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Estimation and validation of contemporaneous PARMA models for streamflow simulation

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Abstract. Seasonal streamflow series generally exhibit periodicity in the autocovariance structure. Such periodicity can be represented by PARMA models, i.e., autoregressive moving average (ARMA) models with parameters that vary with the seasons. Statistical properties of low-order models such as the PARMA(2,2) model are examined. The periodic moment equations are derived; they can be used to compute the periodic covariance structure of a given model. Simulation of streamflow at several sites can be done using the contemporaneous PARMA model. The main problem in using such models is to determine the covariance matrices of innovations. Traditionally, this has been done by the method of maximum likelihood. However, this method generally leads to significant underestimation of the cross correlation of flows. A moment estimator is developed herein for the contemporaneous PARMA(2,2) model along with three approximate moment-based estimators for those cases where a feasible moment solution cannot be obtained. The applicability of the proposed methods is illustrated by fitting PARMA models to weekly flow data for two catchments in the Ottawa River basin.

Introduction

Time series models are frequently used in hydrology to generate synthetic streamflow sequences which serve as input to the analysis of complex water resources systems. Typical applications are the design of reservoirs, testing of reservoir policies, risk and reliability assessment, planning of hydropower production, and flood and drought analysis. The timescale is an important component of time series models. In some situations, only annual flows are of interest. This may be the case if the purpose of the analysis is to design a large reservoir with inflows that possess interannual persistence. In other cases, it may be desirable to use smaller timescales such as months or weeks, when designing reservoirs for within-the-year regulation, for instance.

When dealing with monthly or weekly flows, it may be necessary to use a model that has seasonally varying properties. If the seasonality of the flow data under consideration appears to be only in the mean and the variance, then such seasonality can be removed by simple seasonal standardization, and a stationary model can be applied. However, if the autocorrelation structure of observed data exhibits significant periodicity, then seasonal models that explicitly incorporate a periodic dependence structure must be used. In operational hydrology, two approaches have gained popularity for modeling seasonal streamflows. The first is a direct approach in which a model with periodic parameters is fitted directly to the seasonal flows. The periodic autoregressive moving average (PARMA) model is an example of this category [*Salas et al.*, 1980, 1982; *Vecchia et al.*, 1983]. The second approach is disaggregation, in which the seasonal flows are generated in two or more levels. For instance, annual flows are modeled and generated first and then disaggregated into seasonal flows based on a linear model [*Valencia and Shaake*, 1973; *Mejia and Rousselle*, 1976; *Lane*, 1979; *Hoshi and Burges*, 1979; *Santos and Salas*, 1983, 1992; *Stedinger and Vogel*, 1984; *Stedinger et al.*, 1985a, b; *Grygier and Stedinger*, 1988]. Comparisons of the two approaches have been made by *Curry and Bras* [1978], *Srikanthan and McMahon* [1982], *Stedinger et al.* [1985a], and others.

This paper deals with contemporaneous PARMA models, that is, simplified versions of the fully parameterized multivariate PARMA model [*Salas et al.*, 1980, 1985; *Haltiner and Salas*, 1988]. The major difficulties pertaining to the application of multivariate PARMA models are parameter estimation and model validation, issues that we shall address in the following.

Figure 1 shows the lag-1 to lag-4 week-to-week correlations of weekly flows from the northeast part of the Ottawa River basin. The figure has been constructed from a flow record of 29 years. Although part of the observed fluctuations can be attributed to sampling variation, it is clear that the lagged correlations vary within the hydrological year. While the lag-1 correlation is relatively high for the weekly timescale, there are distinct drops in higher order lagged correlations in the spring flood season. This characteristic can be readily explained. In Quebec the largest annual flood usually occurs in spring, around April–May, as a result of snowmelt, and it extends over a period of a few days. Therefore the observed weekly means in the spring season are based on flood and nonflood flows. In the flood season of a given year, the week in which the spring flood occurs will have a large flow compared to its historical mean, while the remaining weeks will have relatively small flows compared to their means. This effect leads to the negative correlations of weekly flows at higher lags in the flood season. If a realistic generation of extreme flows is desired,

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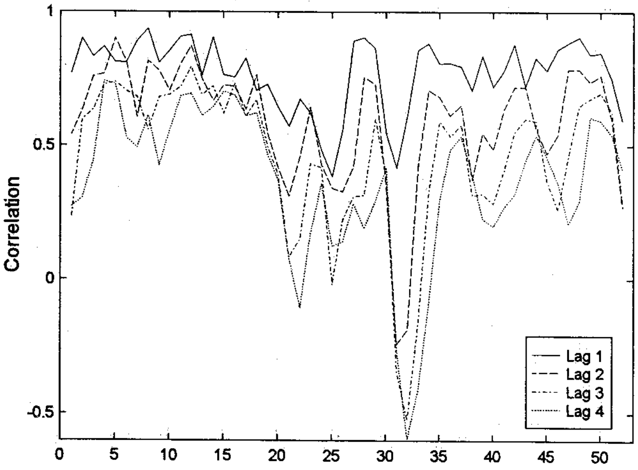

Figure 1. Week-to-week correlation of flows from the northeastern part of the Ottawa River basin.

then care must be exercised when modeling the correlations during the flood season. In fact, the negative correlations that appear in the first lagged week-to-week correlations should be reasonably well reproduced in order to generate realistic flood scenarios. This may imply severe requirements to the flow generation model. For example, certain low-order PARMA models, such as PARMA(1,0), PARMA(2,0), or PARMA(1,1), may not be able to reproduce satisfactorily the correlation structure in the flood season. Thus the use of higher order models, such as PARMA(2,1) and PARMA(2,2), may be necessary to obtain an adequate reproduction of week-to-week correlations beyond the first few lags.

When fitting a stochastic model, such as the ARMA model, to actual data, three stages of model development can be followed [*Box and Jenkins*, 1976; *Hipel et al.*, 1977; *McLeod et al.*, 1977]: identification, estimation, and verification. *Hipel and McLeod* [1994] provide details on how to identify the appropriate model order when the ARMA class of models is considered. Essentially, the procedure is based on examining the shapes of the autocorrelation and the partial autocorrelation functions which may suggest a particular model. However, when dealing with the class of PARMA models, the problem of model identification is more complex. Because of the periodicity and the sampling variability, it would be very cumbersome to deduce a certain model type from plots of periodic correlations and partial correlations. Some periods of the year may call for low-order models, while other periods call for higher order models. In principle, one could choose to let each period have its own model order [*Vecchia*, 1985], but the common approach is to use the same model order throughout the year. Because of these difficulties, a pragmatic approach to model identification is usually adopted in practice. It must reflect considerations to both the principle of parsimony and to satisfactory reproduction of the periodic autocorrelations during critical seasons such as the spring flood season (as referred to in the foregoing example). A pragmatic approach for PARMA model identification is by trial-and-error. One fixes the model order, estimates the parameters, and verifies to what extent selected historical statistics are reproduced by the model. The verification also involves a test for the whiteness of the residuals. One can examine various options and choose the simplest, acceptable model. Usually, one would like the model to preserve the periodic means, variances, covariances, and cross covariances. While traditionally, model characteristics are obtained by simulation, certain statistics may be obtained in closed form for PARMA models.

When modeling seasonal streamflow processes, one generally needs to transform the original flow data into normally distributed data. Hence one may consider statistical properties in the original data space and properties in the transformed data space. However, for model identification it may be sufficient to examine only statistical properties of the transformed data. Generally, one cannot expect a model to perform well in real space if it fails to perform well in the transformed space [*Stedinger*, 1981]. Once a model has been identified, it should be thoroughly validated by considering its statistical properties in the real space.

The first part of the paper deals with the determination of periodic properties, in particular, seasonal variances and autocorrelations, of single-site PARMA models. Analytical results based on the periodic moment (Yule-Walker) equations are presented, and various issues related to model building are discussed. In the second part of the paper, the problem of estimating the cross covariances of residuals in multisite contemporaneous PARMA models is addressed. In a study on contemporaneous ARMA(1,1) models, *Stedinger et al.* [1985a] found that maximum likelihood estimates of the cross covariances of innovations, **G**, lead to downward bias in the cross covariance of flows and recommended instead the use of the method of moments. However, in the case of contemporaneous PARMA models, estimation of **G** by the method of moments is not straightforward, and a solution is not guaranteed. In this paper the mathematical framework for moment estimation of the cross covariances of innovations for the contemporaneous PARMA(2,2) model is presented. Approximate moment estimators are suggested for the case where an exact solution does not exist.

