a coupled fiber-loop system enables the simulation of controlled discrete-time quantum walks (DTQWs). Recently, this platform has been employed for tasks such as bi-photon state preparation, manipulation, and the realization of time-bin two- and four-level interference between two photons [74]. Building on these studies, we developed both theoretical and experimental models for implementing quantum phase estimation using an SPL with time-bin entangled states as probe states, while performing optimal POVMs through DTQWs. Our results establish the SPL as a promising platform for various quantum information protocols, including multi-parameter phase estimation [77], quantum state tomography [78], and phenomena related to QWs in quasi-PT-symmetry, such as quantum interference [79, 80], quantum state discrimination [81], and ultrafast quantum transformations [82, 83].

Further controllable inhomogeneous quantum walk dynamics can be constructed to investigate the effect of coherent disorder on the quantum correlations between two indistinguishable photons. We demonstrated that using a suitable configuration of the disorder, quantum correlations of biphotons are enhanced between two modes of the network in comparison with an ordered quantum walk. As an outlook, it paves the way for numerous avenues of further exploration, including different configurations of disorder to different such enhancements, allowing spatial and temporal manipulation of quantum correlations. Additionally, whether disorder can enhance quantum correlations among more than two particles remains an open question, with the potential to provide a benchmarking framework within the context of quantum resource theories.

In Part II of my thesis, three research papers are presented. **Chapter 6** introduces a novel witness of non-Markovianity, inspired by the observation that non-Markovian effects can accelerate system dynamics and that quantum statistical speed quantifiers set limits on evolution time. This witness is based on the positive rate of change of the Hilbert-Schmidt speed

(HSS), a specific case of quantum statistical speed. A key advantage of this witness is that it eliminates the need to diagonalize the system's evolved density matrix. The efficiency of the witness is demonstrated through several examples of open quantum systems, including a qubit subjected to phase-covariant noise and Pauli channels, two independent qubits interacting locally with leaky cavities, and  $V$ -type and  $\Lambda$ -type three-level atoms (qutrits) in dissipative cavities. The HSS-based witness consistently identifies memory effects, fully aligning with the trace distance-based BLP witness [120] and effectively detecting system-environment information backflows [144]. In **Chapter 7**, we explore the efficacy of this witness in detecting non-Markovianity within various high-dimensional and multipartite open quantum systems. Our analysis demonstrates that the temporal oscillations of the HSS-based witness align remarkably well with those of quantum negativity and other quantum correlation measures. Consequently, the proposed witness serves as a reliable and accurate tool for characterizing memory effects in the quantum evolution of high-dimensional systems. Finally, given the significance of correlated non-Markovian channels in enhancing the quantum speed limit [146], increasing channel capacity, and mitigating noise in quantum error correction [147], we extend our analysis in **Chapter 8** to demonstrate that the Hilbert-Schmidt speed (HSS), as an effective measure of non-Markovianity, serves as a reliable tool for assessing correlations in both unital and non-unital channels across varying noisy spectral densities. This study delves deep into the behavior of multiqubit systems under various noisy channels, particularly unital ones. As the number of qubits increases, the influence of classical correlations diminishes. This phenomenon can be likened to noise-induced fluctuations among individual qubits averaging out, thereby reducing the overall impact on the HSS dynamics. In a physical sense, this behavior parallels that of large macroscopic systems where local disturbances are mitigated by the systems size. Furthermore, this interplay between quantum coherence and classical noise underscores the intricate dynamics within quantum systems. The inhibition of HSS sensitivity to classical correlations in high-qubit systems suggests the potential for more stable and robust quantum information processing. As qubits increase, the system exhibits a form of noise averaging out, leading to a more coherent quantum state overall. This complex interplay between quantum coherence and classical noise offers fresh perspectives for optimizing quantum information processing in large-scale quantum systems. It highlights the potential for noise resilience and stability in advanced quantum computing applications.

In conclusion, this thesis makes significant contributions to the field of quantum information processing by deepening our understanding of the role of indistinguishability in quantum information tasks. Additionally, it presents a novel, easily computable non-Markovianity witness that does not require diagonalization of the density matrix, offering practical tools for characterizing memory effects in various realistic systems, including quantum optical systems, quantum control, and quantum optics. Collectively, these findings provide fundamental insights into quantum dynamics and pave the way for advancing future quantum technologies.



# References

## Bibliography

- [1] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, "Quantum computers," *Nature*, vol. 464, no. 7285, pp. 45–53, 2010.
- [2] A. D. Cronin, J. Schmiedmayer, and D. E. Pritchard, "Optics and interferometry with atoms and molecules," *Reviews of Modern Physics*, vol. 81, no. 3, p. 1051, 2009.
- [3] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, "Quantum-enhanced measurements without entanglement," *Reviews of Modern Physics*, vol. 90, no. 3, p. 035006, 2018.
- [4] R. Lo Franco and G. Compagno, "Quantum entanglement of identical particles by standard information-theoretic notions," *Sci. Rep.*, vol. 6, p. 20603, 2016.
- [5] G. Compagno, A. Castellini, and R. Lo Franco, "Dealing with indistinguishable particles and their entanglement," *Phil. Trans. R. Soc. A*, vol. 376, no. 2123, p. 20170317, 2018.
- [6] F. Herbut and M. Vujicic, "Irrelevance of the pauli principle in distant correlations between identical fermions," *Journal of Physics A: Mathematical and General*, vol. 20, p. 5555, nov 1987.
- [7] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein, and D. Loss, "Quantum correlations in two-fermion systems," *Phys. Rev. A*, vol. 64, p. 022303, Jul 2001.
- [8] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum mechanics, volume 3: fermions, bosons, photons, correlations, and entanglement*. John Wiley & Sons, 2019.
- [9] K. Eckert, J. Schliemann, D. BruSS, and M. Lewenstein, "Quantum correlations in systems of indistinguishable particles," *Annals of Physics*, vol. 299, no. 1, pp. 88–127, 2002.
- [10] M. Tichy, F. de Melo, M. Ku, F. Mintert, and A. Buchleitner, "Entanglement of identical particles and the detection process," *Fortschritte der Physik*, vol. 61, no. 2-3, pp. 225–237, 2013.
- [11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Rev. Mod. Phys.*, vol. 81, no. 2, p. 865, 2009.
- [12] R. Lo Franco and G. Compagno, "Indistinguishability of elementary systems as a resource for quantum information processing," *Phys. Rev. Lett.*, vol. 120, no. 24, p. 240403, 2018.
- [13] K. Sun, Y. Wang, Z.-H. Liu, X.-Y. Xu, J.-S. Xu, C.-F. Li, G.-C. Guo, A. Castellini, F. Nosrati, G. Compagno, *et al.*, "Experimental quantum entanglement and teleportation by tuning remote spatial indistinguishability of independent photons," *Opt. Lett.*, vol. 45, no. 23, pp. 6410–6413, 2020.
- [14] M. R. Barros, S. Chin, T. Pramanik, H.-T. Lim, Y.-W. Cho, J. Huh, and Y.-S. Kim, "Entangling bosons through particle indistinguishability and spatial overlap," *Opt. Express*, vol. 28, no. 25, pp. 38083–38092, 2020.
- [15] Y. Zhou, Y. Wang, L. Jian, G. Chen, and S. Liu, "Double-server blind quantum computation based on the ghz state," *J. App. Phys.*, vol. 131, no. 21, p. 214401, 2022.
- [16] M. Gimeno-Segovia, P. Shadbolt, D. E. Browne, and T. Rudolph, "From three-photon greenberger-horne-zeilinger states to ballistic universal quantum computation," *Phys. Rev. Lett.*, vol. 115, p. 020502, Jul 2015.

- [17] N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, “Experimental observation of four-photon entangled dicke state with high fidelity,” *Phys. Rev. Lett.*, vol. 98, p. 063604, Feb 2007.
- [18] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, “Experimental realization of Dicke states of up to six qubits for multiparty quantum networking,” *Phys. Rev. Lett.*, vol. 103, p. 020503, Jul 2009.
- [19] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, “Experimental entanglement of a six-photon symmetric Dicke state,” *Phys. Rev. Lett.*, vol. 103, p. 020504, Jul 2009.
- [20] G. Tóth and I. Apellaniz, “Quantum metrology from a quantum information science perspective,” *Journal of Physics A: Mathematical and Theoretical*, vol. 47, p. 424006, oct 2014.
- [21] J. Joo, Y.-J. Park, S. Oh, and J. Kim, “Quantum teleportation via a W state,” *New J. Phys.*, vol. 5, pp. 136–136, oct 2003.
- [22] K. Mahdavi pour, F. Nosrati, S. Sciara, R. Morandotti, and R. L. Franco, “Generation of genuine multipartite entangled states via indistinguishability of identical particles,” *arXiv preprint arXiv:2403.17171*, 2024.
- [23] F. Bouchard, A. Sit, Y. Zhang, R. Fickler, F. M. Miatto, Y. Yao, F. Sciarrino, and E. Karimi, “Two-photon interference: the hongoumandel effect,” *Reports on Progress in Physics*, vol. 84, p. 012402, dec 2020.
- [24] E. Polino, M. Valeri, N. Spagnolo, and F. Sciarrino, “Photonic quantum metrology,” *AVS Quantum Science*, vol. 2, p. 024703, 06 2020.
- [25] J. P. Dowling, “Quantum optical metrology—the lowdown on high-n00n states,” *Contemporary physics*, vol. 49, no. 2, pp. 125–143, 2008.
- [26] J. A. Jones, S. D. Karlen, J. Fitzsimons, A. Ardavan, S. C. Benjamin, G. A. D. Briggs, and J. J. L. Morton, “Magnetic field sensing beyond the standard quantum limit using 10-spin noon states,” *Science*, vol. 324, no. 5931, pp. 1166–1168, 2009.
- [27] J. Joo, W. J. Munro, and T. P. Spiller, “Quantum metrology with entangled coherent states,” *Phys. Rev. Lett.*, vol. 107, p. 083601, Aug 2011.
- [28] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, “Quantum metrology with nonclassical states of atomic ensembles,” *Rev. Mod. Phys.*, vol. 90, p. 035005, Sep 2018.
- [29] V. Giovannetti, S. Lloyd, and L. Maccone, “Advances in quantum metrology,” *Nature Photonics*, vol. 5, pp. 222–229, Apr 2011.
- [30] M. J. Holland and K. Burnett, “Interferometric detection of optical phase shifts at the heisenberg limit,” *Phys. Rev. Lett.*, vol. 71, pp. 1355–1358, Aug 1993.
- [31] C. W. Helstrom, “Quantum detection and estimation theory,” *Journal of Statistical Physics*, vol. 1, pp. 231–252, Jun 1969.
- [32] A. M. Childs, “Universal computation by quantum walk,” *Physical Review Letters*, vol. 102, no. 18, p. 180501, 2009.
- [33] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, “Universal quantum computation using the discrete-time quantum walk,” *Physical Review A*, vol. 81, no. 4, p. 042330, 2010.
- [34] G. S. Engel, T. R. Calhoun, E. L. Read, T.-K. Ahn, T. Mančal, Y.-C. Cheng, R. E. Blankenship, and G. R. Fleming, “Evidence for wavelike energy transfer through quantum coherence in photosynthetic systems,” *Nature*, vol. 446, no. 7137, pp. 782–786, 2007.
- [35] M. B. Plenio and S. F. Huelga, “Dephasing-assisted transport: quantum networks and biomolecules,” *New J. Phys.*, vol. 10, no. 11, p. 113019, 2008.
- [36] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, “Environment-assisted quantum walks in photosynthetic energy transfer,” *The J CHEM PHYS*, vol. 129, no. 17, p. 11B603, 2008.
- [37] A. M. Childs, “Universal computation by quantum walk,” *Phys. Rev. Lett.*, vol. 102, p. 180501, May 2009.
- [38] P. Kurzyński and A. Wójcik, “Quantum walk as a generalized measuring device,” *Phys. Rev. Lett.*, vol. 110, p. 200404, May 2013.
- [39] A. Aspuru-Guzik and P. Walther, “Photonic quantum simulators,” *Nature Physics*, vol. 8, pp. 285–291, Apr 2012.

