

1 **Index-flood based multivariate regional frequency analysis**

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## 1 **Abstract**

2           Because of their multivariate nature, several hydrological phenomena can be described by  
3 more than one correlated characteristics. These characteristics are generally not independent and  
4 should be jointly considered. Consequently, univariate regional frequency analysis (FA) cannot  
5 provide complete assessment of true probabilities of occurrence. The objective of the present  
6 paper is to propose a procedure for regional flood FA in a multivariate framework. In the present  
7 paper, the focus is on the estimation step of regional FA. The proposed procedure represents a  
8 multivariate version of the index-flood model and is based on copulas and a multivariate quantile  
9 version with a focus on the bivariate case. The model offers increased flexibility to designers by  
10 leading to several scenarios associated to the same risk. The univariate quantiles represent special  
11 cases corresponding to the extreme scenarios. A simulation study is carried out to evaluate the  
12 performance of the model in a bivariate framework. Simulation results show that bivariate FA  
13 provides the univariate quantiles with equivalent accuracy. Similarity is observed between results  
14 of the bivariate model and those of the univariate one in terms of the behaviour of the  
15 corresponding performance criteria. The procedure performs better when the regional  
16 homogeneity is high. Furthermore, the impacts of small variations in the record length at gauged  
17 sites and the region size on the performance of the proposed procedure are not significant.

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## 1 **1. Introduction and literature review**

2 Extreme events, such as floods, storms and droughts have serious economic,  
3 environmental and social consequences. It is hence of high importance to develop the appropriate  
4 models for the prediction of such events both at gauged and ungauged sites. Local and regional  
5 frequency analysis (FA) procedures are commonly used tools for the analysis of extreme  
6 hydrological events. The objective of regional frequency analysis (RFA) is to transfer  
7 information from gauged sites to an ungauged target site within a homogeneous region.

8 Generally, hydrological events are characterized by several correlated variables. For  
9 instance, floods are described through their volume, peak and duration (Ashkar, 1980; Yue et al.,  
10 1999; Ouarda et al., 2000; Yue, 2001; Shiau, 2003; De Michele et al., 2005; Zhang and Singh,  
11 2006; Chebana and Ouarda, 2008b). These studies have pointed out the importance of jointly  
12 considering all these variables. Depending on data sources and the number of variables that  
13 characterize the event, frequency analysis can be divided into four classes: univariate-local,  
14 univariate-regional, multivariate-local and multivariate-regional. The first two classes have been  
15 extensively studied, see e.g. Stedinger and Tasker (1986), Burn (1990), Hosking and Wallis  
16 (1993), Durrans and Tomic (1996), Nguyen and Pandey (1996), Alila (1999, 2000), Ouarda et al.  
17 (2001, 2006) and Chebana and Ouarda (2008a). Recently, increasing attention has been given to  
18 multivariate-local FA e.g., by Yue et al. (1999), Yue (2001), Shiau (2003), De Michele et al.  
19 (2005), Zhang and Singh (2006) and Chebana and Ouarda (2008b). However, much less attention  
20 is given to multivariate-regional FA. In this category, we find few references such as Ouarda et  
21 al. (2000) and Chebana and Ouarda (2007).

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1 Justifications for adopting the multivariate framework to treat extreme events were  
2 discussed in several references. In bivariate FA, Yue et al. (1999) concluded that single-variable  
3 hydrological FA can only provide limited assessment of extreme events. A better understanding  
4 of the probabilistic characteristics of such events requires the study of their joint distribution. It  
5 was also outlined in Shiau (2003) that multivariate FA requires considerably more data and more  
6 sophisticated mathematical analysis. Univariate FA can be useful when only one random variable  
7 is significant for design purposes or when the two random variables are less dependent. However,  
8 a separate analysis of random variables cannot reveal the significant relationship between them if  
9 the correlation is an important information in the design criteria. Therefore, it is of importance to  
10 jointly consider all the random variables that characterize the hydrological event.

11 Three main elements are treated in multivariate-local FA literature: (1) explaining the  
12 usefulness and importance of considering the multivariate framework, (2) modeling extreme  
13 events by fitting the appropriate copula and marginal distributions, and estimating the  
14 corresponding parameters, and (3) defining bivariate return periods. However, despite the  
15 importance of quantiles in FA, the literature on multivariate-local FA did not specifically address  
16 the estimation of multivariate quantiles. Recently Chebana and Ouarda (2008b) introduced the  
17 notion of multivariate quantile in hydrological FA.

18 Regional FA is generally composed by two main steps: regional delineation and extreme  
19 quantile estimation (see e.g., GREHYS, 1996a). In the multivariate context, the delineation step  
20 was treated in Chebana and Ouarda (2007) where multivariate discordancy and homogeneity  
21 statistical tests were proposed. In univariate-regional FA different quantile estimation methods  
22 were proposed in the literature, such as the index-flood method and regressive models (see  
23 GREHYS, 1996a,b). As a natural continuation of the study by Chebana and Ouarda (2007) and in  
24 order to present a complete multivariate-regional FA framework, an estimation procedure in the

1 multivariate context is presented in this paper. The present procedure is an extension of the  
2 index-flood model to the multivariate context.

3         The multivariate index-flood model is based on two main concepts: multivariate quantile  
4 curves and the notion of copulas. The univariate index-flood model aims to obtain an estimation  
5 of quantiles at ungauged sites using data (and hence quantiles) from sites within a specified  
6 region. The objective of FA is the quantile estimation which can be obtained through the  
7 cumulative distribution function or the density function. The multivariate quantile version  
8 adopted in this paper is a curve composed of combinations of the variables corresponding to the  
9 same risk. Copulas are employed in order to model the dependence between the variables  
10 describing the event.