Univariate PARMA Models

Let us consider a periodic dependent process, $X_{\nu,\tau}$, which has mean zero and variance $m_{0,\tau}$. Such a process may be represented by a univariate PARMA(p,q) model, which can be expressed as

$$\phi_\tau(B) X_{\nu,\tau} = \theta_\tau(B) \varepsilon_{\nu,\tau} \tag{1}$$

where $X_{\nu,\tau}$ is the flow, possibly transformed, in season τ of year ν, and $\phi_\tau(B)$ and $\theta_\tau(B)$ are periodic polynomials in the backward shift operator B:

$$\phi_\tau(B) = 1 - \phi_{1,\tau} B - \phi_{2,\tau} B^2 - \cdots - \phi_{p,\tau} B^p \tag{2a}$$

$$\theta_\tau(B) = 1 - \theta_{1,\tau} B - \theta_{2,\tau} B^2 - \cdots - \theta_{q,\tau} B^q \tag{2b}$$

For periodic processes, the backward shift operator is defined as $B^j X_{\nu,\tau} = X_{\nu,\tau-j}$. Throughout the paper, it is understood that if $\tau - j \leq 0$, then $X_{\nu,\tau-j} = X_{\nu-l\omega, l\omega+\tau-j}$ where $l = [(j-\tau)/\omega] + 1$ and ω is the number of seasons per year. The constants p and q are the number of autoregressive and moving average terms, respectively. The ϕ and the θ are periodic autoregressive and moving average coefficients, respectively, and $\varepsilon_{\nu,\tau}$ is a normally distributed noise term with mean zero and periodic variance g_τ. The above model may alternatively be written as

$$X_{\nu,\tau} = \sum_{i=1}^{p} \phi_{i,\tau} X_{\nu,\tau-i} + \varepsilon_{\nu,\tau} - \sum_{j=1}^{q} \theta_{j,\tau} \varepsilon_{\nu,\tau-j} \tag{3}$$

Popular models are the PARMA(1,0), PARMA(2,0), and PARMA(1,1), which in many cases provide a satisfactory description of the statistical properties of seasonal streamflows. However, when the timescale is short, such as in the case of weekly flows, the periodic autocorrelation structure may exhibit variations that cannot be adequately captured by these low-order models. In such cases, higher order models like PARMA(2,1), PARMA(1,2), and PARMA(2,2) should be considered. Moreover, these models may be able to accommodate dependence structures at timescales beyond weeks, e.g., months, seasons, and years [*Bartolini and Salas*, 1993]. In addition to the stochastic compatibility at various timescales, there are also conceptual (physically based) reasons for choosing these models, especially the PARMA(2,1) and PARMA(2,2) models [e.g., *Salas and Obeysekera*, 1992]. Therefore, in the following we adopt and elaborate on the PARMA(2,2) model and its submodels, i.e., we consider PARMA(p,q) models with $\{p,q\} \le 2$.

Estimation of PARMA Model Parameters

The moment equations (periodic Yule-Walker equations) for the PARMA(2,2) model read (see the appendix)

$$m_{0,\tau} = \phi_{1,\tau} m_{1,\tau} + \phi_{2,\tau} m_{2,\tau} + g_\tau - \theta_{1,\tau} g_{\tau-1}(\phi_{1,\tau} - \theta_{1,\tau}) - \theta_{2,\tau} g_{\tau-2}(\phi_{1,\tau}\phi_{1,\tau-1} - \phi_{1,\tau}\theta_{1,\tau-1} + \phi_{2,\tau} - \theta_{2,\tau}) \tag{4a}$$

$$m_{1,\tau} = \phi_{1,\tau} m_{0,\tau-1} + \phi_{2,\tau} m_{1,\tau-1} - \theta_{1,\tau} g_{\tau-1} - \theta_{2,\tau} g_{\tau-2}(\phi_{1,\tau-1} - \theta_{1,\tau-1}) \tag{4b}$$

$$m_{2,\tau} = \phi_{1,\tau} m_{1,\tau-1} + \phi_{2,\tau} m_{0,\tau-2} - \theta_{2,\tau} g_{\tau-2} \tag{4c}$$

$$m_{k,\tau} = \phi_{1,\tau} m_{k-1,\tau-1} + \phi_{2,\tau} m_{k-2,\tau-2} \qquad k > 2 \tag{4d}$$

where the periodic autocovariance function is defined as $m_{k,\tau} = E(X_{\nu,\tau} X_{\nu,\tau-k})$, since $X_{\nu,\tau}$ has mean zero. Note that in the case where $X_{\nu,\tau}$ has standard deviation one for every τ (seasonally standardized process), the periodic autocovariance function is equal to the periodic autocorrelation function $\rho_{k,\tau}$. The above equations are also valid for any model of lower order than PARMA(2,2) by setting appropriate parameters equal to zero. For instance, the periodic moment equations corresponding to a PARMA(2,1) model are obtained by setting $\theta_{2,\tau}$ equal to zero. In the case of a PARMA(2,2) model, there are 5ω parameters to be estimated, namely, 2ω autoregressive parameters, 2ω moving average parameters, and ω noise variances. The parameters of PARMA models may be estimated by either the method of moments or by the method of least squares, which is approximately equivalent to the method of maximum likelihood. Moment estimators of PARMA model parameters may be obtained from the above equations. However, they may not be feasible solutions, since there is no guarantee that moment-based estimates of the noise variances are positive. The autoregressive parameters can generally be obtained without much difficulty; the troublesome components are the moving average parameters and the noise variances. For a discussion of moment estimators of low-order PARMA models, see *Salas et al.* [1982] and *Bartolini and Salas* [1993].

Least squares (LS) estimators are based on the minimization of the sum of squared residuals (prediction errors). For models involving moving average components, the LS method seems to be the most straightforward approach for parameter estimation, although in the case of ω large (e.g., weekly data) and for high-order PARMA models, the search for a minimum may require considerable computer time. Moreover, the parameter estimates may be unstable, that is, sensitive to starting values and stop criteria used in the optimization program, mainly due to the flatness of the objective function around the optimum. The instability of the solution naturally raises the concern whether the properties of the corresponding model (periodic moments and autocorrelations) also are unstable.

Evaluation of the Model Covariance Structure

In addition to defining the method-of-moment estimators of PARMA parameters, the periodic moment equations (4a)–(4d) serve for calculating the periodic variance and periodic autocorrelation structure of a given model. Combining (4a) and (4c), and rearranging (4b) lead to the following equations:

$$-\phi_{2,\tau}^2 m_{0,\tau-2} + m_{0,\tau} - \phi_{1,\tau}\phi_{2,\tau} m_{1,\tau-1} - \phi_{1,\tau} m_{1,\tau} = g_\tau - \theta_{1,\tau} g_{\tau-1}(\phi_{1,\tau} - \theta_{1,\tau}) - \theta_{2,\tau} g_{\tau-2}(\phi_{1,\tau}\phi_{1,\tau-1} - \phi_{1,\tau}\theta_{1,\tau-1} + 2\phi_{2,\tau} - \theta_{2,\tau}) \tag{5a}$$

$$-\hat{\phi}_{1,\tau} m_{0,\tau-1} - \hat{\phi}_{2,\tau} m_{1,\tau-1} + m_{1,\tau} = -\hat{\theta}_{1,\tau}\hat{g}_{\tau-1} - \hat{\theta}_{2,\tau}\hat{g}_{\tau-2}(\hat{\phi}_{1,\tau-1} - \hat{\theta}_{1,\tau-1}) \tag{5b}$$

which, for known parameter estimates, constitute a linear system of 2ω equations with 2ω unknowns, namely, $m_{0,\tau}$ and $m_{1,\tau}$ for $\tau = 1, 2, \cdots, \omega$. This system is readily solved, and the complete periodic autocovariance function can then be computed from (4c) and (4d). It constitutes a means to quickly test and compare different models. For example, one may wish to compare the PARMA(1,1), PARMA(2,1), and PARMA(2,2) models for modeling weekly data at a given location. The first step would be to obtain parameter estimates for each model considered. Then the periodic variances and autocovariances could be determined from (5a) and (5b) and compared with historical properties. Figures 2 and 3 show such results for two catchments in the Ottawa River basin. The various PARMA models were estimated by the method of least squares.

Multivariate PARMA Models

Frequently, one is interested in generating concurrent flows at several sites. Multivariate periodic autoregressive (PAR) models and multivariate disaggregation models have been suggested for this kind of problem [*Salas et al.*, 1980; *Loucks et al.*, 1981; *Bras and Rodríguez-Iturbe*, 1985]. Multivariate PARMA models have also been considered for multisite streamflow generation [*Salas et al.*, 1980], but relatively little experience with this class of models is available today. In the following, various issues related to parameter estimation and testing of multivariate PARMA models are discussed.