- [40] S. E. Venegas-Andraca, “Quantum walks: a comprehensive review,” *Quantum Information Processing*, vol. 11, no. 5, pp. 1015–1106, 2012.
- [41] D. S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, “Localization of light in a disordered medium,” *Nature*, vol. 390, pp. 671–673, Dec 1997.
- [42] T. Schwartz, G. Bartal, S. Fishman, and M. Segev, “Transport and anderson localization in disordered two-dimensional photonic lattices,” *Nature*, vol. 446, pp. 52–55, Mar 2007.
- [43] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, “Environment-assisted quantum walks in photosynthetic energy transfer,” *The Journal of Chemical Physics*, vol. 129, p. 174106, 11 2008.
- [44] A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, I. Jex, and C. Silberhorn, “Decoherence and disorder in quantum walks: From ballistic spread to localization,” *Phys. Rev. Lett.*, vol. 106, p. 180403, 2011.
- [45] M. A. Broome, A. Fedrizzi, B. P. Lanyon, I. Kassal, A. Aspuru-Guzik, and A. G. White, “Discrete single-photon quantum walks with tunable decoherence,” *Phys. Rev. Lett.*, vol. 104, p. 153602, Apr 2010.
- [46] R. Metzler and J. Klafter, “The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics,” *Journal of Physics A: Mathematical and General*, vol. 37, p. R161, jul 2004.
- [47] R. Klages, G. Radons, and I. M. Sokolov, *Anomalous transport*. Wiley Online Library, 2008.
- [48] Y. Wang, Y. Li, Z.-q. Yin, and B. Zeng, “16-qubit ibm universal quantum computer can be fully entangled,” *npj Quant. Inf.*, vol. 4, p. 46, Sep 2018.
- [49] I. R. Berchera and I. P. Degiovanni, “Quantum imaging with sub-poissonian light: challenges and perspectives in optical metrology,” *Metrologia*, vol. 56, no. 2, p. 024001, 2019.
- [50] M. Karski, L. Förster, J.-M. Choi, A. Steffen, W. Alt, D. Meschede, and A. Widera, “Quantum walk in position space with single optically trapped atoms,” *Science*, vol. 325, no. 5937, pp. 174–177, 2009.
- [51] H. Schmitz, R. Matjeschk, C. Schneider, J. Glueckert, M. Enderlein, T. Huber, and T. Schaetz, “Quantum walk of a trapped ion in phase space,” *Phys. Rev. Lett.*, vol. 103, p. 090504, Aug 2009.
- [52] F. Zähringer, G. Kirchmair, R. Gerritsma, E. Solano, R. Blatt, and C. F. Roos, “Realization of a quantum walk with one and two trapped ions,” *Phys. Rev. Lett.*, vol. 104, p. 100503, Mar 2010.
- [53] V. V. Ramasesh, E. Flurin, M. Rudner, I. Siddiqi, and N. Y. Yao, “Direct probe of topological invariants using bloch oscillating quantum walks,” *Phys. Rev. Lett.*, vol. 118, p. 130501, Mar 2017.
- [54] Z. Yan, Y.-R. Zhang, M. Gong, Y. Wu, Y. Zheng, S. Li, C. Wang, F. Liang, J. Lin, Y. Xu, C. Guo, L. Sun, C.-Z. Peng, K. Xia, H. Deng, H. Rong, J. Q. You, F. Nori, H. Fan, X. Zhu, and J.-W. Pan, “Strongly correlated quantum walks with a 12-qubit superconducting processor,” *Science*, vol. 364, no. 6442, pp. 753–756, 2019.
- [55] M. Gong, S. Wang, C. Zha, M.-C. Chen, H.-L. Huang, Y. Wu, Q. Zhu, Y. Zhao, S. Li, S. Guo, H. Qian, Y. Ye, F. Chen, C. Ying, J. Yu, D. Fan, D. Wu, H. Su, H. Deng, H. Rong, K. Zhang, S. Cao, J. Lin, Y. Xu, L. Sun, C. Guo, N. Li, F. Liang, V. M. Bastidas, K. Nemoto, W. J. Munro, Y.-H. Huo, C.-Y. Lu, C.-Z. Peng, X. Zhu, and J.-W. Pan, “Quantum walks on a programmable two-dimensional 62-qubit superconducting processor,” *Science*, vol. 372, no. 6545, pp. 948–952, 2021.
- [56] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou, and R. Han, “Experimental implementation of the quantum random-walk algorithm,” *Phys. Rev. A*, vol. 67, p. 042316, Apr 2003.
- [57] C. A. Ryan, M. Laforest, J. C. Boileau, and R. Laflamme, “Experimental implementation of a discrete-time quantum random walk on an nmr quantum-information processor,” *Phys. Rev. A*, vol. 72, p. 062317, Dec 2005.
- [58] S. Barkhofen, L. Lorz, T. Nitsche, C. Silberhorn, and H. Schomerus, “Supersymmetric polarization anomaly in photonic discrete-time quantum walks,” *Phys. Rev. Lett.*, vol. 121, p. 260501, Dec 2018.
- [59] L. Lorz, E. Meyer-Scott, T. Nitsche, V. Potoček, A. Gábris, S. Barkhofen, I. Jex, and C. Silberhorn, “Photonic quantum walks with four-dimensional coins,” *Phys. Rev. Res.*, vol. 1, p. 033036, Oct 2019.
- [60] X.-Y. Xu, Q.-Q. Wang, W.-W. Pan, K. Sun, J.-S. Xu, G. Chen, J.-S. Tang, M. Gong, Y.-J. Han, C.-F. Li, and G.-C. Guo, “Measuring the winding number in a large-scale chiral quantum walk,” *Phys. Rev. Lett.*, vol. 120, p. 260501, Jun 2018.

- [61] X. Zhan, L. Xiao, Z. Bian, K. Wang, X. Qiu, B. C. Sanders, W. Yi, and P. Xue, “Detecting topological invariants in nonunitary discrete-time quantum walks,” *Phys. Rev. Lett.*, vol. 119, p. 130501, Sep 2017.
- [62] P. Xue, R. Zhang, H. Qin, X. Zhan, Z. H. Bian, J. Li, and B. C. Sanders, “Experimental quantum-walk revival with a time-dependent coin,” *Phys. Rev. Lett.*, vol. 114, p. 140502, Apr 2015.
- [63] X. Qiang, Y. Wang, S. Xue, R. Ge, L. Chen, Y. Liu, A. Huang, X. Fu, P. Xu, T. Yi, F. Xu, M. Deng, J. B. Wang, J. D. A. Meinecke, J. C. F. Matthews, X. Cai, X. Yang, and J. Wu, “Implementing graph-theoretic quantum algorithms on a silicon photonic quantum walk processor,” *Science Advances*, vol. 7, no. 9, p. eabb8375, 2021.
- [64] H. Tang, C. Di Franco, Z.-Y. Shi, T.-S. He, Z. Feng, J. Gao, K. Sun, Z.-M. Li, Z.-Q. Jiao, T.-Y. Wang, M. S. Kim, and X.-M. Jin, “Experimental quantum fast hitting on hexagonal graphs,” *Nature Photonics*, vol. 12, pp. 754–758, Dec 2018.
- [65] A. Crespi, R. Osellame, R. Ramponi, V. Giovannetti, R. Fazio, L. Sansoni, F. De Nicola, F. Sciarrino, and P. Mataloni, “Anderson localization of entangled photons in an integrated quantum walk,” *Nature Photonics*, vol. 7, pp. 322–328, Apr 2013.
- [66] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, “Two-particle bosonic-fermionic quantum walk via integrated photonics,” *Phys. Rev. Lett.*, vol. 108, p. 010502, Jan 2012.
- [67] A. Peruzzo, M. Lobino, J. C. F. Matthews, N. Matsuda, A. Politi, K. Poulios, X.-Q. Zhou, Y. Lahini, N. Ismail, K. Wörhoff, Y. Bromberg, Y. Silberberg, M. G. Thompson, and J. L. O’Brien, “Quantum walks of correlated photons,” *Science*, vol. 329, no. 5998, pp. 1500–1503, 2010.
- [68] K. Poulios, R. Keil, D. Fry, J. D. A. Meinecke, J. C. F. Matthews, A. Politi, M. Lobino, M. Gräfe, M. Heinrich, S. Nolte, A. Szameit, and J. L. O’Brien, “Quantum walks of correlated photon pairs in two-dimensional waveguide arrays,” *Phys. Rev. Lett.*, vol. 112, p. 143604, Apr 2014.
- [69] K. Wang, X. Qiu, L. Xiao, X. Zhan, Z. Bian, W. Yi, and P. Xue, “Simulating dynamic quantum phase transitions in photonic quantum walks,” *Phys. Rev. Lett.*, vol. 122, p. 020501, Jan 2019.
- [70] P. Zhang, B.-H. Liu, R.-F. Liu, H.-R. Li, F.-L. Li, and G.-C. Guo, “Implementation of one-dimensional quantum walks on spin-orbital angular momentum space of photons,” *Phys. Rev. A*, vol. 81, p. 052322, May 2010.
- [71] S. K. Goyal, F. S. Roux, A. Forbes, and T. Konrad, “Implementing quantum walks using orbital angular momentum of classical light,” *Phys. Rev. Lett.*, vol. 110, p. 263602, Jun 2013.
- [72] A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, P. J. Mosley, E. Andersson, I. Jex, and C. Silberhorn, “Photons walking the line: A quantum walk with adjustable coin operations,” *Phys. Rev. Lett.*, vol. 104, p. 050502, Feb 2010.
- [73] L. Yuan, Q. Lin, M. Xiao, and S. Fan, “Synthetic dimension in photonics,” *Optica*, vol. 5, pp. 1396–1405, Nov 2018.
- [74] M. Monika, F. Nosrati, A. George, S. Sciara, R. Fazili, A. L. Marques Muniz, A. Bisianov, R. Lo Franco, W. J. Munro, M. Chemnitz, U. Peschel, and R. Morandotti, “Quantum state processing through controllable synthetic temporal photonic lattices,” *Nature Photonics*, Oct 2024.
- [75] A. Laneve, F. Nosrati, A. Gerdali, K. Mahdavi pour, F. Pegoraro, M. Khazaei Shadfar, R. Lo Franco, and P. Mataloni, “Enhancing nonclassical bosonic correlations in a quantum walk network through experimental control of disorder,” *Phys. Rev. Res.*, vol. 3, p. 033235, Sep 2021.
- [76] Y. Wang, M. Piccolini, Z.-Y. Hao, Z.-H. Liu, K. Sun, J.-S. Xu, C.-F. Li, G.-C. Guo, R. Morandotti, G. Compagno, and R. Lo Franco, “Proof-of-principle direct measurement of particle statistical phase,” *Phys. Rev. Appl.*, vol. 18, p. 064024, Dec 2022.
- [77] S. Hong, J. ur Rehman, Y.-S. Kim, Y.-W. Cho, S.-W. Lee, H. Jung, S. Moon, S.-W. Han, and H.-T. Lim, “Quantum enhanced multiple-phase estimation with multi-mode n00n states,” *Nature Communications*, vol. 12, no. 1, p. 5211, 2021.
- [78] Z. Hou, J.-F. Tang, J. Shang, H. Zhu, J. Li, Y. Yuan, K.-D. Wu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, “Deterministic realization of collective measurements via photonic quantum walks,” *Nature Communications*, vol. 9, p. 1414, Apr 2018.
- [79] F. Klauck, L. Teuber, M. Ornigotti, M. Heinrich, S. Scheel, and A. Szameit, “Observation of pt-symmetric quantum interference,” *Nature Photonics*, vol. 13, no. 12, pp. 883–887, 2019.

- [80] Y. Qin, H. Chen, D. Luo, C. Pan, H. Hu, Y. Zhang, and D. Wei, “Quantum interference in anti-parity-time symmetric coupled waveguide system,” *Opt. Express*, vol. 29, pp. 29175–29185, Aug 2021.
- [81] C. M. Bender, D. C. Brody, J. Caldeira, U. Günther, B. K. Meister, and B. F. Samsonov, “Pt-symmetric quantum state discrimination,” *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 371, no. 1989, p. 20120160, 2013.
- [82] S. Ibáñez, S. Martínez-Garaot, X. Chen, E. Torrontegui, and J. G. Muga, “Shortcuts to adiabaticity for non-hermitian systems,” *Phys. Rev. A*, vol. 84, p. 023415, Aug 2011.
- [83] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, “Faster than hermitian quantum mechanics,” *Phys. Rev. Lett.*, vol. 98, p. 040403, Jan 2007.
- [84] V. Kendon, “Decoherence in quantum walks—a review,” *Math. Struct. Comput. Sci.*, vol. 17, no. 6, pp. 1169–1220, 2007.
- [85] V. Kendon and B. Tregenna, “Decoherence can be useful in quantum walks,” *Phys. Rev. A*, vol. 67, no. 4, p. 042315, 2003.
- [86] R. Vieira, E. P. Amorim, and G. Rigolin, “Dynamically disordered quantum walk as a maximal entanglement generator,” *Phys. Rev. Lett.*, vol. 111, no. 18, p. 180503, 2013.
- [87] R. Vieira, E. P. Amorim, and G. Rigolin, “Entangling power of disordered quantum walks,” *Phys. Rev. A*, vol. 89, no. 4, p. 042307, 2014.
- [88] M. Zeng and E. H. Yong, “Discrete-time quantum walk with phase disorder: localization and entanglement entropy,” *Scientific reports*, vol. 7, no. 1, pp. 1–9, 2017.
- [89] Q.-Q. Wang, X.-Y. Xu, W.-W. Pan, K. Sun, J.-S. Xu, G. Chen, Y.-J. Han, C.-F. Li, and G.-C. Guo, “Dynamic-disorder-induced enhancement of entanglement in photonic quantum walks,” *Optica*, vol. 5, no. 9, pp. 1136–1140, 2018.
- [90] J.-S. Tang, C.-F. Li, Y.-L. Li, X.-B. Zou, G.-C. Guo, H.-P. Breuer, E.-M. Laine, and J. Piilo, “Measuring non-markovianity of processes with controllable system-environment interaction,” *EPL (Europhysics Letters)*, vol. 97, p. 10002, jan 2012.
- [91] H. M. Wiseman and J. M. Gambetta, “Pure-state quantum trajectories for general non-markovian systems do not exist,” *Phys. Rev. Lett.*, vol. 101, p. 140401, 2008.
- [92] J. Ying-Hua and H. Ju-Ju, “Entanglement and decoherence of coupled superconductor qubits in a non-markovian environment,” *Chinese Phys. B*, vol. 19, p. 060304, jun 2010.
- [93] U. Hoeppe, C. Wolff, J. Küchenmeister, J. Niegemann, M. Drescher, H. Benner, and K. Busch, “Direct observation of non-markovian radiation dynamics in 3d bulk photonic crystals,” *Phys. Rev. Lett.*, vol. 108, p. 043603, Jan 2012.
- [94] A. W. Chin, A. Datta, F. Caruso, S. F. Huelga, and M. B. Plenio, “Noise-assisted energy transfer in quantum networks and light-harvesting complexes,” *New J. Phys.*, vol. 12, p. 065002, jun 2010.
- [95] B. Bellomo, R. Lo Franco, S. Maniscalco, and G. Compagno, “Entanglement trapping in structured environments,” *Phys. Rev. A*, vol. 78, p. 060302, Dec 2008.
- [96] J. Shao, “Decoupling quantum dissipation interaction via stochastic fields,” *J. Chem. Phys.*, vol. 120, no. 11, pp. 5053–5056, 2004.
- [97] A. Pomyalov and D. J. Tannor, “The non-markovian quantum master equation in the collective-mode representation: Application to barrier crossing in the intermediate friction regime,” *J. Chem. Phys.*, vol. 123, no. 20, p. 204111, 2005.
- [98] S. F. Huelga, A. Rivas, and M. B. Plenio, “Non-markovianity-assisted steady state entanglement,” *Phys. Rev. Lett.*, vol. 108, p. 160402, Apr 2012.
- [99] R. Vasile, S. Olivares, M. A. Paris, and S. Maniscalco, “Continuous-variable quantum key distribution in non-markovian channels,” *Phys. Rev. A*, vol. 83, p. 042321, Apr 2011.
- [100] A. W. Chin, S. F. Huelga, and M. B. Plenio, “Quantum metrology in non-markovian environments,” *Phys. Rev. Lett.*, vol. 109, p. 233601, Dec 2012.
- [101] E.-M. Laine, H.-P. Breuer, and J. Piilo, “Nonlocal memory effects allow perfect teleportation with mixed states,” *Sci. Rep.*, vol. 4, p. 4620, 2014.

- [102] Y. Dong, Y. Zheng, S. Li, C.-C. Li, X.-D. Chen, G.-C. Guo, and F.-W. Sun, “Non-markovianity-assisted high-fidelity deutsch–jozsa algorithm in diamond,” *npj Quantum Information*, vol. 4, no. 1, pp. 1–6, 2018.
- [103] E. B. Davies, “Quantum theory of open systems,” 1976.
- [104] R. Alicki and K. Lendi, *Quantum dynamical semigroups and applications*, vol. 717. Springer Science & Business Media, 2007.
- [105] H. Breuer, F. Petruccione, and S. Petruccione, *The Theory of Open Quantum Systems*. Oxford University Press, 2002.
- [106] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*. Cambridge university press, 2010.
- [107] K. Kraus, “Lecture notes in physics,” *States, Effects and Operations. Fundamental Notions of Quantum Theory*, vol. 190, 1983.
- [108] D. Chruściński, A. Rivas, and E. Størmer, “Divisibility and information flow notions of quantum markovianity for noninvertible dynamical maps,” *Phys. Rev. Lett.*, vol. 121, p. 080407, Aug 2018.
- [109] M. M. Wolf and J. I. Cirac, “Dividing quantum channels,” *Communications in Mathematical Physics*, vol. 279, pp. 147–168, Apr 2008.
- [110] K. Vahala, “Handbook of stochastic methods for physics, chemistry and the natural sciences,” *IEEE Journal of Quantum Electronics*, vol. 22, no. 9, pp. 1922–1922, 1986.
- [111] N. Van Kampen, *Stochastic Processes in Physics and Chemistry*. North-Holland Publishing Co, 1992.
- [112] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, “Markovianity and non-markovianity in quantum and classical systems,” *New Journal of Physics*, vol. 13, p. 093004, sep 2011.
- [113] G. Lindblad, “On the generators of quantum dynamical semigroups,” *Communications in Mathematical Physics*, vol. 48, pp. 119–130, Jun 1976.
- [114] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, “Completely positive dynamical semigroups of n-level systems,” *Journal of Mathematical Physics*, vol. 17, no. 5, pp. 821–825, 1976.
- [115] Á. Rivas, S. F. Huelga, and M. B. Plenio, “Quantum non-markovianity: characterization, quantification and detection,” *Reports on Progress in Physics*, vol. 77, p. 094001, aug 2014.
- [116] M. M. Wolf and D. Perez-Garcia, “Assessing quantum dimensionality from observable dynamics,” *Phys. Rev. Lett.*, vol. 102, p. 190504, May 2009.
- [117] A. Rivas, S. F. Huelga, and M. B. Plenio, “Entanglement and non-markovianity of quantum evolutions,” *Phys. Rev. Lett.*, vol. 105, p. 050403, Jul 2010.
- [118] M.-D. Choi, “Completely positive linear maps on complex matrices,” *Linear Algebra and its Applications*, vol. 10, no. 3, pp. 285–290, 1975.
- [119] A. Jamiokowski, “Linear transformations which preserve trace and positive semidefiniteness of operators,” *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 275–278, 1972.
- [120] H.-P. Breuer, E.-M. Laine, and J. Piilo, “Measure for the degree of non-markovian behavior of quantum processes in open systems,” *Phys. Rev. Lett.*, vol. 103, p. 210401, Nov 2009.
- [121] E.-M. Laine, J. Piilo, and H.-P. Breuer, “Measure for the non-markovianity of quantum processes,” *Phys. Rev. A*, vol. 81, p. 062115, Jun 2010.
- [122] P. Haikka, J. D. Cresser, and S. Maniscalco, “Comparing different non-markovianity measures in a driven qubit system,” *Phys. Rev. A*, vol. 83, p. 012112, Jan 2011.
- [123] H.-S. Zeng, N. Tang, Y.-P. Zheng, and G.-Y. Wang, “Equivalence of the measures of non-markovianity for open two-level systems,” *Phys. Rev. A*, vol. 84, p. 032118, 2011.
- [124] Z. Y. Xu, W. L. Yang, and M. Feng, “Proposed method for direct measurement of the non-markovian character of the qubits coupled to bosonic reservoirs,” *Phys. Rev. A*, vol. 81, p. 044105, Apr 2010.
- [125] E.-M. Laine, H.-P. Breuer, J. Piilo, C.-F. Li, and G.-C. Guo, “Nonlocal memory effects in the dynamics of open quantum systems,” *Phys. Rev. Lett.*, vol. 108, p. 210402, May 2012.

- [126] W.-j. Gu and G.-x. Li, “Non-markovian behavior for spontaneous decay of a v-type three-level atom with quantum interference,” *Phys. Rev. A*, vol. 85, p. 014101, Jan 2012.
- [127] B.-H. Liu, S. Wißmann, X.-M. Hu, C. Zhang, Y.-F. Huang, C.-F. Li, G.-C. Guo, A. Karlsson, J. Piilo, and H.-P. Breuer, “Locality and universality of quantum memory effects,” *Scientific Reports*, vol. 4, p. 6327, Sep 2014.
- [128] B.-H. Liu, L. Li, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, “Experimental control of the transition from Markovian to non-Markovian dynamics of open quantum systems,” *Nat. Phys.*, vol. 7, pp. 931–934, 2011.
- [129] B.-H. Liu, D.-Y. Cao, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, “Photonic realization of nonlocal memory effects and non-markovian quantum probes,” *Scientific Reports*, vol. 3, no. 1, p. 1781, 2013.
- [130] M. Hayashi, *Quantum information*. Springer, 2006.
- [131] D. Spehner, “Quantum correlations and distinguishability of quantum states,” *Journal of Mathematical Physics*, vol. 55, p. 075211, 07 2014.
- [132] E.-M. Laine, J. Piilo, and H.-P. Breuer, “Witness for initial system-environment correlations in open-system dynamics,” *Europhysics Letters*, vol. 92, p. 60010, jan 2011.
- [133] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, “Colloquium: Non-markovian dynamics in open quantum systems,” *Rev. Mod. Phys.*, vol. 88, p. 021002, Apr 2016.
- [134] M. Cianciaruso, S. Maniscalco, and G. Adesso, “Role of non-markovianity and backflow of information in the speed of quantum evolution,” *Phys. Rev. A*, vol. 96, p. 012105, Jul 2017.
- [135] V. Giovannetti, S. Lloyd, and L. Maccone, “Quantum limits to dynamical evolution,” *Phys. Rev. A*, vol. 67, p. 052109, May 2003.
- [136] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, “Quantum speed limit for physical processes,” *Phys. Rev. Lett.*, vol. 110, p. 050402, Jan 2013.
- [137] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, “Quantum speed limits in open system dynamics,” *Phys. Rev. Lett.*, vol. 110, p. 050403, Jan 2013.
- [138] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, “Generalized geometric quantum speed limits,” *Phys. Rev. X*, vol. 6, p. 021031, Jun 2016.
- [139] I. de Vega and D. Alonso, “Dynamics of non-markovian open quantum systems,” *Rev. Mod. Phys.*, vol. 89, p. 015001, Jan 2017.
- [140] and, “Quantum speed limit of a photon under non-markovian dynamics,” *Chinese Physics Letters*, vol. 31, p. 020301, feb 2014.
- [141] Z. Sun, J. Liu, J. Ma, and X. Wang, “Quantum speed limits in open systems: Non-markovian dynamics without rotating-wave approximation,” Feb 2015.
- [142] N. Mirkin, F. Toscano, and D. A. Wisniacki, “Quantum-speed-limit bounds in an open quantum evolution,” *Phys. Rev. A*, vol. 94, p. 052125, Nov 2016.
- [143] Y.-J. Zhang, Y.-J. Xia, and H. Fan, “Control of quantum dynamics: Non-markovianity and the speedup of the open system evolution,” *Europhysics Letters*, vol. 116, p. 30001, dec 2016.
- [144] H. Rangani Jahromi, K. Mahdavi-pour, M. Khazaei Shadfar, and R. Lo Franco, “Witnessing non-markovian effects of quantum processes through hilbert-schmidt speed,” *Phys. Rev. A*, vol. 102, p. 022221, Aug 2020.
- [145] C. Addis, G. Karpat, C. Macchiavello, and S. Maniscalco, “Dynamical memory effects in correlated quantum channels,” *Phys. Rev. A*, vol. 94, p. 032121, Sep 2016.
- [146] K. Xu, G.-F. Zhang, and W.-M. Liu, “Quantum dynamical speedup in correlated noisy channels,” *Phys. Rev. A*, vol. 100, p. 052305, Nov 2019.
- [147] V. B. Sabale, N. R. Dash, A. Kumar, and S. Banerjee, “Facets of Correlated Non-Markovian Channels,” *Annalen der Physik*, p. 2400151, 2024.
- [148] D. Lee, T. Pramanik, S. Hong, Y.-W. Cho, H.-T. Lim, S. Chin, and Y.-S. Kim, “Entangling three identical particles via spatial overlap,” *Opt. Express*, vol. 30, pp. 30525–30535, Aug 2022.

- [149] A. Castellini, R. Lo Franco, L. Lami, A. Winter, G. Adesso, and G. Compagno, “Indistinguishability-enabled coherence for quantum metrology,” *Phys. Rev. A*, vol. 100, no. 1, p. 012308, 2019.
- [150] K. Sun, Z.-H. Liu, Y. Wang, Z.-Y. Hao, X.-Y. Xu, J.-S. Xu, C.-F. Li, G.-C. Guo, A. Castellini, L. Lami, A. Winter, G. Adesso, G. Compagno, and R. L. Franco, “Activation of indistinguishability-based quantum coherence for enhanced metrological applications with particle statistics imprint,” *Proceedings of the National Academy of Sciences*, vol. 119, no. 21, p. e2119765119, 2022.
- [151] B. Bellomo, R. Lo Franco, and G. Compagno, “N identical particles and one particle to entangle them all,” *Phys. Rev. A*, vol. 96, no. 2, p. 022319, 2017.
- [152] A. Castellini, B. Bellomo, G. Compagno, and R. Lo Franco, “Activating remote entanglement in a quantum network by local counting of identical particles,” *Phys. Rev. A*, vol. 99, p. 062322, Jun 2019.
- [153] S. Sciara, C. Reimer, M. Kues, P. Roztock, A. Cino, D. J. Moss, L. Caspani, W. J. Munro, and R. Morandotti, “Universal  $n$ -partite  $d$ -level pure-state entanglement witness based on realistic measurement settings,” *Phys. Rev. Lett.*, vol. 122, p. 120501, Mar 2019.
- [154] X. Gu, L. Chen, A. Zeilinger, and M. Krenn, “Quantum experiments and graphs. iii. high-dimensional and multi-particle entanglement,” *Phys. Rev. A*, vol. 99, p. 032338, Mar 2019.
- [155] S. Chin, Y.-S. Kim, and S. Lee, “Graph picture of linear quantum networks and entanglement,” *Quantum*, vol. 5, p. 611, 2021.
- [156] K. Mahdavi-pour, M. Khazaei Shadfar, H. Rangani Jahromi, R. Morandotti, and R. Lo Franco, “Memory Effects in High-Dimensional Systems Faithfully Identified by Hilbert-Schmidt Speed-Based Witness,” *Entropy*, vol. 24, no. 3, 2022.
- [157] P. Shor, “Algorithms for quantum computation: discrete logarithms and factoring,” in *Proceedings 35th Annual Symposium on Foundations of Computer Science*, pp. 124–134, 1994.
- [158] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. S. L. Brandao, D. A. Buell, B. Burkett, Y. Chen, Z. Chen, B. Chiaro, R. Collins, W. Courtney, A. Dunsworth, E. Farhi, B. Foxen, A. Fowler, C. Gidney, M. Giustina, R. Graff, K. Guerin, S. Habegger, M. P. Harrigan, M. J. Hartmann, A. Ho, M. Hoffmann, T. Huang, T. S. Humble, S. V. Isakov, E. Jeffrey, Z. Jiang, D. Kafri, K. Kechedzhi, J. Kelly, P. V. Klimov, S. Knysh, A. Korotkov, F. Kostritsa, D. Landhuis, M. Lindmark, E. Lucero, D. Lyakh, S. Mandrà, J. R. McClean, M. McEwen, A. Megrant, X. Mi, K. Michielsen, M. Mohseni, J. Mutus, O. Naaman, M. Neeley, C. Neill, M. Y. Niu, E. Ostby, A. Petukhov, J. C. Platt, C. Quintana, E. G. Rieffel, P. Roushan, N. C. Rubin, D. Sank, K. J. Satzinger, V. Smelyanskiy, K. J. Sung, M. D. Trevithick, A. Vainsencher, B. Villalonga, T. White, Z. J. Yao, P. Yeh, A. Zalcman, H. Neven, and J. M. Martinis, “Quantum supremacy using a programmable superconducting processor,” *Nature*, vol. 574, no. 7779, pp. 505–510, 2019.
- [159] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, *et al.*, “Quantum computational advantage using photons,” *Science*, vol. 370, no. 6523, pp. 1460–1463, 2020.
- [160] A. K. Ekert, “Quantum cryptography and bells theorem,” in *Quantum Measurements in Optics*, pp. 413–418, Springer, 1992.
- [161] D. P. Nadlinger, P. Drmota, B. C. Nichol, G. Araneda, D. Main, R. Srinivas, D. M. Lucas, C. J. Ballance, K. Ivanov, E. Y.-Z. Tan, P. Sekatski, R. L. Urbanke, R. Renner, N. Sangouard, and J.-D. Bancal, “Experimental quantum key distribution certified by Bell’s theorem,” *Nature*, vol. 607, pp. 682–686, Jul 2022.
- [162] A. Peres, *Quantum theory: concepts and methods*. Springer, 1997.
- [163] W. Pauli, *General principles of quantum mechanics*. Springer Science & Business Media, 2012.
- [164] C.-K. Hong, Z.-Y. Ou, and L. Mandel, “Measurement of subpicosecond time intervals between two photons by interference,” *Physical review letters*, vol. 59, no. 18, p. 2044, 1987.
- [165] Y. Wang, Z.-Y. Hao, Z.-H. Liu, K. Sun, J.-S. Xu, C.-F. Li, G.-C. Guo, A. Castellini, B. Bellomo, G. Compagno, and R. Lo Franco, “Remote entanglement distribution in a quantum network via multinode indistinguishability of photons,” *Phys. Rev. A*, vol. 106, p. 032609, Sep 2022.
- [166] A. Perez-Leija, D. Guzmán-Silva, R. d. J. León-Montiel, M. Gräfe, M. Heinrich, H. Moya-Cessa, K. Busch, and A. Szameit, “Endurance of quantum coherence due to particle indistinguishability in noisy quantum networks,” *npj Quantum Inf.*, vol. 4, no. 1, pp. 1–12, 2018.