The univariate PARMA model in (3) can be generalized to an m site model as

$$\mathbf{X}_{\nu,\tau} = \sum_{i=1}^{p} \mathbf{\Phi}_{i,\tau} \mathbf{X}_{\nu,\tau-i} + \boldsymbol{\varepsilon}_{\nu,\tau} - \sum_{j=1}^{q} \mathbf{\Theta}_{j,\tau} \boldsymbol{\varepsilon}_{\nu,\tau-j} \tag{6}$$

where $\mathbf{X}_{\nu,\tau}$ is a $(m \times 1)$ vector of flows, $\mathbf{\Phi}_{i,\tau}$ and $\mathbf{\Theta}_{j,\tau}$ are $(m \times m)$ matrices of autoregressive and moving average pa-

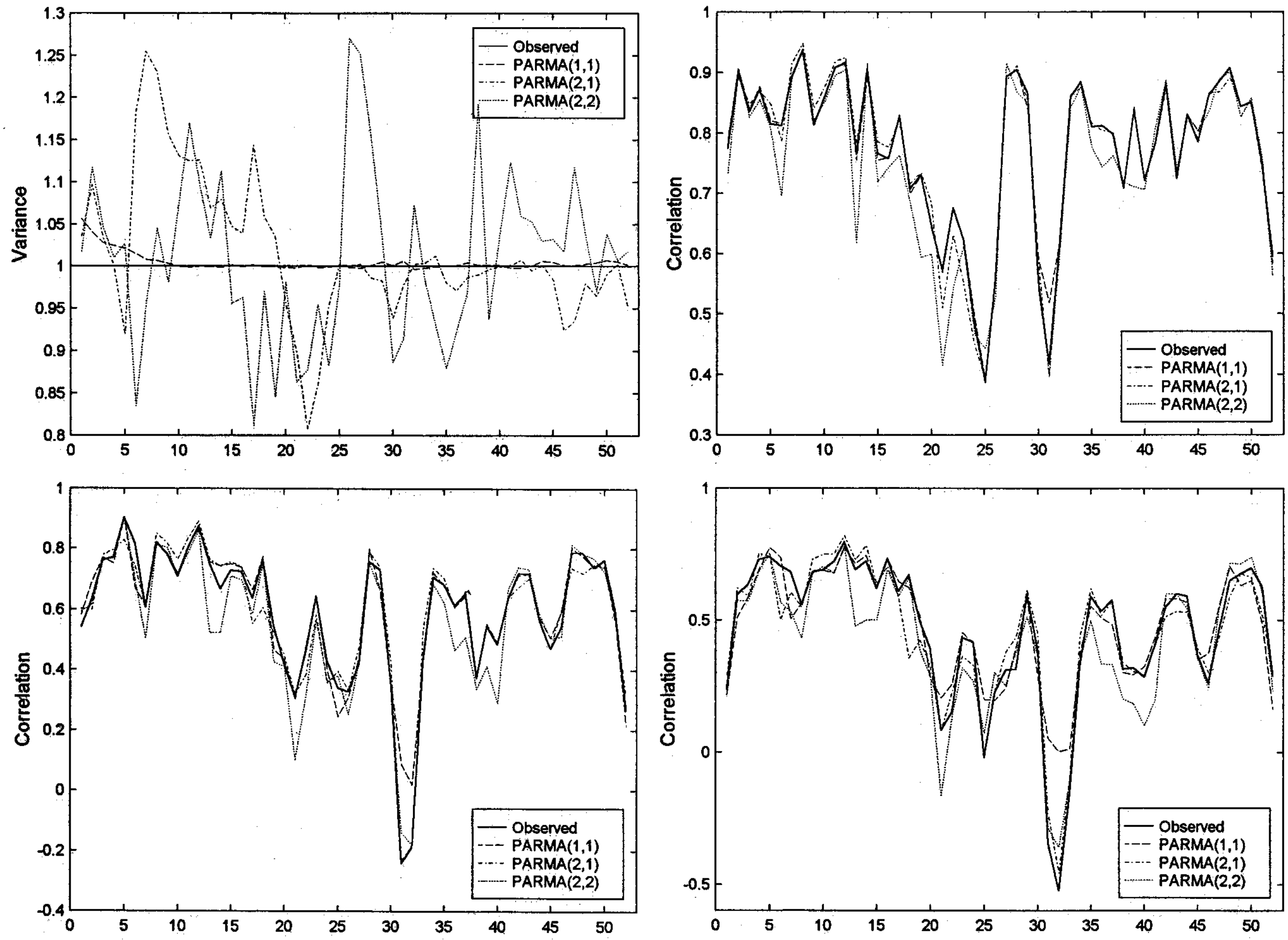

Figure 2. Observed moments and moments of PARMA models of order (1,1), (2,1), and (2,2) for the NE region of the Ottawa River basin. (a) Variance and periodic correlations of (b) lag 1, (c) lag 2, and (d) lag 3.

rameters, respectively, and $\boldsymbol{\varepsilon}_{\nu,\tau}$ is a $(m \times 1)$ vector of innovations, independent in time, but correlated in space with covariance matrix denoted by $\mathbf{G}_\tau$. As before, ν denotes the year and τ denotes the period. Variables and parameters in bold indicate vectors or matrices. In its most general form, the model parameters are considered to be full matrices. However, except for very simple cases, the estimation problem becomes overwhelming if full matrices are considered. A substantial simplification is obtained if the two parameter matrices, $\boldsymbol{\Phi}_{i,\tau}$ and $\boldsymbol{\Theta}_{j,\tau}$, are considered to be diagonal. This implies decomposing the problem into that of calibrating m univariate models, followed by the determination of the cross-covariance matrix of residuals $\mathbf{G}_\tau$. This latter approach, suggested by *Salas et al.* [1980], is called contemporaneous modeling because innovations at different sites are assumed to be cross correlated only at lag zero. While several papers have dealt with contemporaneous models in the stationary case [e.g., *Matalas*, 1967; *Pegram and James*, 1972; *Camacho et al.*, 1985, 1987; *Salas et al.*, 1985; *Stedinger et al.*, 1985a], only a few have addressed contemporaneous PARMA models [*Haltiner and Salas*, 1988; *Bartolini et al.*, 1988]. In the next section, we consider the problem of estimating $\mathbf{G}_\tau$ in the case of the contemporaneous PARMA(2,2) model, that is, we assume univariate PARMA(2,2) models at each of the m sites. It is understood that the derivations and procedures also apply to any model of order smaller than (2,2).

Estimation of $\mathbf{G}_\tau$

Generally, two methods can be used to estimate the cross-covariance matrices of the innovations: the method of maximum likelihood (ML) and the method of moments (MOM). In the ML method the estimated autoregressive and moving average parameters of the single-site models, i.e., the $\boldsymbol{\Phi}_{i,\tau}$ and the $\boldsymbol{\Theta}_{j,\tau}$ matrices in (6), are first used to derive the series of residuals at each site. Since *Box and Jenkins'* [1976] back forecast method cannot be applied to PARMA models, the first q residuals are usually set to zero. This introduces a transient error in the first few residuals, and it may be preferable to disregard them in the subsequent computations. The concurrent series of residuals at each site are then used to estimate the matrices of residual cross covariances $\mathbf{G}_\tau$, $\tau = 1, 2, \cdots, \omega$. However, the ML estimator of $\mathbf{G}_\tau$ does not ensure that the corresponding model will produce (transformed) flows whose cross covariances are equal or even close to the observed cross covariances. On the other hand, the method of moments is designed to yield $\mathbf{G}_\tau$ matrices that will lead to exact reproduction of the observed cross covariance $\hat{\mathbf{M}}_{0,\tau}$. (In the following, it will be necessary to make a clear distinction between moments estimated from the data and moments associated with a given model. Hence sample moments are provided with a circumflex.)