- [167] F. Nosrati, A. Castellini, G. Compagno, and R. Lo Franco, “Robust entanglement preparation against noise by controlling spatial indistinguishability,” *npj Quantum Inf.*, vol. 6, no. 1, pp. 1–7, 2020.
- [168] F. Nosrati, A. Castellini, G. Compagno, and R. Lo Franco, “Dynamics of spatially indistinguishable particles and quantum entanglement protection,” *Phys. Rev. A*, vol. 102, no. 6, p. 062429, 2020.
- [169] M. Piccolini, F. Nosrati, G. Compagno, P. Livreri, R. Morandotti, and R. Lo Franco, “Entanglement robustness via spatial deformation of identical particle wave functions,” *Entropy*, vol. 23, no. 6, p. 708, 2021.
- [170] M. Piccolini, F. Nosrati, R. Morandotti, and R. Lo Franco, “Indistinguishability-enhanced entanglement recovery by spatially localized operations and classical communication,” *Open Sys. Inf. Dyn.*, vol. 28, no. 04, p. 2150020, 2021.
- [171] M. Piccolini, V. Giovannetti, and R. Lo Franco, “Robust engineering of maximally entangled states by identical particle interferometry,” *Adv. Quant. Tech.*, vol. 6, no. 9, p. 2300146, 2023.
- [172] M. Piccolini, V. Giovannetti, and R. Lo Franco, “Asymptotically deterministic robust preparation of maximally entangled bosonic states,” *Phys. Rev. Res.*, vol. 6, p. 013073, Jan 2024.
- [173] M. Piccolini, M. Karczewski, A. Winter, and R. L. Franco, “Robust generation of  $n$ -partite  $n$ -level singlet states by identical particle interferometry,” *arXiv preprint arXiv:2312.17184*, 2023.
- [174] F. Nosrati, B. Bellomo, G. D. Chiara, G. Compagno, R. Morandotti, and R. L. Franco, “Indistinguishability-assisted two-qubit entanglement distillation,” *Quantum Science and Technology*, vol. 9, p. 015027, jan 2024.
- [175] O. Gühne and G. Tóth, “Entanglement detection,” *Physics Reports*, vol. 474, no. 1-6, pp. 1–75, 2009.
- [176] W. Dür, G. Vidal, and J. I. Cirac, “Three qubits can be entangled in two inequivalent ways,” *Phys. Rev. A*, vol. 62, p. 062314, Nov 2000.
- [177] N. Kiesel, C. Schmid, U. Weber, G. Tóth, O. Gühne, R. Ursin, and H. Weinfurter, “Experimental analysis of a four-qubit photon cluster state,” *Phys. Rev. Lett.*, vol. 95, p. 210502, Nov 2005.
- [178] R. Raussendorf and H. J. Briegel, “A one-way quantum computer,” *Phys. Rev. Lett.*, vol. 86, pp. 5188–5191, May 2001.
- [179] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, “Measurement-based quantum computation,” *Nat. Phys.*, vol. 5, pp. 19–26, Jan 2009.
- [180] H. J. Briegel and R. Raussendorf, “Persistent entanglement in arrays of interacting particles,” *Phys. Rev. Lett.*, vol. 86, pp. 910–913, Jan 2001.
- [181] W. Dür, “Multipartite entanglement that is robust against disposal of particles,” *Phys. Rev. A*, vol. 63, p. 020303, Jan 2001.
- [182] L. DiCarlo, M. D. Reed, L. Sun, B. R. Johnson, J. M. Chow, J. M. Gambetta, L. Frunzio, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, “Preparation and measurement of three-qubit entanglement in a superconducting circuit,” *Nature*, vol. 467, pp. 574–578, Sep 2010.
- [183] C. Song, K. Xu, W. Liu, C.-p. Yang, S.-B. Zheng, H. Deng, Q. Xie, K. Huang, Q. Guo, L. Zhang, P. Zhang, D. Xu, D. Zheng, X. Zhu, H. Wang, Y.-A. Chen, C.-Y. Lu, S. Han, and J.-W. Pan, “10-qubit entanglement and parallel logic operations with a superconducting circuit,” *Phys. Rev. Lett.*, vol. 119, p. 180511, Nov 2017.
- [184] M. Gong, M.-C. Chen, Y. Zheng, S. Wang, C. Zha, H. Deng, Z. Yan, H. Rong, Y. Wu, S. Li, F. Chen, Y. Zhao, F. Liang, J. Lin, Y. Xu, C. Guo, L. Sun, A. D. Castellano, H. Wang, C. Peng, C.-Y. Lu, X. Zhu, and J.-W. Pan, “Genuine 12-qubit entanglement on a superconducting quantum processor,” *Phys. Rev. Lett.*, vol. 122, p. 110501, Mar 2019.
- [185] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, “Experimental entanglement of four particles,” *Nature*, vol. 404, pp. 256–259, Mar 2000.
- [186] D. Leibfried, E. Knill, S. Seidelin, J. Britton, R. B. Blakestad, J. Chiaverini, D. B. Hume, W. M. Itano, J. D. Jost, C. Langer, R. Ozeri, R. Reichle, and D. J. Wineland, “Creation of a six-atom ‘Schrödinger cat’ state,” *Nature*, vol. 438, pp. 639–642, Dec 2005.
- [187] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, M. Harlander, W. Hänsel, M. Hennrich, and R. Blatt, “14-qubit entanglement: Creation and coherence,” *Phys. Rev. Lett.*, vol. 106, p. 130506, Mar 2011.

- [188] N. Friis, O. Marty, C. Maier, C. Hempel, M. Holzäpfel, P. Jurcevic, M. B. Plenio, M. Huber, C. Roos, R. Blatt, and B. Lanyon, “Observation of entangled states of a fully controlled 20-qubit system,” *Phys. Rev. X*, vol. 8, p. 021012, Apr 2018.
- [189] O. Mandel, M. Greiner, A. Widera, T. Rom, T. W. Hänsch, and I. Bloch, “Controlled collisions for multi-particle entanglement of optically trapped atoms,” *Nature*, vol. 425, pp. 937–940, Oct 2003.
- [190] B. P. Lanyon, P. Jurcevic, M. Zwerger, C. Hempel, E. A. Martinez, W. Dür, H. J. Briegel, R. Blatt, and C. F. Roos, “Measurement-based quantum computation with trapped ions,” *Phys. Rev. Lett.*, vol. 111, p. 210501, Nov 2013.
- [191] H. Kaufmann, T. Ruster, C. T. Schmiegelow, M. A. Luda, V. Kaushal, J. Schulz, D. von Lindenfels, F. Schmidt-Kaler, and U. G. Poschinger, “Scalable creation of long-lived multipartite entanglement,” *Phys. Rev. Lett.*, vol. 119, p. 150503, Oct 2017.
- [192] I. Pogorelov, T. Feldker, C. D. Marciniak, L. Postler, G. Jacob, O. Krieglsteiner, V. Podlesnic, M. Meth, V. Negnevitsky, M. Stadler, B. Höfer, C. Wächter, K. Lakhmanskiy, R. Blatt, P. Schindler, and T. Monz, “Compact ion-trap quantum computing demonstrator,” *PRX Quantum*, vol. 2, p. 020343, Jun 2021.
- [193] P. Neumann, N. Mizuochi, F. Rempp, P. Hemmer, H. Watanabe, S. Yamasaki, V. Jacques, T. Gaebel, F. Jelezko, and J. Wrachtrup, “Multipartite entanglement among single spins in diamond,” *Science*, vol. 320, no. 5881, pp. 1326–1329, 2008.
- [194] H.-S. Zhong, Y. Li, W. Li, L.-C. Peng, Z.-E. Su, Y. Hu, Y.-M. He, X. Ding, W. Zhang, H. Li, L. Zhang, Z. Wang, L. You, X.-L. Wang, X. Jiang, L. Li, Y.-A. Chen, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, “12-photon entanglement and scalable scattershot boson sampling with optimal entangled-photon pairs from parametric down-conversion,” *Phys. Rev. Lett.*, vol. 121, p. 250505, Dec 2018.
- [195] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf, “Sequential generation of entangled multiqubit states,” *Phys. Rev. Lett.*, vol. 95, p. 110503, Sep 2005.
- [196] N. H. Lindner and T. Rudolph, “Proposal for pulsed on-demand sources of photonic cluster state strings,” *Phys. Rev. Lett.*, vol. 103, p. 113602, Sep 2009.
- [197] M. V. Larsen, X. Guo, C. R. Breum, J. S. Neergaard-Nielsen, and U. L. Andersen, “Deterministic generation of a two-dimensional cluster state,” *Science*, vol. 366, no. 6463, pp. 369–372, 2019.
- [198] P. Thomas, L. Ruscio, O. Morin, and G. Rempe, “Efficient generation of entangled multiphoton graph states from a single atom,” *Nature*, vol. 608, pp. 677–681, Aug 2022.
- [199] A. Omran, H. Levine, A. Keesling, G. Semeghini, T. T. Wang, S. Ebadi, H. Bernien, A. S. Zibrov, H. Pichler, S. Choi, J. Cui, M. Rossignolo, P. Rembold, S. Montangero, T. Calarco, M. Endres, M. Greiner, V. Vuleti, and M. D. Lukin, “Generation and manipulation of Schrödinger cat states in Rydberg atom arrays,” *Science*, vol. 365, no. 6453, pp. 570–574, 2019.
- [200] J.-C. Besse, K. Reuer, M. C. Collodo, A. Wulff, L. Wernli, A. Copetudo, D. Malz, P. Magnard, A. Akin, M. Gaburac, G. J. Norris, J. I. Cirac, A. Wallraff, and C. Eichler, “Realizing a deterministic source of multipartite-entangled photonic qubits,” *Nat. Commun.*, vol. 11, p. 4877, Sep 2020.
- [201] Y. Lu, S. Zhang, K. Zhang, W. Chen, Y. Shen, J. Zhang, J.-N. Zhang, and K. Kim, “Global entangling gates on arbitrary ion qubits,” *Nature*, vol. 572, pp. 363–367, Aug 2019.
- [202] D. Maslov and Y. Nam, “Use of global interactions in efficient quantum circuit constructions,” *New J. Phys.*, vol. 20, p. 033018, mar 2018.
- [203] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, “Bells theorem without inequalities,” *American Journal of Physics*, vol. 58, pp. 1131–1143, 12 1990.
- [204] B. Yurke and D. Stoler, “Einstein-podolsky-rosen effects from independent particle sources,” *Phys. Rev. Lett.*, vol. 68, pp. 1251–1254, Mar 1992.
- [205] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, “Observation of three-photon greenberger-horne-zeilinger entanglement,” *Phys. Rev. Lett.*, vol. 82, pp. 1345–1349, Feb 1999.
- [206] M. Krenn, A. Hochrainer, M. Lahiri, and A. Zeilinger, “Entanglement by path identity,” *Phys. Rev. Lett.*, vol. 118, p. 080401, Feb 2017.
- [207] A. Hochrainer, M. Lahiri, M. Erhard, M. Krenn, and A. Zeilinger, “Quantum indistinguishability by path identity and with undetected photons,” *Rev. Mod. Phys.*, vol. 94, p. 025007, Jun 2022.

- [208] M. Krenn, X. Gu, and A. Zeilinger, “Quantum experiments and graphs: Multiparty states as coherent superpositions of perfect matchings,” *Phys. Rev. Lett.*, vol. 119, p. 240403, Dec 2017.
- [209] X. Gu, M. Erhard, A. Zeilinger, and M. Krenn, “Quantum experiments and graphs ii: Quantum interference, computation, and state generation,” *Proceedings of the National Academy of Sciences*, vol. 116, no. 10, pp. 4147–4155, 2019.
- [210] S. Chin, M. Karczewski, and Y.-S. Kim, “From graphs to circuits: Optical heralded generation of  $n$ -partite GHZ and W states,” *arXiv preprint arXiv:2310.10291*, 2023.
- [211] M. Krenn, J. S. Kottmann, N. Tischler, and A. Aspuru-Guzik, “Conceptual understanding through efficient automated design of quantum optical experiments,” *Phys. Rev. X*, vol. 11, p. 031044, Aug 2021.
- [212] C. Ruiz-Gonzalez, S. Arlt, J. Petermann, S. Sayyad, T. Jaouni, E. Karimi, N. Tischler, X. Gu, and M. Krenn, “Digital Discovery of 100 diverse Quantum Experiments with PyTheus,” *Quantum*, vol. 7, p. 1204, Dec. 2023.
- [213] M. Krenn, J. Landgraf, T. Foesel, and F. Marquardt, “Artificial intelligence and machine learning for quantum technologies,” *Phys. Rev. A*, vol. 107, p. 010101, Jan 2023.
- [214] S. Sciara, R. Lo Franco, and G. Compagno, “Universality of Schmidt decomposition and particle identity,” *Sci. Rep.*, vol. 7, p. 44675, Mar 2017.
- [215] G. Compagno, A. Castellini, and R. Lo Franco, “Dealing with indistinguishable particles and their entanglement,” *Phil. Trans. R. Soc. A*, vol. 376, no. 2123, p. 20170317, 2018.
- [216] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, “Experimental realization of any discrete unitary operator,” *Phys. Rev. Lett.*, vol. 73, pp. 58–61, Jul 1994.
- [217] M. Krenn, M. Erhard, and A. Zeilinger, “Computer-inspired quantum experiments,” *Nat. Rev. Phys.*, vol. 2, pp. 649–661, Nov 2020.
- [218] D. B. West *et al.*, *Introduction to graph theory*, vol. 2. Prentice Hall Upper Saddle River, 2001.
- [219] K. Fukuda and T. Matsui, “Finding all the perfect matchings in bipartite graphs,” *Appl. Math. Lett.*, vol. 7, no. 1, pp. 15–18, 1994.
- [220] T. Tassa, “Finding all maximally-matchable edges in a bipartite graph,” *Theor. Comput. Sci.*, vol. 423, pp. 50–58, 2012.
- [221] T. Uno, “Algorithms for enumerating all perfect, maximum and maximal matchings in bipartite graphs,” in *International Symposium on Algorithms and Computation*, pp. 92–101, Springer, 1997.
- [222] D. Poole, *Linear algebra: A modern introduction*. Brooks/Cole/Thomson Learning, 2003.
- [223] P. Blasiak, E. Borsuk, M. Markiewicz, and Y.-S. Kim, “Efficient linear-optical generation of a multipartite  $w$  state,” *Phys. Rev. A*, vol. 104, p. 023701, Aug 2021.
- [224] O. Gühne, C.-Y. Lu, W.-B. Gao, and J.-W. Pan, “Toolbox for entanglement detection and fidelity estimation,” *Phys. Rev. A*, vol. 76, p. 030305, Sep 2007.
- [225] S. Campbell, M. S. Tame, and M. Paternostro, “Characterizing multipartite symmetric Dicke states under the effects of noise,” *New J. Phys.*, vol. 11, p. 073039, jul 2009.
- [226] G. Tóth, “Detection of multipartite entanglement in the vicinity of symmetric Dicke states,” *J. Opt. Soc. Am. B*, vol. 24, pp. 275–282, Feb 2007.
- [227] Y. Tokunaga, T. Yamamoto, M. Koashi, and N. Imoto, “Fidelity estimation and entanglement verification for experimentally produced four-qubit cluster states,” *Phys. Rev. A*, vol. 74, p. 020301, Aug 2006.
- [228] H. Häffner, W. Hänsel, C. F. Roos, J. Benhelm, D. Chek-al kar, M. Chwalla, T. Körber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, and R. Blatt, “Scalable multiparticle entanglement of trapped ions,” *Nature*, vol. 438, pp. 643–646, Dec 2005.
- [229] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, “Experimental detection of multipartite entanglement using witness operators,” *Phys. Rev. Lett.*, vol. 92, p. 087902, Feb 2004.
- [230] R. H. Dicke, “Coherence in spontaneous radiation processes,” *Phys. Rev.*, vol. 93, no. 1, p. 99, 1954.