Haltiner and Salas [1988] considered the moment estimator

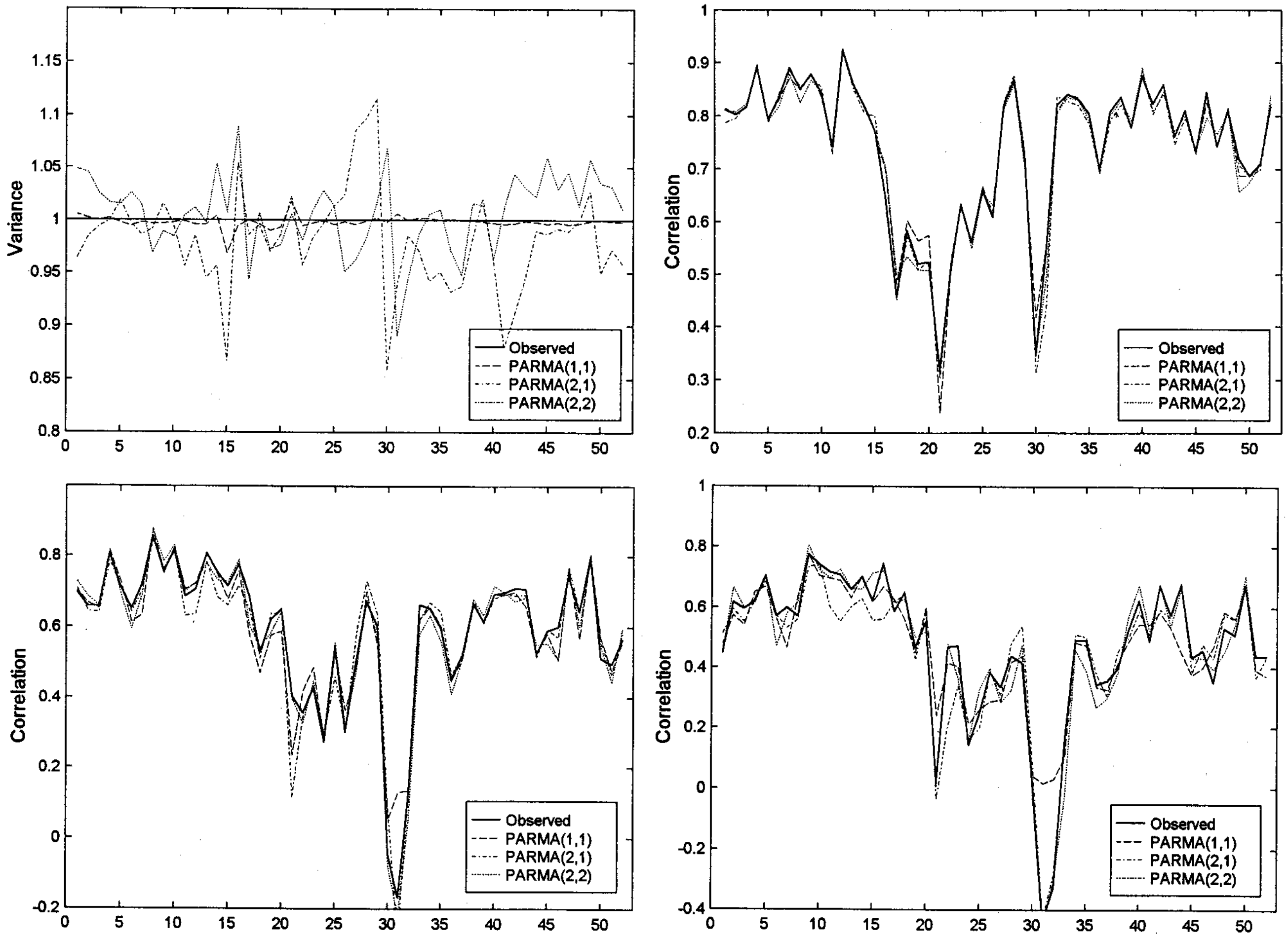

Figure 3. Observed moments and moments of PARMA models of order (1,1), (2,1), and (2,2) for the NW region of the Ottawa River basin. (a) Variance and periodic correlations of (b) lag 1, (c) lag 2, and (d) lag 3.

of $\mathbf{G}_\tau$ for the contemporaneous PARMA(1,1) model. Their estimation of $\mathbf{G}_\tau$ is briefly discussed below, before the estimation of the contemporaneous PARMA(2,2) model is addressed. By considering $p = q = 1$ in (6), multiplying each side by itself, and taking expectation, one obtains

$$\begin{aligned} E[\mathbf{X}_\tau\mathbf{X}_\tau^T] &= E[(\mathbf{\Phi}_\tau\mathbf{X}_{\tau-1} + \boldsymbol{\varepsilon}_\tau - \mathbf{\Theta}_\tau\boldsymbol{\varepsilon}_{\tau-1}) \\ &\quad \cdot (\mathbf{\Phi}_\tau\mathbf{X}_{\tau-1} + \boldsymbol{\varepsilon}_\tau - \mathbf{\Theta}_\tau\boldsymbol{\varepsilon}_{\tau-1})^T] \\ &= \mathbf{\Phi}_\tau E(\mathbf{X}_{\tau-1}\mathbf{X}_{\tau-1}^T)\mathbf{\Phi}_\tau^T + \mathbf{\Phi}_\tau E(\mathbf{X}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T) \\ &\quad - \mathbf{\Phi}_\tau E(\mathbf{X}_{\tau-1}\boldsymbol{\varepsilon}_{\tau-1}^T)\mathbf{\Theta}_\tau^T + E(\boldsymbol{\varepsilon}_\tau\mathbf{X}_{\tau-1}^T)\mathbf{\Phi}_\tau^T + E(\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_\tau^T) \\ &\quad - E(\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_{\tau-1}^T)\mathbf{\Theta}_\tau^T - \mathbf{\Theta}_\tau E(\boldsymbol{\varepsilon}_{\tau-1}\mathbf{X}_{\tau-1}^T)\mathbf{\Phi}_\tau^T \\ &\quad - \mathbf{\Theta}_\tau E(\boldsymbol{\varepsilon}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T) + \mathbf{\Theta}_\tau E(\boldsymbol{\varepsilon}_{\tau-1}\boldsymbol{\varepsilon}_{\tau-1}^T)\mathbf{\Theta}_\tau^T \end{aligned} \quad (7)$$

where the year index has been omitted for notational convenience. Note that the diagonal matrices $\mathbf{\Phi}_\tau$ and $\mathbf{\Theta}_\tau$ are assumed to have been already estimated. Since $E[\mathbf{X}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T] = 0$ and $E[\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_{\tau-1}^T] = 0$, the previous equation simplifies to

$$\begin{aligned} \mathbf{G}_\tau = \hat{\mathbf{M}}_{0,\tau} &- \mathbf{\Phi}_\tau\hat{\mathbf{M}}_{0,\tau-1}\mathbf{\Phi}_\tau^T + \mathbf{\Phi}_\tau\mathbf{G}_{\tau-1}\mathbf{\Theta}_\tau^T \\ &+ \mathbf{\Theta}_\tau\mathbf{G}_{\tau-1}\mathbf{\Phi}_\tau^T - \mathbf{\Theta}_\tau\mathbf{G}_{\tau-1}\mathbf{\Theta}_\tau^T \end{aligned} \quad (8)$$

in which only the $\mathbf{G}_\tau$ matrices are unknown. Note that $\mathbf{G}_\tau$ depends on $\mathbf{G}_{\tau-1}$, etc. *Haltiner and Salas* [1988] devised an iterative procedure for solving this equation. However, as will be shown below, (8) can be written as a linear system of equations for which a solution (not necessarily feasible) can be readily obtained. If a feasible solution exists, then the model will exactly reproduce the variance of the (transformed) flows at each site. However, there is a potential risk in combining LS estimators of the autoregressive and moving average parameters with moment estimators of the residual variances. A combined solution can in some cases lead to periodic autocorrelations that deviate more from the observed correlations than those corresponding to the classical LS estimation. It is therefore important to compute each site's periodic autocovariance structure using the approach described in the previous section. If the moment solution from (8) combined with LS estimates of the autoregressive and moving average parameters produces unsatisfactory results at some sites, one may choose to preserve only the cross correlations between sites exactly, i.e., the off-diagonal elements of $\hat{\mathbf{M}}_{0,\tau}$, and let the diagonal elements of $\mathbf{G}_\tau$ be equal to the estimates obtained from the LS estimation. This approach will be described below in more detail. First, we consider the problem of estimating $\mathbf{G}_\tau$ by the method of moments when the individual sites are represented by PARMA(2,2) models.