- [231] S. Kasture, “Scalable approach to generation of large symmetric dicke states,” *Phys. Rev. A*, vol. 97, p. 043862, Apr 2018.
- [232] G. Tóth and O. Gühne, “Detecting genuine multipartite entanglement with two local measurements,” *Phys. Rev. Lett.*, vol. 94, p. 060501, Feb 2005.
- [233] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, “Experimental one-way quantum computing,” *Nature*, vol. 434, pp. 169–176, Mar 2005.
- [234] M. Hein, W. Dür, and H.-J. Briegel, “Entanglement properties of multipartite entangled states under the influence of decoherence,” *Phys. Rev. A*, vol. 71, p. 032350, Mar 2005.
- [235] N. Somaschi, V. Giesz, L. De Santis, J. C. Loredó, M. P. Almeida, G. Hornecker, S. L. Portalupi, T. Grange, C. Antón, J. Demory, C. Gómez, I. Sagnes, N. D. Lanzillotti-Kimura, A. Lemaître, A. Auffeves, A. G. White, L. Lanco, and P. Senellart, “Near-optimal single-photon sources in the solid state,” *Nat. Photonics*, vol. 10, pp. 340–345, May 2016.
- [236] P. Schnauber, J. Schall, S. Bounouar, T. Höhne, S.-I. Park, G.-H. Ryu, T. Heindel, S. Burger, J.-D. Song, S. Rodt, and S. Reitzenstein, “Deterministic integration of quantum dots into on-chip multimode interference beamsplitters using in situ electron beam lithography,” *Nano Letters*, vol. 18, pp. 2336–2342, Apr 2018.
- [237] A. Kuhn, M. Hennrich, and G. Rempe, “Deterministic single-photon source for distributed quantum networking,” *Phys. Rev. Lett.*, vol. 89, p. 067901, Jul 2002.
- [238] H. Wang, Q. Zeng, H. Ma, and Z. Yuan, “Progress on chip-based spontaneous four-wave mixing quantum light sources,” *Advanced Devices & Instrumentation*, vol. 5, p. 0032, 2024.
- [239] A. Peruzzo, A. Laing, A. Politi, T. Rudolph, and J. L. O’Brien, “Multimode quantum interference of photons in multiport integrated devices,” *Nat. Commun.*, vol. 2, p. 224, Mar 2011.
- [240] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, “Circuit quantum electrodynamics,” *Rev. Mod. Phys.*, vol. 93, p. 025005, May 2021.
- [241] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, “Quantum-enhanced measurements without entanglement,” *Rev. Mod. Phys.*, vol. 90, p. 035006, Sep 2018.
- [242] R. P. M. J. W. Notermans, R. J. Rengelink, and W. Vassen, “Comparison of spectral linewidths for quantum degenerate bosons and fermions,” *Phys. Rev. Lett.*, vol. 117, p. 213001, Nov 2016.
- [243] F. V. Gubarev, I. V. Dyakonov, M. Y. Saygin, G. I. Struchalin, S. S. Straupe, and S. P. Kulik, “Improved heralded schemes to generate entangled states from single photons,” *Phys. Rev. A*, vol. 102, p. 012604, Jul 2020.
- [244] S. Kumar, D. Bhatti, A. E. Jones, and S. Barz, “Experimental entanglement generation using multiport beam splitters,” *New J. Phys.*, vol. 25, p. 063027, jun 2023.
- [245] P. Blasiak and M. Markiewicz, “Entangling three qubits without ever touching,” *Scientific Reports*, vol. 9, p. 20131, Dec 2019.
- [246] C. Zhang, Y.-F. Huang, B.-H. Liu, C.-F. Li, and G.-C. Guo, “Experimental generation of a high-fidelity four-photon linear cluster state,” *Phys. Rev. A*, vol. 93, p. 062329, Jun 2016.
- [247] R. W. Boyd, A. L. Gaeta, and E. Giese, *Nonlinear Optics*, pp. 1097–1110. Cham: Springer International Publishing, 2023.
- [248] L. Pezze and A. Smerzi, “Quantum theory of phase estimation,” in *Atom interferometry*, pp. 691–741, IOS Press, 2014.
- [249] J. P. Dowling, “Quantum optical metrology the lowdown on high-n00n states,” *Contemporary Physics*, vol. 49, no. 2, pp. 125–143, 2008.
- [250] M. A. Taylor and W. P. Bowen, “Quantum metrology and its application in biology,” *Physics Reports*, vol. 615, pp. 1–59, 2016. Quantum metrology and its application in biology.
- [251] H. Katori, “Optical lattice clocks and quantum metrology,” *Nature Photonics*, vol. 5, no. 4, pp. 203–210, 2011.
- [252] J. Borregaard and A. S. Sørensen, “Near-heisenberg-limited atomic clocks in the presence of decoherence,” *Phys. Rev. Lett.*, vol. 111, p. 090801, 2013.

- [253] A. D. Ludlow, M. M. Boyd, J. Ye, E. Peik, and P. O. Schmidt, “Optical atomic clocks,” *Rev. Mod. Phys.*, vol. 87, pp. 637–701, 2015.
- [254] P.-A. Moreau, E. Toninelli, T. Gregory, and M. J. Padgett, “Imaging with quantum states of light,” *Nature Reviews Physics*, vol. 1, no. 6, pp. 367–380, 2019.
- [255] S. Pirandola, B. R. Bardhan, T. Gehring, C. Weedbrook, and S. Lloyd, “Advances in photonic quantum sensing,” *Nature Photonics*, vol. 12, no. 12, pp. 724–733, 2018.
- [256] B. J. Lawrie, P. D. Lett, A. M. Marino, and R. C. Pooser, “Quantum sensing with squeezed light,” *ACS Photonics*, vol. 6, no. 6, pp. 1307–1318, 2019.
- [257] J.-S. Lee, S.-J. Yoon, H. Rah, M. Tame, C. Rockstuhl, S. H. Song, C. Lee, and K.-G. Lee, “Quantum plasmonic sensing using single photons,” *Opt. Express*, vol. 26, no. 22, pp. 29272–29282, 2018.
- [258] P. K. Hwang Lee and J. P. Dowling, “A quantum rosetta stone for interferometry,” *Journal of Modern Optics*, vol. 49, no. 14–15, pp. 2325–2338, 2002.
- [259] F. Flamini, N. Spagnolo, and F. Sciarrino, “Photonic quantum information processing: a review,” *Reports on Progress in Physics*, vol. 82, no. 1, p. 016001, 2018.
- [260] S. Slussarenko, M. M. Weston, H. M. Chrzanowski, L. K. Shalm, V. B. Verma, S. W. Nam, and G. J. Pryde, “Unconditional violation of the shot-noise limit in photonic quantum metrology,” *Nature Photonics*, vol. 11, no. 11, pp. 700–703, 2017.
- [261] M. Hiekkamäki, F. Bouchard, and R. Fickler, “Photonic angular superresolution using twisted  $n00n$  states,” *Phys. Rev. Lett.*, vol. 127, no. 26, p. 263601, 2021.
- [262] H. Takesue and K. Inoue, “Generation of  $1.5\text{--}\mu\text{m}$  band time-bin entanglement using spontaneous fiber four-wave mixing and planar light-wave circuit interferometers,” *Phys. Rev. A*, vol. 72, no. 4, p. 041804, 2005.
- [263] H. de Riedmatten, I. Marcikic, J. A. W. van Houwelingen, W. Tittel, H. Zbinden, and N. Gisin, “Long-distance entanglement swapping with photons from separated sources,” *Phys. Rev. A*, vol. 71, no. 5, p. 050302, 2005.
- [264] R. Valivarthi, M. G. Puigibert, Q. Zhou, G. H. Aguilar, V. B. Verma, F. Marsili, M. D. Shaw, S. W. Nam, D. Oblak, and W. Tittel, “Quantum teleportation across a metropolitan fibre network,” *Nature Photonics*, vol. 10, no. 10, pp. 676–680, 2016.
- [265] J.-H. Kim, J.-W. Chae, Y.-C. Jeong, and Y.-H. Kim, “Quantum communication with time-bin entanglement over a wavelength-multiplexed fiber network,” *APL Photonics*, vol. 7, no. 1, p. 016106, 2022.
- [266] A. Schreiber, F. Katzschmann, A. Gábris, P. P. Rohde, K. Laiho, M. Štefaňák, V. Potoček, C. Hamilton, I. Jex, and C. Silberhorn, “Simulations of two particle dynamics employing dynamic coin control in 2d quantum walks,” in *2013 Conference on Lasers and Electro-Optics - International Quantum Electronics Conference*, p. IB\_2\_3, Optica Publishing Group, 2013.
- [267] Z. Li, H. Zhang, and H. Zhu, “Implementation of generalized measurements on a qudit via quantum walks,” *Phys. Rev. A*, vol. 99, no. 6, p. 062342, 2019.
- [268] X. Wang, X. Zhan, Y. Li, L. Xiao, G. Zhu, D. Qu, Q. Lin, Y. Yu, and P. Xue, “Generalized quantum measurements on a higher-dimensional system via quantum walks,” *Phys. Rev. Lett.*, vol. 131, no. 15, p. 150803, 2023.
- [269] Y.-y. Zhao, N.-k. Yu, P. Kurzyński, G.-y. Xiang, C.-F. Li, and G.-C. Guo, “Experimental realization of generalized qubit measurements based on quantum walks,” *Phys. Rev. A*, vol. 91, no. 4, p. 042101, 2015.
- [270] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, “Parity–time synthetic photonic lattices,” *Nature*, vol. 488, pp. 167–171, Aug 2012.
- [271] Y. Aharonov, L. Davidovich, and N. Zagury, “Quantum random walks,” *Phys. Rev. A*, vol. 48, no. 2, p. 1687, 1993.
- [272] S. M. Barnett, “On single-photon and classical interference,” *Physica Scripta*, vol. 97, no. 11, p. 114004, 2022.
- [273] W. H. Zurek, “Decoherence, einselection, and the quantum origins of the classical,” *Rev. Mod. Phys.*, vol. 75, no. 3, p. 715, 2003.
- [274] M. A. Nielsen and I. Chuang, “Quantum computation and quantum information,” 2002.
- [275] A. Streltsov, G. Adesso, and M. B. Plenio, “Colloquium: Quantum coherence as a resource,” *Rev. Mod. Phys.*, vol. 89, no. 4, p. 041003, 2017.

- [276] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, “The classical-quantum boundary for correlations: Discord and related measures,” *Rev. Mod. Phys.*, vol. 84, pp. 1655–1707, 2012.
- [277] M. T. Quintino, T. Vértesi, and N. Brunner, “Joint measurability, einstein-podolsky-rosen steering, and bell nonlocality,” *Phys. Rev. Lett.*, vol. 113, no. 16, p. 160402, 2014.
- [278] H. M. Wiseman, S. J. Jones, and A. C. Doherty, “Steering, entanglement, nonlocality, and the einstein-podolsky-rosen paradox,” *Phys. Rev. Lett.*, vol. 98, no. 14, p. 140402, 2007.
- [279] N. H. Y. Ng and M. P. Woods, “Resource theory of quantum thermodynamics: Thermal operations and second laws,” in *Thermodynamics in the Quantum Regime*, pp. 625–650, Springer, 2018.
- [280] H. M. Wiseman and J. A. Vaccaro, “Entanglement of indistinguishable particles shared between two parties,” *Phys. Rev. Lett.*, vol. 91, no. 9, p. 097902, 2003.
- [281] Z. Ou, C. Hong, and L. Mandel, “Relation between input and output states for a beam splitter,” *Opt. Commun.*, vol. 63, no. 2, pp. 118–122, 1987.
- [282] T. Giordani, F. Flamini, M. Pompili, N. Viggianiello, N. Spagnolo, A. Crespi, R. Osellame, N. Wiebe, M. Walschaers, A. Buchleitner, *et al.*, “Experimental statistical signature of many-body quantum interference,” *Nat. Photonics*, vol. 12, no. 3, pp. 173–178, 2018.
- [283] F. Benatti, R. Floreanini, F. Franchini, and U. Marzolino, “Entanglement in indistinguishable particle systems,” *Phys. Rep.*, 2020.
- [284] S. Bose and D. Home, “Generic entanglement generation, quantum statistics, and complementarity,” *Phys. Rev. Lett.*, vol. 88, no. 5, p. 050401, 2002.
- [285] L. Mandel, “Coherence and indistinguishability,” *Opt. Lett.*, vol. 16, no. 23, pp. 1882–1883, 1991.
- [286] J. Sperling, A. Perez-Leija, K. Busch, and I. A. Walmsley, “Quantum coherences of indistinguishable particles,” *Phys. Rev. A*, vol. 96, no. 3, p. 032334, 2017.
- [287] H.-K. Lo, M. Curty, and B. Qi, “Measurement-device-independent quantum key distribution,” *Phys. Rev. Lett.*, vol. 108, no. 13, p. 130503, 2012.
- [288] Y. Liu, T.-Y. Chen, L.-J. Wang, H. Liang, G.-L. Shentu, J. Wang, K. Cui, H.-L. Yin, N.-L. Liu, L. Li, *et al.*, “Experimental measurement-device-independent quantum key distribution,” *Phys. Rev. Lett.*, vol. 111, no. 13, p. 130502, 2013.
- [289] S. Aaronson and A. Arkhipov, “The computational complexity of linear optics,” in *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pp. 333–342, 2011.
- [290] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, “One-dimensional quantum walks,” in *Proceedings of the thirty-third annual ACM symposium on Theory of computing*, pp. 37–49, 2001.
- [291] V. B. Bulchandani, C. Karrasch, and J. E. Moore, “Superdiffusive transport of energy in one-dimensional metals,” *PNAS*, vol. 117, no. 23, pp. 12713–12718, 2020.
- [292] A. Schreiber, K. Cassemiro, V. Potoček, A. Gábris, I. Jex, and C. Silberhorn, “Decoherence and disorder in quantum walks: from ballistic spread to localization,” *Phys. Rev. Lett.*, vol. 106, no. 18, p. 180403, 2011.
- [293] A. Crespi, R. Osellame, R. Ramponi, V. Giovannetti, R. Fazio, L. Sansoni, F. De Nicola, F. Sciarrino, and P. Mataloni, “Anderson localization of entangled photons in an integrated quantum walk,” *Nat. Photonics*, vol. 7, no. 4, pp. 322–328, 2013.
- [294] F. De Nicola, L. Sansoni, A. Crespi, R. Ramponi, R. Osellame, V. Giovannetti, R. Fazio, P. Mataloni, and F. Sciarrino, “Quantum simulation of bosonic-fermionic noninteracting particles in disordered systems via a quantum walk,” *Phys. Rev. A*, vol. 89, no. 3, p. 032322, 2014.
- [295] A. Gerdardi, A. Laneve, L. D. Bonavena, L. Sansoni, J. Ferraz, A. Fratalocchi, F. Sciarrino, Á. Cuevas, and P. Mataloni, “Experimental investigation of superdiffusion via coherent disordered quantum walks,” *Phys. Rev. Lett.*, vol. 123, no. 14, p. 140501, 2019.
- [296] N. Lambert, Y.-N. Chen, Y.-C. Cheng, C.-M. Li, G.-Y. Chen, and F. Nori, “Quantum biology,” *Nature Physics*, vol. 9, no. 1, pp. 10–18, 2013.
- [297] Y. Bromberg, Y. Lahini, R. Morandotti, and Y. Silberberg, “Quantum and classical correlations in waveguide lattices,” *Phys. Rev. Lett.*, vol. 102, no. 25, p. 253904, 2009.