The derivation of (8) was fairly straightforward because $\mathbf{G}_\tau$ could be expressed, although implicitly, as a function of the property to be preserved, $\hat{\mathbf{M}}_{0,\tau}$. However, in the case of the contemporaneous PARMA(2,2) model, note that if the left- and right-hand sides of (6), with $p = q = 2$, are squared, then lagged cross correlations will appear. Contemporaneous PARMA models do not preserve lagged cross correlations explicitly. Preservation of the symmetric $\hat{\mathbf{M}}_{0,\tau}$ matrices imposes $\omega m(m + 1)/2$ constraints, which is exactly the number of degrees of freedom in the ω $\mathbf{G}_\tau$ matrices. To estimate $\mathbf{G}_\tau$, we shall make use of the periodic multivariate moment equations. First define $E(\mathbf{X}_\tau \mathbf{X}_{\tau-k}^T) = \mathbf{M}_{k,\tau}$. Note that for $k \neq 0$, $E(\mathbf{X}_t \mathbf{X}_{t-k}^T) \neq E(\mathbf{X}_{t-k} \mathbf{X}_t^T)$, i.e., $\mathbf{M}_{k,\tau}$ is nonsymmetric. The multivariate moment equations are (see appendix)

$$\mathbf{M}_{0,\tau} = E(\mathbf{X}_\tau \mathbf{X}_\tau^T)$$
$$= \mathbf{M}_{1,\tau}\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{2,\tau}\mathbf{\Phi}_{2,\tau}^T + \mathbf{G}_\tau - (\mathbf{\Phi}_{1,\tau} - \mathbf{\Theta}_{1,\tau})\mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T$$
$$- (\mathbf{\Phi}_{1,\tau}\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Phi}_{1,\tau}\mathbf{\Theta}_{1,\tau-1} + \mathbf{\Phi}_{2,\tau} - \mathbf{\Theta}_{2,\tau})\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T \quad (9a)$$

$$\mathbf{M}_{1,\tau}^T = E(\mathbf{X}_{\tau-1}\mathbf{X}_\tau^T)$$
$$= \mathbf{M}_{0,\tau-1}\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{1,\tau-1}\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T$$
$$- (\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Theta}_{1,\tau-1})\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T \quad (9b)$$

$$\mathbf{M}_{2,\tau}^T = E(\mathbf{X}_{\tau-2}\mathbf{X}_\tau^T)$$
$$= \mathbf{M}_{1,\tau-1}^T\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{0,\tau-2}\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T \quad (9c)$$

where the parameter $\mathbf{\Phi}_{i,\tau}$ and $\mathbf{\Theta}_{j,\tau}$ matrices are assumed diagonal. Inspection of the above equations shows that the relationships corresponding to the diagonal elements are simply the univariate cases given in (4a)–(4c). In fact, the variance terms of $\mathbf{G}_\tau$ can be estimated one at a time, and the covariances, i.e., the off-diagonal elements, can be estimated independently of the variances. Moreover, since the particular elements of $\mathbf{G}_\tau$ that correspond to site i and site j are unaffected by the parameters associated with other sites, one can determine the elements of $\mathbf{G}_\tau$ sequentially, considering all different pairs of sites. Hence in the following, the estimation procedure is developed for any two sites i and j.

First, (9c) is used to eliminate $\mathbf{M}_{2,\tau}$ from (9a). Then, after some manipulations, one can write the off-diagonal elements of $\mathbf{M}_{0,\tau}$ and $\mathbf{M}_{1,\tau}$ of (9a) and (9b), respectively, as

$$-[(\phi_{1,\tau}^{(i)}\phi_{1,\tau-1}^{(i)} - \phi_{1,\tau}^{(i)}\theta_{1,\tau-1}^{(i)} + \phi_{2,\tau}^{(i)} - \theta_{2,\tau}^{(i)})\theta_{2,\tau}^{(j)} + \theta_{2,\tau}^{(i)}\phi_{2,\tau}^{(j)}]G_{\tau-2}^{(ij)}$$
$$- (\phi_{1,\tau}^{(i)} - \theta_{1,\tau}^{(i)})\theta_{1,\tau}^{(j)}G_{\tau-1}^{(ij)} + G_\tau^{(ij)} + \phi_{1,\tau}^{(i)}\phi_{2,\tau}^{(j)}M_{1,\tau-1}^{(ij)} + \phi_{1,\tau}^{(j)}M_{1,\tau}^{(ij)}$$
$$= \hat{M}_{0,\tau}^{(ij)} - \phi_{2,\tau}^{(i)}\phi_{2,\tau}^{(j)}\hat{M}_{0,\tau-2}^{(ij)} \quad (10a)$$

$$\theta_{2,\tau}^{(j)}(\phi_{1,\tau-1}^{(i)} - \hat{\theta}_{1,\tau-1}^{(i)})G_{\tau-2}^{(ij)} + \theta_{1,\tau}^{(j)}G_{\tau-1}^{(ij)} - \phi_{2,\tau}^{(j)}M_{1,\tau-1}^{(ij)} + M_{1,\tau}^{(ji)}$$
$$= \phi_{1,\tau}^{(j)}\hat{M}_{0,\tau-1}^{(ij)} \quad (10b)$$

As noted above, the circumflex is used to distinguish sample moments from model moments, i.e., the terms that are to be estimated from the above equations. $G_\tau^{(ij)}$ is the covariance between the innovations of sites i and j at time τ, and $\hat{M}_{0,\tau}^{(ij)} = \hat{M}_{0,\tau}^{(ji)}$ is the lag zero cross covariance at time τ, estimated from the data. For standardized data, the cross covariances are equal to the cross correlations. $M_{1,\tau}^{(ij)} = E(X_\tau^{(i)}X_{\tau-1}^{(j)})$ is the lag-1 cross covariance between flows at sites i and j where, in general, $M_{1,\tau}^{(ij)} \neq M_{1,\tau}^{(ji)}$. The above expressions have been obtained from (9a) and (9b) by considering the entry (i, j). Two other sets of equations can be derived from (9a) and (9b) by considering the entry (j, i). Due to symmetry, these equations can be readily obtained by simply switching the site indices in (10a) and (10b); however, the element $G_\tau^{(ij)}$ remains the same, since $G_\tau^{(ij)} = G_\tau^{(ji)}$. The above considerations lead to a system of 4ω linear equations with 3ω unknowns, namely, $G_\tau^{(ij)}$, $M_{1,\tau}^{(ij)}$, and $M_{1,\tau}^{(ji)}$, for $\tau = 1, \cdots, \omega$. However, since $\hat{M}_{0,\tau}^{(ij)} = \hat{M}_{0,\tau}^{(ji)}$, one set of equations is superfluous and can be omitted. Eventually, it can be used to check that the matrices of cross covariances are indeed symmetric as they should be, or if errors have occurred during the computations. The system of linear equations can be readily solved for $G_\tau^{(ij)}$, $M_{1,\tau}^{(ij)}$, and $M_{1,\tau}^{(ji)}$.

If the variance at each site is not preserved exactly by the individual univariate models (this will typically be the case if one uses the LS estimator of noise variances), then $\hat{M}_{0,\tau}^{(ij)}$ on the right-hand side of (10a) and (10b) should be adjusted with a factor $(M_{0,\tau}^{(ii)}, M_{0,\tau}^{(jj)})^{1/2}/(\hat{M}_{0,\tau}^{(ii)}\hat{M}_{0,\tau}^{(jj)})^{1/2}$, where the terms in the nominator are the variances obtained from the individual models, and $\hat{M}_{0,\tau}^{(ii)}$ and $\hat{M}_{0,\tau}^{(jj)}$ in the denominator are observed variances. This adjustment is necessary in order to reproduce the correlation of flows. More specifically, we want $\rho_{0,\tau}^{(ij)} = M_{0,\tau}^{(ij)}/(M_{0,\tau}^{(ii)}M_{0,\tau}^{(jj)})^{1/2}$ to be equal to $\hat{\rho}_{0,\tau}^{(ij)} = \hat{M}_{0,\tau}^{(ij)}/(\hat{M}_{0,\tau}^{(ii)}\hat{M}_{0,\tau}^{(jj)})^{1/2}$; hence $M_{0,\tau}^{(ij)} = \hat{M}_{0,\tau}^{(ij)}(M_{0,\tau}^{(ii)}M_{0,\tau}^{(jj)})^{1/2}/(\hat{M}_{0,\tau}^{(ii)}\hat{M}_{0,\tau}^{(jj)})^{1/2}$.

It should be emphasized, however, that the complex structure of the individual site models may impose constraints on each series so that the exact preservation of the spatial cross correlation of flows may not be feasible. For example, some of the estimated correlations between innovations may fall outside the range $(-1; 1)$. The general requirement is that the $\mathbf{G}_\tau$ matrices be at least positive semidefinite. Hence, if a given $\mathbf{G}_\tau$ matrix is nonpositive definite, an adjustment must be made. Adjustments generally imply that the cross correlation of flows will no longer be preserved exactly. Although the problem of nonpositive definite matrices frequently arises when modeling multivariate data, the literature on the subject is rather limited, and there seems to be no standard solution to the problem. The disaggregation softwares SPIGOT [*Grygier and Stedinger*, 1990] and LAST [*Lane*, 1979] propose different procedures for dealing with nonpositive definite matrices. Other references are *Crosby and Maddock* [1970] and *Bras and Rodriguez-Iturbe* [1985]. Adjustment procedures often depend on the type of problem being considered. In the following, we present three techniques to overcome the problem of nonpositive definite matrices. The three options represent different degrees of complexity. Their performances are illustrated by an example in a subsequent section.