- [298] A. Peruzzo, M. Lobino, J. C. Matthews, N. Matsuda, A. Politi, K. Poullos, X.-Q. Zhou, Y. Lahini, N. Ismail, K. Wörhoff, *et al.*, “Quantum walks of correlated photons,” *Science*, vol. 329, no. 5998, pp. 1500–1503, 2010.
- [299] Y. Lahini, M. Verbin, S. D. Huber, Y. Bromberg, R. Pugatch, and Y. Silberberg, “Quantum walk of two interacting bosons,” *Phys. Rev. A*, vol. 86, no. 1, p. 011603, 2012.
- [300] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, “Two-particle bosonic-fermionic quantum walk via integrated photonics,” *Phys. Rev. Lett.*, vol. 108, no. 1, p. 010502, 2012.
- [301] K. Poullos, R. Keil, D. Fry, J. D. Meinecke, J. C. Matthews, A. Politi, M. Lobino, M. Gräfe, M. Heinrich, S. Nolte, *et al.*, “Quantum walks of correlated photon pairs in two-dimensional waveguide arrays,” *Phys. Rev. Lett.*, vol. 112, no. 14, p. 143604, 2014.
- [302] Y. Lahini, Y. Bromberg, D. N. Christodoulides, and Y. Silberberg, “Quantum correlations in two-particle anderson localization,” *Phys. Rev. Lett.*, vol. 105, no. 16, p. 163905, 2010.
- [303] A. Gerdardi, S. De, A. Laneve, S. Barkhofen, J. Sperling, P. Mataloni, and C. Silberhorn, “Subdiffusion via disordered quantum walks,” *arXiv preprint arXiv:2007.12526*, 2020.
- [304] A. Gerdardi, L. D. Bonavena, C. Liorni, P. Mataloni, and Á. Cuevas, “A novel bulk-optics scheme for quantum walk with high phase stability,” *Condens. Matter*, vol. 4, no. 1, p. 14, 2019.
- [305] A. Schreiber, K. Cassemiro, V. Potoček, A. Gábris, I. Jex, and C. Silberhorn, “Decoherence and disorder in quantum walks: from ballistic spread to localization,” *Physical review letters*, vol. 106, no. 18, p. 180403, 2011.
- [306] A. Fedrizzi, T. Herbst, A. Poppe, T. Jennewein, and A. Zeilinger, “A wavelength-tunable fiber-coupled source of narrowband entangled photons,” *Optics Express*, vol. 15, no. 23, pp. 15377–15386, 2007.
- [307] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, “Colloquium: Non-markovian dynamics in open quantum systems,” *Rev. Mod. Phys.*, vol. 88, no. 2, p. 021002, 2016.
- [308] Á. Rivas, S. F. Huelga, and M. B. Plenio, “Quantum non-markovianity: characterization, quantification and detection,” *Rep. Prog. Phys.*, vol. 77, no. 9, p. 094001, 2014.
- [309] R. L. Franco, B. Bellomo, S. Maniscalco, and G. Compagno, “Dynamics of quantum correlations in two-qubit systems within non-markovian environments,” *Int. J. Mod. Phys. B*, vol. 27, no. 01n03, p. 1345053, 2013.
- [310] “Coherence and entanglement dynamics of vibrating qubits,” *Optics Communications*, vol. 424, pp. 26–31, 2018.
- [311] F. Caruso, V. Giovannetti, C. Lupo, and S. Mancini, “Quantum channels and memory effects,” *Rev. Mod. Phys.*, vol. 86, pp. 1203–1259, Dec 2014.
- [312] “Quantumness and memory of one qubit in a dissipative cavity under classical control,” *Annals of Physics*, vol. 414, p. 168073, 2020.
- [313] A. D’Arrigo, R. Lo Franco, G. Benenti, E. Paladino, and G. Falci, “Recovering entanglement by local operations,” *Ann. Phys.*, vol. 350, pp. 211–224, 2014.
- [314] J.-S. Xu, K. Sun, C.-F. Li, X.-Y. Xu, G.-C. Guo, E. Andersson, R. L. Franco, and G. Compagno, “Experimental recovery of quantum correlations in absence of system-environment back-action,” *Nature communications*, vol. 4, no. 1, pp. 1–7, 2013.
- [315] A. Smirne, L. Mazzola, M. Paternostro, and B. Vacchini, “Interaction-induced correlations and non-markovianity of quantum dynamics,” *Phys. Rev. A*, vol. 87, p. 052129, 2013.
- [316] L. Mazzola, C. A. Rodríguez-Rosario, K. Modi, and M. Paternostro, “Dynamical role of system-environment correlations in non-markovian dynamics,” *Phys. Rev. A*, vol. 86, p. 010102, 2012.
- [317] A. Orioux, A. d’Arrigo, G. Ferranti, R. L. Franco, G. Benenti, E. Paladino, G. Falci, F. Sciarrino, and P. Mataloni, “Experimental on-demand recovery of entanglement by local operations within non-markovian dynamics,” *Sci. Rep.*, vol. 5, no. 1, pp. 1–8, 2015.
- [318] N. K. Bernardes, A. Cuevas, A. Orioux, C. H. Monken, P. Mataloni, F. Sciarrino, and M. F. Santos, “Experimental observation of weak non-markovianity,” *Scientific Reports*, vol. 5, no. 1, p. 17520, 2015.
- [319] N. K. Bernardes, J. P. S. Peterson, R. S. Sarthour, A. M. Souza, C. H. Monken, I. Roditi, I. S. Oliveira, and M. F. Santos, “High resolution non-markovianity in nmr,” *Scientific Reports*, vol. 6, no. 1, p. 33945, 2016.

- [320] A. Chiuri, C. Greganti, L. Mazzola, M. Paternostro, and P. Mataloni, “Linear optics simulation of quantum non-markovian dynamics,” *Scientific Reports*, vol. 2, no. 1, p. 968, 2012.
- [321] B. Bylicka, D. Chruściński, and S. Maniscalco, “Non-markovianity and reservoir memory of quantum channels: a quantum information theory perspective,” *Sci. Rep.*, vol. 4, no. 1, pp. 1–7, 2014.
- [322] G.-Y. Xiang, Z.-B. Hou, C.-F. Li, G.-C. Guo, H.-P. Breuer, E.-M. Laine, and J. Piilo, “Entanglement distribution in optical fibers assisted by nonlocal memory effects,” *EPL (Europhysics Letters)*, vol. 107, no. 5, p. 54006, 2014.
- [323] N. Mirkin, P. Poggi, and D. Wisniacki, “Information backflow as a resource for entanglement,” *Phys. Rev. A*, vol. 99, p. 062327, Jun 2019.
- [324] N. Mirkin, P. Poggi, and D. Wisniacki, “Entangling protocols due to non-markovian dynamics,” *Phys. Rev. A*, vol. 99, p. 020301, 2019.
- [325] G. Thomas, N. Siddharth, S. Banerjee, and S. Ghosh, “Thermodynamics of non-markovian reservoirs and heat engines,” *Physical Review E*, vol. 97, no. 6, p. 062108, 2018.
- [326] J. Teittinen, H. Lyyra, B. Sokolov, and S. Maniscalco, “Revealing memory effects in phase-covariant quantum master equations,” *New J. Phys.*, vol. 20, no. 7, p. 073012, 2018.
- [327] J. Naikoo, S. Dutta, and S. Banerjee, “Facets of quantum information under non-markovian evolution,” *Phys. Rev. A*, vol. 99, no. 4, p. 042128, 2019.
- [328] Á. Rivas, S. F. Huelga, and M. B. Plenio, “Entanglement and non-markovianity of quantum evolutions,” *Phys. Rev. Lett.*, vol. 105, no. 5, p. 050403, 2010.
- [329] H.-P. Breuer, E.-M. Laine, and J. Piilo, “Measure for the degree of non-markovian behavior of quantum processes in open systems,” *Phys. Rev. Lett.*, vol. 103, no. 21, p. 210401, 2009.
- [330] E.-M. Laine, J. Piilo, and H.-P. Breuer, “Measure for the non-markovianity of quantum processes,” *Phys. Rev. A*, vol. 81, no. 6, p. 062115, 2010.
- [331] M. J. Hall, J. D. Cresser, L. Li, and E. Andersson, “Canonical form of master equations and characterization of non-markovianity,” *Phys. Rev. A*, vol. 89, no. 4, p. 042120, 2014.
- [332] S. Luo, S. Fu, and H. Song, “Quantifying non-markovianity via correlations,” *Phys. Rev. A*, vol. 86, no. 4, p. 044101, 2012.
- [333] Z. He, C. Yao, Q. Wang, and J. Zou, “Measuring non-markovianity based on local quantum uncertainty,” *Phys. Rev. A*, vol. 90, no. 4, p. 042101, 2014.
- [334] H. S. Dhar, M. N. Bera, and G. Adesso, “Characterizing non-markovianity via quantum interferometric power,” *Phys. Rev. A*, vol. 91, no. 3, p. 032115, 2015.
- [335] D. Girolami, A. M. Souza, V. Giovannetti, T. Tufarelli, J. G. Filgueiras, R. S. Sarthour, D. O. Soares-Pinto, I. S. Oliveira, and G. Adesso, “Quantum discord determines the interferometric power of quantum states,” *Phys. Rev. Lett.*, vol. 112, no. 21, p. 210401, 2014.
- [336] F. F. Fanchini, D. d. O. S. Pinto, and G. Adesso, *Lectures on General Quantum Correlations and their Applications*. Springer, 2017.
- [337] H. Rangani Jahromi, M. Amini, and M. Ghanaatian, “Multiparameter estimation, lower bound on quantum fisher information, and non-markovianity witnesses of noisy two-qubit systems,” *Quantum Inf. Process.*, vol. 18, no. 11, p. 338, 2019.
- [338] T. Chanda and S. Bhattacharya, “Delineating incoherent non-markovian dynamics using quantum coherence,” *Ann. Phys.*, vol. 366, pp. 1–12, 2016.
- [339] Z. He, H.-S. Zeng, Y. Li, Q. Wang, and C. Yao, “Non-markovianity measure based on the relative entropy of coherence in an extended space,” *Phys. Rev. A*, vol. 96, no. 2, p. 022106, 2017.
- [340] A. Rajagopal, A. U. Devi, and R. Rendell, “Kraus representation of quantum evolution and fidelity as manifestations of markovian and non-markovian forms,” *Phys. Rev. A*, vol. 82, no. 4, p. 042107, 2010.
- [341] B. Farajollahi, M. Jafarzadeh, H. Rangani Jahromi, and M. Amniat-Talab, “Estimation of temperature in micromaser-type systems,” *Quantum Inf. Process.*, vol. 17, no. 6, p. 119, 2018.