Method 1: MOM1. In this approach, initial moment estimates of $\mathbf{G}_\tau$, $\tau = 1, 2, \cdots, \omega$, are obtained by solving (10) for all possible combinations of sites. A $\mathbf{G}$ matrix which is nonpositive semidefinite (i.e., has one or more negative eigenvalues) is made positive semidefinite through the following steps.

1. Decompose the $\mathbf{G}$ matrix into eigenvectors and eigenvalues, $\mathbf{P}$ and $\mathbf{\Lambda}$, where the columns of $\mathbf{P}$ contain the eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues on the diagonal. Hence $\mathbf{G} = \mathbf{P\Lambda P^T}$.

2. Set all negative eigenvalues in $\mathbf{\Lambda}$ equal to zero. This defines a new matrix $\mathbf{\Lambda}^*$.

3. Compute the matrix $\mathbf{G}^* = \mathbf{P\Lambda^*P^T}$, which is positive semidefinite.

4. In order to preserve the variance terms of the original $\mathbf{G}$ matrix (i.e., the diagonal elements), perform the following computation:

$$\mathbf{G}_{\text{adj}} = \mathbf{U}\mathbf{G}^*\mathbf{U}$$

where $\mathbf{U} = \text{diag}\,(G^{(ii)}/G^{*(ii)})^{1/2}$.

The implication of this adjustment is best understood by considering the two-site case. If a (2×2) **G** matrix, estimated from (10), is negative definite, then $G^{(11)}G^{(22)} - (G^{(12)})^2 < 0$, implying that the correlation between the innovations of the two sites exceeds the feasible range $(-1, 1)$. The above adjustment procedure essentially sets the correlation equal to 1 or -1, depending on the sign of the original moment estimate of the cross correlation. For the two-site case, this represents the minimum adjustment needed to obtain a feasible **G** matrix without changing the diagonal elements.

Method 2: MOM2. The MOM1 method can be criticized for decoupling the initial moment estimation and the subsequent adjustments. When a matrix for a particular period is adjusted, there is no compensation for this in the estimation of the remaining matrices. It is obvious that the correction of matrix **G** at time τ will imply that the cross correlation of flows in that period will not be preserved exactly. However, since there is an interaction between consecutive **G** matrices, the correction in period τ is likely to also deteriorate the preservation of cross correlations in subsequent periods.

In order to preserve as many periodic cross correlations of flows as possible, one may, in the two-site case, proceed as follows. First, obtain an initial moment solution from (10a) and (10b) and find the first period, $\tau = k$, with nonpositive definite **G** matrix. Use the MOM1 technique to adjust that matrix, i.e., set $\hat{G}_k^{(12)} = (\hat{G}_k^{(11)}\hat{G}_k^{(22)})^{1/2}$. This implies that there will be at least one period, for example, period k, in which flows at the two sites will not have the specified cross correlation. Consider (10a) and (10b) with the additional constraint $\hat{G}_k^{(12)} = (\hat{G}_k^{(11)}\hat{G}_k^{(22)})^{1/2}$, that is, the cross correlation of innovations in period k is assumed known. The number of constraints now exceeds the number of degrees of freedom by 1. To obtain a feasible solution of the system of equations, one may slack the requirement that all flow cross correlations be exactly preserved, which anyway is impossible within the model framework. Specifically, we remove the constraint that $M_{0,k}^{(12)}$ equal its historical value. Hence we solve (10) again, but consider this time $G_k^{(12)}$ known and $M_{0,k}^{(12)}$ an unknown model property to be estimated. In practice, this is done by moving the terms which correspond to $G_k^{(12)}$ (including coefficients) to the right-hand side of (10a) and (10b), and those corresponding to $\hat{M}_{0,k}^{(12)}$, now representing a degree of freedom, to the left-hand side. This allows for a unique solution to be obtained. The procedure is repeated until all the $\hat{G}_\tau^{(12)}$ matrices are positive semidefinite. The generalization of the MOM2 estimation procedure to the case of more than two sites is straightforward.

Method 3: LS-MOM. The result of applying the MOM2 approach for correcting for nonpositive definite **G** matrices is that in the periods where an adjustment is made, the cross correlation of flows will be badly preserved, while in the remaining periods it will be preserved exactly. One may find it desirable to sacrifice the exact preservation of cross correlations in certain weeks to obtain a better overall agreement of historical and simulated statistics, for example, measured in terms of average mean square error. Such considerations naturally lead to suggesting a least squares moment (LS-MOM) estimator. In the two-site case, we define the LS-MOM estimator of the cross correlations of innovations to be the solution to the constrained optimization problem defined by

$$\min_{\mathbf{G}^{(12)}} f(\mathbf{G}^{(12)}) = \sum_{\tau=1}^{\omega} (M_{0,\tau}^{(12)} - \hat{M}_{0,\tau}^{(12)})^2 \tag{11}$$

subject to $|G_\tau^{(12)}| \leq (G_\tau^{(11)}G_\tau^{(22)})^{1/2}$, $\tau = 1, \cdots, \omega$, where $\mathbf{G}^{(12)} = (G_1^{(12)}, G_2^{(12)}, \cdots, G_\omega^{(12)})$, $\hat{M}_{0,\tau}^{(12)}$ is the historical cross correlation of flows, possibly corrected for unpreserved variances, and $M_{0,\tau}^{(12)}$ is the cross covariance of flows obtained from the model. The latter property can be obtained as described in the next section. Standard algorithms are available for solving this kind of optimization problem. In the general case of n sites, the objective function must be modified to include the average squared error for all $n(n-1)/2$ combinations of sites. The constraints should be formulated so as to ensure that no $\mathbf{G}_\tau$ matrix is nonpositive definite. It should be noted that if an exact moment solution exists, the LS-MOM estimator should, in principle, converge to the MOM solution.

Evaluation of the Model Cross-Covariance Structure

In order to verify to what degree the cross covariance of the transformed flows are reproduced given the estimates of $G_\tau^{(ij)}$, (9a) and (9b) (with $\mathbf{M}_{2,\tau}$ eliminated from (9a) by means of (9c)) may be reformulated as a system of 4ω linear equations in which $M_{0,\tau}^{(ij)}$, $M_{0,\tau}^{(ji)}$, $M_{1,\tau}^{(ij)}$, and $M_{1,\tau}^{(ji)}$ are the unknowns. The resulting equations are

$$\begin{aligned}
-\phi_{2,\tau}^{(i)}\phi_{2,\tau}^{(j)}M_{0,\tau-2}^{(ij)} &+ M_{0,\tau}^{(ij)} - \phi_{1,\tau}^{(i)}\phi_{2,\tau}^{(j)}M_{1,\tau-1}^{(ij)} - \phi_{1,\tau}^{(j)}M_{1,\tau}^{(ij)} \\
&= -\theta_{2,\tau}^{(i)}G_{\tau-2}^{(ij)}\phi_{2,\tau}^{(j)} + G_\tau^{(ij)} - (\phi_{1,\tau}^{(i)} - \theta_{1,\tau}^{(i)})G_{\tau-1}^{(ij)}\theta_{1,\tau}^{(j)} \\
&\quad - (\phi_{1,\tau}^{(i)}\phi_{1,\tau-1}^{(i)} - \phi_{1,\tau}^{(i)}\theta_{1,\tau-1}^{(i)} + \phi_{2,\tau}^{(i)} - \theta_{2,\tau}^{(i)})G_{\tau-2}^{(ij)}\theta_{2,\tau}^{(j)}
\end{aligned} \tag{12a}$$

$$\begin{aligned}
\phi_{1,\tau}^{(j)}M_{0,\tau-1}^{(ij)} + \phi_{2,\tau}^{(j)}M_{1,\tau-1}^{(ij)} - M_{1,\tau}^{(ji)} &= G_{\tau-1}^{(ij)}\theta_{1,\tau}^{(j)} + (\phi_{1,\tau-1}^{(i)} \\
&\quad - \theta_{1,\tau-1}^{(i)})G_{\tau-2}^{(ij)}\theta_{2,\tau}^{(j)}
\end{aligned} \tag{12b}$$

Two additional equations result by reversing site indices. Since $M_{0,\tau}^{(ij)} = M_{0,\tau}^{(ji)}$, only three equations are needed. However, the fourth equation can be used to check that $M_{0,\tau}^{(ij)}$ is indeed equal to $M_{0,\tau}^{(ji)}$. Otherwise, this could indicate a programming error or lack of precision in the computations.