- [342] S. Lorenzo, F. Plastina, and M. Paternostro, “Geometrical characterization of non-markovianity,” *Phys. Rev. A*, vol. 88, no. 2, p. 020102, 2013.
- [343] X.-M. Lu, X. Wang, and C. Sun, “Quantum fisher information flow and non-markovian processes of open systems,” *Phys. Rev. A*, vol. 82, no. 4, p. 042103, 2010.
- [344] H. Rangani Jahromi, “Relation between quantum probe and entanglement in n-qubit systems within markovian and non-markovian environments,” *J. Mod. Opt.*, vol. 64, no. 14, pp. 1377–1385, 2017.
- [345] W.-M. Zhang, P.-Y. Lo, H.-N. Xiong, M. W.-Y. Tu, and F. Nori, “General non-markovian dynamics of open quantum systems,” *Phys. Rev. Lett.*, vol. 109, no. 17, p. 170402, 2012.
- [346] P. Strasberg and M. Esposito, “Non-markovianity and negative entropy production rates,” *Phys. Rev. E*, vol. 99, no. 1, p. 012120, 2019.
- [347] H. Rangani Jahromi and M. Amniat-Talab, “Precision of estimation and entropy as witnesses of non-markovianity in the presence of random classical noises,” *Ann. Phys.*, vol. 360, pp. 446–461, 2015.
- [348] D. De Santis, M. Johansson, B. Bylicka, N. K. Bernardes, and A. Acín, “Correlation measure detecting almost all non-markovian evolutions,” *Phys. Rev. A*, vol. 99, no. 1, p. 012303, 2019.
- [349] X. Zheng, S.-Q. Ma, and G.-F. Zhang, “Detecting non-markovianity via linear entropy of choi state,” *Ann. Phys.-Berlin*, vol. 532, no. 1, p. 1900320, 2020.
- [350] S. Deffner and E. Lutz, “Quantum speed limit for non-markovian dynamics,” *Phys. Rev. Lett.*, vol. 111, no. 1, p. 010402, 2013.
- [351] Z.-Y. Xu, S. Luo, W. Yang, C. Liu, and S. Zhu, “Quantum speedup in a memory environment,” *Phys. Rev. A*, vol. 89, no. 1, p. 012307, 2014.
- [352] K. Xu, Y.-J. Zhang, Y.-J. Xia, Z. Wang, and H. Fan, “Hierarchical-environment-assisted non-markovian speedup dynamics control,” *Phys. Rev. A*, vol. 98, no. 2, p. 022114, 2018.
- [353] D. Chruściński and F. Mukhamedov, “Dissipative generators, divisible dynamical maps, and the kadison-schwarz inequality,” *Phys. Rev. A*, vol. 100, p. 052120, Nov 2019.
- [354] G. Guarnieri, A. Smirne, and B. Vacchini, “Quantum regression theorem and non-markovianity of quantum dynamics,” *Phys. Rev. A*, vol. 90, p. 022110, Aug 2014.
- [355] H.-B. Liu, W. L. Yang, J.-H. An, and Z.-Y. Xu, “Mechanism for quantum speedup in open quantum systems,” *Phys. Rev. A*, vol. 93, p. 020105, Feb 2016.
- [356] B. Ahansaz and A. Ektesabi, “Quantum speedup, non-markovianity and formation of bound state,” *Sci. Rep.*, vol. 9, p. 14946, 2019.
- [357] S. L. Braunstein and C. M. Caves, “Statistical distance and the geometry of quantum states,” *Physical Review Letters*, vol. 72, no. 22, p. 3439, 1994.
- [358] M. Gessner and A. Smerzi, “Statistical speed of quantum states: Generalized quantum fisher information and schatten speed,” *Phys. Rev. A*, vol. 97, no. 2, p. 022109, 2018.
- [359] V. Giovannetti, S. Lloyd, and L. Maccone, “Quantum metrology,” *Physical review letters*, vol. 96, no. 1, p. 010401, 2006.
- [360] J. Lankinen, H. Lyyra, B. Sokolov, J. Teittinen, B. Ziaei, and S. Maniscalco, “Complete positivity, finite-temperature effects, and additivity of noise for time-local qubit dynamics,” *Phys. Rev. A*, vol. 93, no. 5, p. 052103, 2016.
- [361] D. Chruściński and F. A. Wudarski, “Non-markovian random unitary qubit dynamics,” *Phys. Lett. A*, vol. 377, no. 21-22, pp. 1425–1429, 2013.
- [362] D. Chruściński and F. A. Wudarski, “Non-markovianity degree for random unitary evolution,” *Phys. Rev. A*, vol. 91, no. 1, p. 012104, 2015.
- [363] J. Teittinen, H. Lyyra, and S. Maniscalco, “There is no general connection between the quantum speed limit and non-markovianity,” *New J. Phys.*, vol. 21, no. 12, p. 123041, 2019.
- [364] H. Song and Y. Mao, “Dynamics of rényi entropy and applications in detecting quantum non-markovianity,” *Phys. Rev. A*, vol. 96, no. 3, p. 032115, 2017.

- [365] X. Wang and S. Schirmer, “Contractivity of the hilbert-schmidt distance under open-system dynamics,” *Phys. Rev. A*, vol. 79, no. 5, p. 052326, 2009.
- [366] A. Smirne, J. Kołodyński, S. F. Huelga, and R. Demkowicz-Dobrzański, “Ultimate precision limits for noisy frequency estimation,” *Phys. Rev. Lett.*, vol. 116, p. 120801, Mar 2016.
- [367] J. Teittinen, H. Lyyra, B. Sokolov, and S. Maniscalco, “Revealing memory effects in phase-covariant quantum master equations,” *New Journal of Physics*, vol. 20, p. 073012, jul 2018.
- [368] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- [369] Z. He, H.-S. Zeng, Y. Chen, and C. Yao, “Non-markovian dynamics of a dephasing model in a squeezed thermal bath,” *Laser Phys. Lett.*, vol. 16, no. 6, p. 065204, 2019.
- [370] M. Jiang and S. Luo, “Comparing quantum markovianities: Distinguishability versus correlations,” *Phys. Rev. A*, vol. 88, no. 3, p. 034101, 2013.
- [371] B. Bellomo, R. L. Franco, and G. Compagno, “Non-markovian effects on the dynamics of entanglement,” *Phys. Rev. Lett.*, vol. 99, no. 16, p. 160502, 2007.
- [372] D. Wang, W.-N. Shi, R. D. Hoehn, F. Ming, W.-Y. Sun, L. Ye, and S. Kais, “Probing entropic uncertainty relations under a two-atom system coupled with structured bosonic reservoirs,” *Quantum Inf. Process.*, vol. 17, no. 12, p. 335, 2018.
- [373] M. O. Scully and M. S. Zubairy, *Quantum Optics*. Cambridge University Press, 1997.
- [374] W.-j. Gu and G.-x. Li, “Non-markovian behavior for spontaneous decay of a v-type three-level atom with quantum interference,” *Phys. Rev. A*, vol. 85, no. 1, p. 014101, 2012.
- [375] B. P. Lanyon, T. J. Weinhold, N. K. Langford, J. L. O’Brien, K. J. Resch, A. Gilchrist, and A. G. White, “Manipulating biphotonic qutrits,” *Phys. Rev. Lett.*, vol. 100, p. 060504, 2008.
- [376] K. S. Kumar, A. Vepsäläinen, S. Danilin, and G. S. Paraoanu, “Stimulated raman adiabatic passage in a three-level superconducting circuit,” *Nature Communications*, vol. 7, no. 1, p. 10628, 2016.
- [377] N. Behzadi, B. Ahansaz, A. Ektesabi, and E. Faizi, “Effect of two-qutrit entanglement on quantum speed limit time of a bipartite v-type open system,” *Ann. Phys.*, vol. 378, pp. 407–417, 2017.
- [378] J. Piilo, K. Härkönen, S. Maniscalco, and K.-A. Suominen, “Open system dynamics with non-markovian quantum jumps,” *Phys. Rev. A*, vol. 79, no. 6, p. 062112, 2009.
- [379] I. de Vega and D. Alonso, “Dynamics of non-Markovian open quantum systems,” *Rev. Mod. Phys.*, vol. 89, p. 015001, 2017.
- [380] H. P. Breuer, E. M. Laine, J. Piilo, and B. Vacchini, “Colloquium: Non-Markovian dynamics in open quantum systems,” *Rev. Mod. Phys.*, vol. 88, p. 021002, 2016.
- [381] M. Banacki, M. Marciniak, K. Horodecki, and P. Horodecki, “Information backflow may not indicate quantum memory,” *arXiv preprint arXiv:2008.12638*, 2020.
- [382] B.-H. Liu, D.-Y. Cao, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, “Photonic realization of nonlocal memory effects and non-Markovian quantum probes,” *Sci. Rep.*, vol. 3, p. 1781, 2013.
- [383] A. Chiuri, C. Greganti, L. Mazzola, M. Paternostro, and P. Mataloni, “Linear optics simulation of quantum non-Markovian dynamics,” *Sci. Rep.*, vol. 2, p. 968, 2012.
- [384] G. A. L. White, C. D. Hill, F. A. Pollock, L. C. L. Hollenberg, and K. Modi, “Demonstration of non-markovian process characterisation and control on a quantum processor,” *Nat. Comm.*, vol. 11, no. 1, p. 6301, 2020.
- [385] V. S. Ferreira, J. Banker, A. Sipahigil, M. H. Matheny, A. J. Keller, E. Kim, M. Mirhosseini, and O. Painter, “Collapse and revival of an artificial atom coupled to a structured photonic reservoir,” *Phys. Rev. X*, vol. 11, p. 041043, 2021.
- [386] A. Burgess and M. Florescu, “Modelling non-markovian dynamics in photonic crystals with recurrent neural networks,” *Opt. Mater. Express*, vol. 11, pp. 2037–2048, 2021.
- [387] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, “Quantifying non-markovianity of continuous-variable gaussian dynamical maps,” *Phys. Rev. A*, vol. 84, p. 052118, Nov 2011.

- [388] A. K. Rajagopal, A. R. Usha Devi, and R. W. Rendell, “Kraus representation of quantum evolution and fidelity as manifestations of markovian and non-markovian forms,” *Phys. Rev. A*, vol. 82, p. 042107, Oct 2010.
- [389] A. R. Usha Devi, A. K. Rajagopal, and Sudha, “Open-system quantum dynamics with correlated initial states, not completely positive maps, and non-markovianity,” *Phys. Rev. A*, vol. 83, p. 022109, Feb 2011.
- [390] X.-M. Lu, X. Wang, and C. P. Sun, “Quantum fisher information flow and non-markovian processes of open systems,” *Phys. Rev. A*, vol. 82, p. 042103, Oct 2010.
- [391] B. Bylicka, D. Chruściński, and S. Maniscalco, “Non-markovianity as a resource for quantum technologies,” *arXiv preprint arXiv:1301.2585*, 2013.
- [392] C. Benedetti, M. G. A. Paris, and S. Maniscalco, “Non-markovianity of colored noisy channels,” *Phys. Rev. A*, vol. 89, p. 012114, Jan 2014.
- [393] C. Addis, G. Brebner, P. Haikka, and S. Maniscalco, “Coherence trapping and information backflow in dephasing qubits,” *Phys. Rev. A*, vol. 89, p. 024101, Feb 2014.
- [394] S. Lorenzo, F. Plastina, and M. Paternostro, “Geometrical characterization of non-Markovianity,” *Phys. Rev. A*, vol. 88, p. 020102, Aug 2013.
- [395] T. Tufarelli, M. S. Kim, and F. Ciccarello, “Non-markovianity of a quantum emitter in front of a mirror,” *Phys. Rev. A*, vol. 90, p. 012113, Jul 2014.
- [396] T. J. G. Apollaro, S. Lorenzo, C. Di Franco, F. Plastina, and M. Paternostro, “Competition between memory-keeping and memory-erasing decoherence channels,” *Phys. Rev. A*, vol. 90, p. 012310, Jul 2014.
- [397] T. Baumgratz, M. Cramer, and M. B. Plenio, “Quantifying coherence,” *Phys. Rev. Lett.*, vol. 113, p. 140401, Sep 2014.
- [398] A. Winter and D. Yang, “Operational resource theory of coherence,” *Phys. Rev. Lett.*, vol. 116, p. 120404, Mar 2016.
- [399] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, “Assisted distillation of quantum coherence,” *Phys. Rev. Lett.*, vol. 116, p. 070402, Feb 2016.
- [400] A. Streltsov, G. Adesso, and M. B. Plenio, “Colloquium: Quantum coherence as a resource,” *Rev. Mod. Phys.*, vol. 89, p. 041003, Oct 2017.
- [401] S. Luo, S. Fu, and H. Song, “Quantifying non-markovianity via correlations,” *Phys. Rev. A*, vol. 86, p. 044101, 2012.
- [402] C. Uchiyama, “Exploring initial correlations in a gibbs state by application of external field,” *Phys. Rev. A*, vol. 85, p. 052104, May 2012.
- [403] S. Lorenzo, F. Plastina, and M. Paternostro, “Role of environmental correlations in the non-markovian dynamics of a spin system,” *Phys. Rev. A*, vol. 84, p. 032124, Sep 2011.
- [404] R. Vasile, F. Galve, and R. Zambrini, “Spectral origin of non-markovian open-system dynamics: A finite harmonic model without approximations,” *Phys. Rev. A*, vol. 89, p. 022109, Feb 2014.
- [405] R. Lo Franco, B. Bellomo, E. Andersson, and G. Compagno, “Revival of quantum correlations without system-environment back-action,” *Phys. Rev. A*, vol. 85, p. 032318, Mar 2012.
- [406] A. D’Arrigo, G. Benenti, R. Lo Franco, G. Falci, and E. Paladino, “Hidden entanglement, system-environment information flow and non-markovianity,” *Int. J. Quant. Infor.*, vol. 12, p. 1461005, 2014.
- [407] A. Datta, A. Shaji, and C. M. Caves, “Quantum discord and the power of one qubit,” *Phys. Rev. Lett.*, vol. 100, p. 050502, Feb 2008.
- [408] A. Streltsov, H. Kampermann, and D. Bruß, “Behavior of quantum correlations under local noise,” *Phys. Rev. Lett.*, vol. 107, p. 170502, Oct 2011.
- [409] F. Ciccarello and V. Giovannetti, “Creating quantum correlations through local nonunitary memoryless channels,” *Phys. Rev. A*, vol. 85, p. 010102, Jan 2012.
- [410] J. Maziero, L. C. Céleri, R. M. Serra, and V. Vedral, “Classical and quantum correlations under decoherence,” *Phys. Rev. A*, vol. 80, p. 044102, Oct 2009.

- [411] B. Bellomo, G. Compagno, R. Lo Franco, A. Ridolfo, and S. Savasta, “Dynamics and extraction of quantum discord in a multipartite open system,” *Int. J. Quant. Infor.*, vol. 9, no. 07n08, pp. 1665–1676, 2011.
- [412] T. Yu and J. Eberly, “Sudden death of entanglement,” *Science*, vol. 323, no. 5914, pp. 598–601, 2009.
- [413] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. Walborn, P. S. Ribeiro, and L. Davidovich, “Environment-induced sudden death of entanglement,” *Science*, vol. 316, no. 5824, pp. 579–582, 2007.
- [414] H. Ollivier and W. H. Zurek, “Quantum discord: a measure of the quantumness of correlations,” *Physical review letters*, vol. 88, no. 1, p. 017901, 2001.
- [415] L. Henderson and V. Vedral, “Classical, quantum and total correlations,” *J. Phys. A: Math. Gen.*, vol. 34, no. 35, p. 6899, 2001.
- [416] S. Luo, “Using measurement-induced disturbance to characterize correlations as classical or quantum,” *Phys. Rev. A*, vol. 77, p. 022301, Feb 2008.
- [417] H. Rangani Jahromi and R. Lo Franco, “Searching for exceptional points and inspecting non-contractivity of trace distance in (anti-)pt-symmetric systems,” *arXiv preprint arXiv:2101.04663*, 2021.
- [418] H. Rangani Jahromi, “Remote sensing and faithful quantum teleportation through non-localized qubits,” *Phys. Lett. A*, vol. 424, p. 127850, 2022.
- [419] H. Rangani Jahromi and S. Haseli, “Quantum memory and quantum correlations of majorana qubits used for magnetometry,” *Quant. Inf. Comput.*, vol. 20, p. 935, 2020.
- [420] H. Rangani Jahromi and R. Lo Franco, “Hilbert–schmidt speed as an efficient figure of merit for quantum estimation of phase encoded into the initial state of open n-qubit systems,” *Sci. Rep.*, vol. 11, p. 7128, Mar 2021.
- [421] H. Rangani Jahromi, R. Radgozar, S. M. Hosseiny, and M. Amniat-Talab, “Estimating energy levels of a three-level atom in single and multi-parameter metrological schemes,” *arXiv preprint arXiv:2110.10256*, 2021.
- [422] D. Bruß and C. Macchiavello, “Optimal eavesdropping in cryptography with three-dimensional quantum states,” *Phys. Rev. Lett.*, vol. 88, p. 127901, Mar 2002.
- [423] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, “Security of quantum key distribution using  $d$ -level systems,” *Phys. Rev. Lett.*, vol. 88, p. 127902, Mar 2002.
- [424] D. Gottesman, “Fault-tolerant quantum computation with higher-dimensional systems,” in *Quantum Computing and Quantum Communications* (C. P. Williams, ed.), (Berlin, Heidelberg), pp. 302–313, Springer Berlin Heidelberg, 1999.
- [425] Y. Ji and J. Hu, “Control of quantum correlation of high dimensional system in squeezed vacuum reservoir,” *Optik*, vol. 208, p. 164088, 2020.
- [426] M. Tchoffo, A. T. Tsokeng, O. M. Tiokang, P. N. Nganyo, and L. C. Fai, “Frozen entanglement and quantum correlations of one-parameter qubit-qutrit states under classical noise effects,” *Phys. Lett. A*, vol. 383, no. 16, pp. 1856 – 1864, 2019.
- [427] H.-P. Breuer, “Foundations and measures of quantum non-markovianity,” *J. Phys. B*, vol. 45, no. 15, p. 154001, 2012.
- [428] S. Wißmann, H.-P. Breuer, and B. Vacchini, “Generalized trace-distance measure connecting quantum and classical non-markovianity,” *Phys. Rev. A*, vol. 92, p. 042108, Oct 2015.
- [429] M. Ozawa, “Entanglement measures and the hilbertschmidt distance,” *Physics Letters A*, vol. 268, no. 3, pp. 158–160, 2000.
- [430] S. Luo and Q. Zhang, “Informational distance on quantum-state space,” *Phys. Rev. A*, vol. 69, p. 032106, Mar 2004.
- [431] M. Gessner and A. Smerzi, “Statistical speed of quantum states: Generalized quantum fisher information and schatten speed,” *Phys. Rev. A*, vol. 97, p. 022109, Feb 2018.
- [432] M. B. Plenio, “Logarithmic negativity: A full entanglement monotone that is not convex,” *Phys. Rev. Lett.*, vol. 95, no. 9, p. 090503, 2005.
- [433] M. Nakahara, *Quantum computing: from linear algebra to physical realizations*. CRC press, 2008.
- [434] G. Jaeger, *Quantum information*. Springer, 2007.

- [435] M. M. Wilde, *Quantum Information Theory*. Cambridge University Press, 2013.
- [436] S. Luo, “Using measurement-induced disturbance to characterize correlations as classical or quantum,” *Phys. Rev. A*, vol. 77, no. 2, p. 022301, 2008.
- [437] Q.-J. Tong, J.-H. An, H.-G. Luo, and C. H. Oh, “Mechanism of entanglement preservation,” *Phys. Rev. A*, vol. 81, p. 052330, May 2010.
- [438] B. Bellomo, R. Lo Franco, and G. Compagno, “Non-markovian effects on the dynamics of entanglement,” *Phys. Rev. Lett.*, vol. 99, p. 160502, Oct 2007.
- [439] F. Zi-Long, T. Jing, and Z. Hao-Sheng, “Entanglement and non-markovianity of a spin-s system in a dephasing environment,” *Chinese Phys. B*, vol. 23, no. 6, p. 060303, 2014.
- [440] Y. Ji, Q. Ke, and J. Hu, “Quantum correlation of high dimensional system in a dephasing environment,” *Physica E: Low-dimensional Systems and Nanostructures*, vol. 99, pp. 139–144, 2018.
- [441] L. Mazzola, J. Piilo, and S. Maniscalco, “Sudden transition between classical and quantum decoherence,” *Phys. Rev. Lett.*, vol. 104, p. 200401, May 2010.
- [442] T. R. Bromley, M. Cianciaruso, and G. Adesso, “Frozen quantum coherence,” *Phys. Rev. Lett.*, vol. 114, p. 210401, May 2015.
- [443] B. Aaronson, R. Lo Franco, and G. Adesso, “Comparative investigation of the freezing phenomena for quantum correlations under nondissipative decoherence,” *Phys. Rev. A*, vol. 88, p. 012120, Jul 2013.
- [444] B. Aaronson, R. L. Franco, G. Compagno, and G. Adesso, “Hierarchy and dynamics of trace distance correlations,” *New J. Phys.*, vol. 15, p. 093022, sep 2013.
- [445] M. Cianciaruso, T. R. Bromley, W. Roga, R. Lo Franco, and G. Adesso, “Universal freezing of quantum correlations within the geometric approach,” *Sci. Rep.*, vol. 5, p. 10177, 2015.
- [446] P. Haikka, T. H. Johnson, and S. Maniscalco, “Non-markovianity of local dephasing channels and time-invariant discord,” *Phys. Rev. A*, vol. 87, p. 010103, Jan 2013.
- [447] R. Lo Franco, “Nonlocality threshold for entanglement under general dephasing evolutions: a case study,” *Quant. Inf. Proc.*, vol. 15, pp. 2393–2404, 2016.
- [448] G. Karpat and Z. Gedik, “Correlation dynamics of qubitqutrit systems in a classical dephasing environment,” *Phys. Lett. A*, vol. 375, no. 47, pp. 4166 – 4171, 2011.
- [449] Y. M. Galperin, B. L. Altshuler, J. Bergli, and D. V. Shantsev, “Non-gaussian low-frequency noise as a source of qubit decoherence,” *Phys. Rev. Lett.*, vol. 96, p. 097009, Mar 2006.
- [450] G. Burkard, “Non-markovian qubit dynamics in the presence of  $1/f$  noise,” *Phys. Rev. B*, vol. 79, p. 125317, Mar 2009.
- [451] C. Benedetti, F. Buscemi, P. Bordone, and M. G. A. Paris, “Dynamics of quantum correlations in colored-noise environments,” *Phys. Rev. A*, vol. 87, p. 052328, May 2013.
- [452] L. Faoro and L. B. Ioffe, “Microscopic origin of low-frequency flux noise in josephson circuits,” *Phys. Rev. Lett.*, vol. 100, p. 227005, Jun 2008.
- [453] F. Yoshihara, K. Harrabi, A. O. Niskanen, Y. Nakamura, and J. S. Tsai, “Decoherence of flux qubits due to  $1/f$  flux noise,” *Phys. Rev. Lett.*, vol. 97, p. 167001, Oct 2006.
- [454] A. T. Tsokeng, M. Tchoffo, and L. C. Fai, “Quantum correlations and decoherence dynamics for a qutrit-qutrit system under random telegraph noise,” *Quant. Inf. Proc.*, vol. 16, p. 191, Aug. 2017.
- [455] M. Tchoffo, A. T. Tsokeng, O. M. Tiokang, P. N. Nganyo, and L. C. Fai, “Frozen entanglement and quantum correlations of one-parameter qubit- qutrit states under classical noise effects,” *Phys. Lett. A*, vol. 383, no. 16, pp. 1856–1864, 2019.
- [456] L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway, “Sudden death and sudden birth of entanglement in common structured reservoirs,” *Phys. Rev. A*, vol. 79, p. 042302, Apr 2009.
- [457] F. Benatti, R. Floreanini, and M. Piani, “Environment induced entanglement in markovian dissipative dynamics,” *Phys. Rev. Lett.*, vol. 91, p. 070402, Aug 2003.

- [458] D. Braun, “Creation of entanglement by interaction with a common heat bath,” *Phys. Rev. Lett.*, vol. 89, p. 277901, Dec 2002.
- [459] R. Lo Franco, B. Bellomo, S. Maniscalco, and G. Compagno, “Dynamics of quantum correlations in two-qubit systems within non-Markovian environments,” *Int. J. Mod. Phys. B*, vol. 27, no. 01n03, p. 1345053, 2013.
- [460] R. Lo Franco and G. Compagno, “Overview on the phenomenon of two-qubit entanglement revivals in classical environments,” in *Lectures on general quantum correlations and their applications*, pp. 367–391, Springer, 2017.
- [461] P. M. Poggi, S. Campbell, and S. Deffner, “Diverging quantum speed limits: A herald of classicality,” *PRX Quantum*, vol. 2, p. 040349, Dec 2021.
- [462] K. Goswami, C. Giarmatzi, C. Monterola, S. Shrapnel, J. Romero, and F. Costa, “Experimental characterization of a non-markovian quantum process,” *Phys. Rev. A*, vol. 104, p. 022432, Aug 2021.
- [463] J.-Y. Sun, K. Xu, and Z.-D. Li, “Quantum speedup dynamics process in multiqubit-interacting system,” *Phys. Rev. Res.*, vol. 6, p. 013317, 2024.
- [464] A. Rajabalinia, M. Khazaei Shadfar, F. Nosrati, A. Mortezaipoor, R. Morandotti, and R. Lo Franco, “Quantumness and speedup limit of a qubit under transition-frequency modulation,” *Phys. Rev. A*, vol. 106, p. 062431, 2022.
- [465] A. Motavallibashi, H. Mohammadi, and A. Akhound, “Non-markovianity as a resource for quantum correlation teleportation,” *J. Opt. Soc. Am. B*, vol. 38, pp. 1200–1204, 2021.
- [466] C. Gaikwad, D. Kowsari, C. Brame, X. Song, H. Zhang, M. Esposito, A. Ranadive, G. Cappelli, N. Roch, E. M. Levenson-Falk, and K. W. Murch, “Entanglement assisted probe of the non-markovian to markovian transition in open quantum system dynamics,” *Phys. Rev. Lett.*, vol. 132, p. 200401, 2024.
- [467] R. Gupta, S. Gupta, S. Mal, and A. Sen(De), “Constructive feedback of non-Markovianity on resources in random quantum states,” *Phys. Rev. A*, vol. 105, p. 012424, 2022.
- [468] B. Bylicka, M. Tukiainen, D. Chruściński, J. Piilo, and S. Maniscalco, “Thermodynamic power of non-markovianity,” *Sci. Rep.*, vol. 6, p. 27989, 2016.
- [469] Bogna, D. Chruściński, and S. Maniscalco, “Non-Markovianity and reservoir memory of quantum channels: a quantum information theory perspective,” *Scientific Reports*, vol. 4, p. 5720, Jul 2014.
- [470] M.-L. Hu and H. Fan, “Quantum coherence of multiqubit states in correlated noisy channels,” *arXiv preprint arXiv:1812.04385*, 2018.
- [471] M. Hu and W. Zhou, “Enhancing two-qubit quantum coherence in a correlated dephasing channel,” *Laser Phys. Lett.*, vol. 16, p. 045201, feb 2019.
- [472] M. Rameshkumar, N. Ananth, R. Muthuganesan, and N. Sivanantham, “Dynamics of quantum correlations under correlated noisy channels,” *Opt Quant Electron*, vol. 55, no. 8, p. 732, 2023.
- [473] C. Macchiavello and G. M. Palma, “Entanglement-enhanced information transmission over a quantum channel with correlated noise,” *Phys. Rev. A*, vol. 65, p. 050301, Apr 2002.
- [474] Z.-Y. Peng, F.-L. Wu, J. Li, H.-N. Xue, S.-Y. Liu, and Z.-Y. Wang, “Analytical method of multiqubit entanglement robustness in correlated quantum channels,” *Phys. Rev. A*, vol. 107, p. 022405, Feb 2023.
- [475] Z.-Y. Wang and Z.-Y. Qin, “Quantum teleportation, entanglement, and Bell nonlocality in correlated noisy channels,” *Laser Phys*, vol. 30, p. 055201, apr 2020.
- [476] M.-L. Hu and H.-F. Wang, “Protecting quantum Fisher information in correlated quantum channels,” *Annalen der Physik*, vol. 532, no. 1, p. 1900378, 2020.
- [477] G. Karpat, “Entropic uncertainty relation under correlated dephasing channels,” *Canadian Journal of Physics*, vol. 96, no. 7, pp. 700–704, 2018.
- [478] N. P. Kumar, S. Banerjee, R. Srikanth, V. Jagadish, and F. Petruccione, “Non-Markovian evolution: a quantum walk perspective,” *Open Systems & Information Dynamics*, vol. 25, no. 03, p. 1850014, 2018.
- [479] S.-Q. Tang, “Average steered coherence in correlated amplitude damping channel,” *Results in Physics*, vol. 42, p. 105998, 2022.

- [480] S. M. Hosseiny, H. Rangani Jahromi, and M. Amniat-Talab, “Monitoring variations of refractive index via hilbert–schmidt speed and applying this phenomenon to improve quantum metrology,” *J. Phys. B: At. Mol. Opt. Phys.*, vol. 56, no. 17, p. 175402, 2023.
- [481] S. M. Hosseiny, H. R. Jahromi, B. Farajollahi, and M. Amniat-Talab, “Technique for studying the coalescence of eigenstates and eigenvalues in non-hermitian systems,” *Front. Phys.*, vol. 20, no. 1, p. 014201, 2025.
- [482] S. Daffer, K. Wódkiewicz, J. D. Cresser, and J. K. McIver, “Depolarizing channel as a completely positive map with memory,” *Phys. Rev. A*, vol. 70, p. 010304, Jul 2004.
- [483] S.-x. Wu and C.-s. Yu, “The Precision of Parameter Estimation for Dephasing Model Under Squeezed Reservoir,” *Int J Theor Phys*, vol. 56, pp. 1198–1207, Apr 2017.
- [484] Y. Yeo and A. Skeen, “Time-correlated quantum amplitude-damping channel,” *Phys. Rev. A*, vol. 67, p. 064301, Jun 2003.
- [485] Z. Huang and C. Zhang, “Protecting Quantum Correlation from Correlated Amplitude Damping Channel,” *Braz J Phys*, vol. 47, pp. 400–405, Aug 2017.