With known $\mathbf{M}_{0,\tau}$ and $\mathbf{M}_{1,\tau}$, the matrix $\mathbf{M}_{2,\tau}$ is readily obtained from (9c). Furthermore, one can generalize (4d) to the multivariate case as

$$\mathbf{M}_{k,\tau} = E(\mathbf{X}_\tau\mathbf{X}_{\tau-k}^T) = \mathbf{\Phi}_{1,\tau}\mathbf{M}_{k-1,\tau-1} + \mathbf{\Phi}_{2,\tau}\mathbf{M}_{k-2,\tau-2} \qquad k > 2 \tag{13}$$

which can be used to determine cross correlations beyond lag 2. The extent to which a given model yields lagged cross correlations that resemble the observed correlations can be evaluated by comparing $\mathbf{M}_{k,\tau}$ with the corresponding observed matrices $\hat{\mathbf{M}}_{k,\tau}$.

Example

To illustrate the use of the various tools described above, we consider two catchments located in the Ottawa River basin northwest of Montreal. Twenty-two years of concurrent weekly flows in the two catchments have been analyzed. The data used in the example were obtained by aggregating observed daily flows to weekly flows and by transforming them so as to conform to the hypothesis of normality. The two watersheds are neighboring and of similar size, so one would expect a significant cross correlation of weekly flows. A specific objective

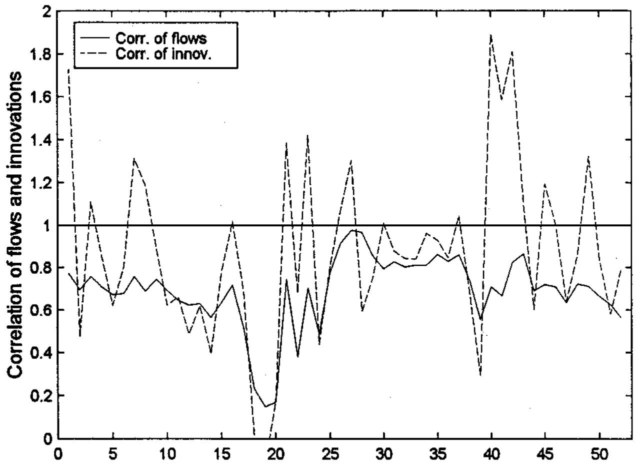

Figure 4. Correlation of innovations required to reproduce the cross correlation of transformed flows. The univariate PARMA models are of order (2,1) for the NE catchment and (2,2) for the NW catchment.

pursued in this illustration was to identify a multivariate model that reproduces the observed periodic autocorrelation structure and the periodic cross correlation of transformed flows in the two catchments satisfactorily. Less priority was given to model parsimony.

The first step in the analysis was to estimate the parameters of potential PARMA models (considering normally transformed data) for the two sites. Three univariate PARMA models, namely, PARMA(1,1), (2,1), and (2,2), were considered. For both sites and for all models considered, the method of moments, which by definition should reproduce the historical moments used in the estimation, led to negative estimates of the noise variances in one or more weeks, indicating that no exact moment solution exists. Hence the LS method was used in all cases. For each model considered, the system of equations given by (5a) and (5b) was applied to obtain the corresponding model variance $m_{0,\tau}$ and model periodic autocorrelation correlation function (ACF), defined by $\rho_{k,\tau} = m_{k,\tau}[m_{0,\tau}m_{0,\tau-k}]^{-1/2}$. The results are presented in Figures 2 and 3.

For the northeast (NE) region, the observed periodic variance (equal to 1 because of the seasonal standardization) is well reproduced by the PARMA(1,1) model, but it is not so well reproduced by the higher order models, which exhibit significant fluctuations around 1. As for the periodic ACF, the PARMA(2,2) shows some instability in many weeks throughout the year, while the PARMA(1,1) model fails to reproduce the important drop in correlations observed during the spring season. Hence, overall the PARMA(2,1) model seems to yield the most satisfactory reproduction of sample statistics. For the northwest (NW) region, the PARMA(1,1) yields the best reproduction of the variance, while the PARMA(2,2) seems slightly better than the PARMA(2,1) model. The periodic ACF is equally well preserved by the PARMA(2,1) and PARMA(2,2) models, whereas the PARMA(1,1) fails to reproduce the spring drop in lagged correlations. Overall, the PARMA(2,2) model seems best.

The method of moments was used to estimate the correlation of innovations. First, the periodic cross correlation of flows, $\hat{M}_{0,\tau}^{(12)}$, estimated from the two concurrent data sets was corrected by a factor $(M_{0,\tau}^{(11)}M_{0,\tau}^{(22)})^{1/2}$ (assuming standardized data), where $M_{0,\tau}^{(ii)}$ is the periodic model variance for site i (denoted $m_{0,\tau}$ in the univariate case). As previously mentioned, this adjustment is needed if the noise variances from the single-site LS estimation are maintained. The solution to (10) is shown in Figure 4 in terms of required periodic correlation of innovations. It is seen that for several weeks the required cross correlations fall outside the feasible range $(-1, 1)$, indicating that no exact moment estimator is available. In fact, 18 out of the 52 cross-correlation matrices turned out to be nonpositive definite.

The MOM1 adjustment technique was used to make all nonpositive definite matrices positive semidefinite. For the two-site case the adjustment essentially corresponds to setting the correlation between innovations equal to 1, that is, $\hat{G}_{\tau}^{(12)} = (\hat{G}_{\tau}^{(11)}\hat{G}_{\tau}^{(22)})^{1/2}$, where $\hat{G}_{\tau}^{(ii)}$ is the estimated variance of innovations for site i in period τ. The system of equations given by (12) was then solved with the corrected moment estimates as input. The result is shown in Figure 5. Owing to the adjustment of nonpositive definite matrices, the observed periodic cross correlation is not exactly reproduced by the model. However, as compared with the maximum likelihood estimates, the MOM1 solution yields cross correlations that are substantially closer to the observed correlations of flows. A closer examination of Figures 4 and 5 reveals that in periods following the weeks where an adjustment was made, the cross correlations of flows are not well reproduced. Changing the value of $\hat{G}^{(12)}$ in period τ introduces a transient error in the cross correlation of flows, which in most cases only slowly dies out. This effect is produced by the autocorrelation structure of the univariate models. For example, with two univariate PARMA(1,0) models there would be no difficulty in preserving the cross correlation of flows.

The application of the MOM2 adjustment procedure to the Ottawa River data led to the model cross correlations shown in Figure 6. In 32 out of 52 weeks, the observed cross correlations could be exactly reproduced. For the remaining weeks, where exact preservation was not possible, one may observe that the MOM2 estimated model generally yielded improved agreement with historical values as compared with the MOM1 estimated model.

Finally, the LS-MOM estimator was used to obtain estimates

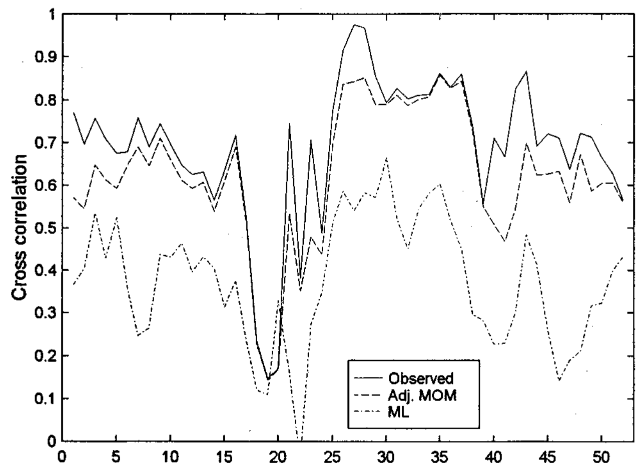

Figure 5. Cross correlation of flows corresponding to MOM1 estimation of the covariance matrices of innovations.

of $G_\tau^{(12)}$. Results are shown in Figure 7. It is seen that there is an overall good agreement between observed and model-produced cross correlations. Note that in the period of weeks 39–44, where both the MOM1 and the MOM2 estimators failed to preserve satisfactorily the cross correlations of flows, a significant improvement is obtained. If the preservation of cross correlations is particularly important in certain weeks, for instance, during the flood season, then one may consider a weighted LS-MOM estimator, which penalizes discrepancies between observed moments and model moments more in certain weeks than in others.

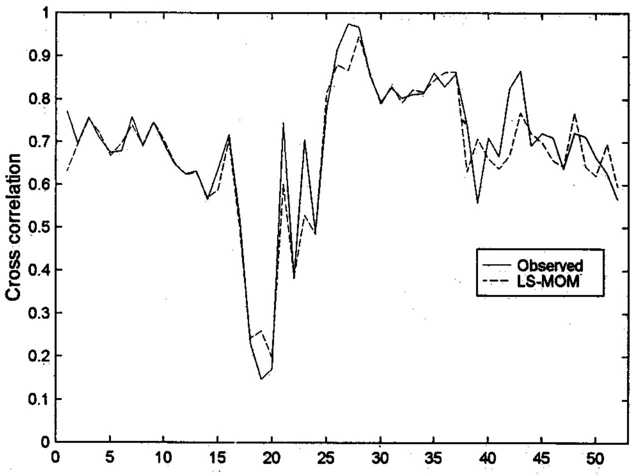

Figure 7. Cross correlation of flows corresponding to LS-MOM estimation of the covariance matrices of innovations.

Conclusions

This paper has focused on the ability of contemporaneous PARMA models to preserve historical statistics, in particular, periodic variance and autocorrelation, and periodic cross correlations. Practitioners are often concerned with an adequate preservation of observed statistics while also considering the parsimony and corresponding robustness of stochastic models. To obtain a satisfactory reproduction of sample statistics, it is frequently necessary to adopt high-order models. The least squares method permits us to estimate the parameters of any PARMA model. However, care must be exercised when fitting high-order models to series of small timescales, such as weekly flows. Parameter estimates may be unstable and lead to poor reproduction of some of the most important statistics, for example, the periodic variance. It is therefore imperative to carefully examine the statistical properties of the model. A set of analytical tools has been presented for computing the main periodic statistics of the PARMA(2,2) model and any model of lower order.

A critical element in the use of contemporaneous PARMA models is the estimation of the covariance matrices of innovations. The maximum likelihood estimator generally leads to a substantial underestimation of the observed cross correlations of flows. A moment estimator has been developed to improve the reproduction of cross correlations. However, due to restrictions imposed by the univariate models, an exact moment estimator may not always be obtainable. Three modifications of the moment estimator have been proposed for the cases where an exact solution is not feasible. In our two-site example, overall best results were obtained with the LS-MOM estimator, which minimizes the averaged (over periods) squared deviation between observed and model-produced cross correlations of flows within the feasible parameter space.

Finally, it should be noted that when there are many sites (more than four) and cross correlations are high (as is typically the case for small timescales), then it may be impossible to obtain a satisfactory reproduction of cross correlations with the multivariate PARMA(2,2) model, even with the approximate moment estimators of $\mathbf{G}_\tau$ developed in this study. If the multivariate PARMA(2,2) model is deemed unacceptable in terms of spatial correlations, one may consider simpler models such as the multivariate PARMA(1,0), PARMA(1,1), or disaggregation models, which generally permit a good representation of spatial correlations. This, however, may be at the expense of a less satisfactory representation of the temporal correlations.

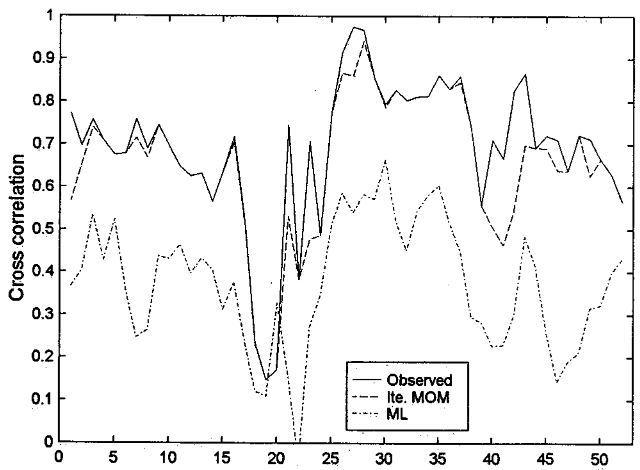

Figure 6. Cross correlation of flows corresponding to MOM2 estimation of the covariance matrices of innovations.

Appendix: Periodic Multivariate Moment Equations for the PARMA(2,2) Model

The periodic multivariate covariance equations can be derived as follows. By definition, we have

$$E(\mathbf{X}_\tau\mathbf{X}_\tau^T) = \mathbf{M}_{0,\tau}$$

$$E(\mathbf{X}_\tau\mathbf{X}_{\tau-1}^T) = \mathbf{M}_{1,\tau} \qquad E[\mathbf{X}_{\tau-1}\mathbf{X}_\tau^T] = \mathbf{M}_{1,\tau}^T$$

$$E(\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_\tau^T) = \mathbf{G}_\tau$$

$$E(\mathbf{X}_{\tau-i}\boldsymbol{\varepsilon}_\tau^T) = \mathbf{0} \qquad i > 0$$

The multivariate PARMA(2,2) model has the form

$$\mathbf{X}_\tau = \mathbf{\Phi}_{1,\tau}\mathbf{X}_{\tau-1} + \mathbf{\Phi}_{2,\tau}\mathbf{X}_{\tau-2} + \boldsymbol{\varepsilon}_\tau - \mathbf{\Theta}_{1,\tau}\boldsymbol{\varepsilon}_{\tau-1} - \mathbf{\Theta}_{2,\tau}\boldsymbol{\varepsilon}_{\tau-2} \qquad (A1)$$

Using the above definitions, it follows that

$$E(\mathbf{X}_\tau\boldsymbol{\varepsilon}_\tau^T) = E(\boldsymbol{\varepsilon}_\tau\mathbf{X}_\tau^T) = \mathbf{G}_\tau$$

$$E(\mathbf{X}_\tau\boldsymbol{\varepsilon}_{\tau-1}^T) = (\mathbf{\Phi}_{1,\tau} - \mathbf{\Theta}_{1,\tau})\mathbf{G}_{\tau-1}$$

$$E(\boldsymbol{\varepsilon}_{\tau-1}\mathbf{X}_\tau^T) = \mathbf{G}_{\tau-1}(\mathbf{\Phi}_{1,\tau}^T - \mathbf{\Theta}_{1,\tau}^T)$$

$$E(\mathbf{X}_\tau\boldsymbol{\varepsilon}_{\tau-2}^T) = (\mathbf{\Phi}_{1,\tau}\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Phi}_{1,\tau}\mathbf{\Theta}_{1,\tau-1} + \mathbf{\Phi}_{2,\tau} - \mathbf{\Theta}_{2,\tau})\mathbf{G}_{\tau-2}$$

$$E(\boldsymbol{\varepsilon}_{\tau-2}\mathbf{X}_\tau^T) = \mathbf{G}_{\tau-2}(\mathbf{\Phi}_{1,\tau-1}^T\mathbf{\Phi}_{1,\tau}^T - \mathbf{\Theta}_{1,\tau-1}^T\mathbf{\Phi}_{1,\tau}^T + \mathbf{\Phi}_{2,\tau}^T - \mathbf{\Theta}_{2,\tau}^T)$$

The covariance matrix of $\mathbf{X}_\tau$ and $\mathbf{X}_{\tau-i}$ is obtained by multiplying (A1) by $\mathbf{X}_{\tau-i}^T$ and taking expectation:

$$\begin{aligned}\mathbf{M}_{0,\tau} &= E(\mathbf{X}_\tau \mathbf{X}_\tau^T)\\ &= \mathbf{M}_{1,\tau}\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{2,\tau}\mathbf{\Phi}_{2,\tau}^T + \mathbf{G}_\tau\\ &\quad - (\mathbf{\Phi}_{1,\tau} - \mathbf{\Theta}_{1,\tau})\mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T - (\mathbf{\Phi}_{1,\tau}\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Phi}_{1,\tau}\mathbf{\Theta}_{1,\tau-1}\\ &\quad + \mathbf{\Phi}_{2,\tau} - \mathbf{\Theta}_{2,\tau})\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T \end{aligned} \quad (A2)$$

$$\begin{aligned}\mathbf{M}_{1,\tau}^T &= E(\mathbf{X}_{\tau-1}\mathbf{X}_\tau^T) = \mathbf{M}_{0,\tau-1}\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{1,\tau-1}\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T\\ &\quad - (\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Theta}_{1,\tau-1})\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T\end{aligned} \quad (A3)$$

$$\begin{aligned}\mathbf{M}_{2,\tau}^T &= E(\mathbf{X}_{\tau-2}\mathbf{X}_\tau^T)\\ &= \mathbf{M}_{1,\tau-1}^T\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{0,\tau-2}\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T\end{aligned} \quad (A4)$$

$$\mathbf{M}_{k,\tau}^T = E(\mathbf{X}_{\tau-k}\mathbf{X}_\tau^T) = \mathbf{M}_{k-1,\tau-1}^T\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{k-2,\tau-2}\mathbf{\Phi}_{2,\tau}^T \quad (A5)$$

$$k > 2$$

The univariate cases (4a)–(4d) correspond to the diagonal elements of the above matrices.

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