11         The paper is organized as follows. In Section 2, we present some background elements  
12 required for the development of the methodology: the index-flood model and multivariate  
13 quantile curves. In Section 3, we present the multivariate-index flood model. In Section 4 a  
14 simulation study is carried out to evaluate the performance of the proposed model with an  
15 adaptation of the procedure to flood events. Results and discussions are reported in Sections 5  
16 and 6 respectively, whereas the conclusions are presented in the last section.

17

## 18 **2. Background**

19         The principal elements required for the development of the proposed estimation procedure  
20 are presented in this section; namely, the univariate index-flood model and bivariate quantile  
21 curves.

22

23

## 2.1. Univariate index-flood model

The index-flood model was first introduced by Dalrymple (1960). Similar models can also be used for other hydrological variables including droughts and storms (Pilon, 1990, Hosking and Wallis, 1997 and Hamza et al., 2001). In this model the region is assumed to be homogeneous. That is, all sites in the region have the same frequency distribution apart from a scale parameter that characterizes each site. Explicitly, for a region where data are available for  $N$  sites, the model gives the quantile  $Q_i(p)$  corresponding to the non-exceedence probability  $p$  at site  $i$  as:

$$Q_i(p) = \mu_i q(p), \quad i = 1, \dots, N \quad \text{and} \quad 0 < p < 1 \quad (1)$$

where  $\mu_i$  represents the index-flood and  $q(\cdot)$  is the regional growth curve.

The index-flood parameter  $\mu_i$  may be estimated, for instance, as the sample mean at site  $i$ . The growth curve  $q(\cdot)$  may be estimated using the standardized data of the whole region. Usually, we assume known the form of  $q(\cdot)$  through a regional distribution  $K(\cdot; \underline{\theta})$  except for some parameters  $\underline{\theta} = (\theta_1, \dots, \theta_s)$ . More details about the index-flood model can be found in Hosking and Wallis (1993), and for a more recent review the reader is referred to Bocchiola et al. (2003).

## 2.2. Multivariate quantiles

In the literature, several studies proposed to extend the well-known univariate quantile to higher dimensions. Serfling (2002) presented a review and a classification of some of these multivariate quantile versions. According to this classification, there are two major categories of multivariate quantiles: vector- and real-valued quantiles. In the vector-valued class, we find multivariate quantiles based on depth functions (Serfling, 2002); multivariate quantiles based on norm minimization defined by Abdous and Theodorecu (1992) and Chaudhuri (1996);

1 multivariate quantiles as inversions of mappings studied by Koltchinskii and Dudley (1996); and  
 2 data-based multivariate quantiles based on gradients developed by Hettmansperger et al. (1992).  
 3 The real-valued quantile class contains the generalized quantile processes introduced by Einmahl  
 4 and Mason (1992).

5 More recently, Belzunce et al. (2007) defined another bivariate vector-valued quantile  
 6 version. This version is not included in the review by Serfling (2002) and is focused on the  
 7 bivariate context. Let  $(X, Y)$  be an absolutely continuous random vector and  $p \in ]0, 1[$ . The  $p$ th  
 8 bivariate quantile set or bivariate quantile curve for the direction  $\varepsilon$  is defined as:

$$9 \quad Q_{X,Y}(p, \varepsilon) = \{(x, y) \in R^2 : F_{\varepsilon}(x, y) = p\} \quad (2)$$

10 where  $F_{\varepsilon}(x, y)$  is one of the four following probabilities:

$$11 \quad F_{\varepsilon_{++}}(x, y) = \Pr\{X \geq x, Y \geq y\}, F_{\varepsilon_{+-}}(x, y) = \Pr\{X \geq x, Y \leq y\}$$

$$11 \quad F_{\varepsilon_{--}}(x, y) = \Pr\{X \leq x, Y \leq y\}, F_{\varepsilon_{-+}}(x, y) = \Pr\{X \leq x, Y \geq y\}$$

12 which represent the probabilities of the events in the four quadrants of the plane.

13 In other words, the bivariate quantile (2) is a curve corresponding to any combination  
 14  $(x, y)$  that satisfies  $F_{\varepsilon}(x, y) = p$  (an infinity of combinations). This definition of the bivariate  
 15 quantile is simple, intuitive and does not require any symmetry assumption. Furthermore, the  
 16 bivariate distribution (copula and margins) appears in its evaluation. A multivariate quantile  
 17 curve can be obtained for the uniform margins and then transformed using the univariate quantile  
 18 function of each component. To this end, we introduce copulas as follows. A copula is a  
 19 description of the dependence structure between two or more random variables. For more details  
 20 on copula functions, the reader is referred for instance to Nelsen (2006) or to Chebana and  
 21 Ouarda (2007). Sklar's (1959) theorem is an important result which provides, for two random  
 22 variables  $X$  and  $Y$ , the relationship between their bivariate joint distribution  $F$ , the corresponding

1 copula  $C$  and marginal distributions  $F_X$  and  $F_Y$ . Sklar's result states that there exists a copula  $C$   
 2 such that:

$$3 \quad F(x, y) = C(F_X(x), F_Y(y)) \text{ for all real } x \text{ and } y \quad (3)$$

4 In addition, if  $F_X$  and  $F_Y$  are continuous, the copula  $C$  is unique. Hence, for the  
 5 event  $\{X \leq x, Y \leq y\}$ , using (2) and (3), the quantile curve can be expressed as follows:

$$6 \quad Q_{X,Y}(p) = \{(x, y) \in R^2 \text{ such that } x = F_X^{-1}(u), y = F_Y^{-1}(v); u, v \in [0, 1] : C(u, v) = p\} \quad (4)$$

7 Consequently, in the present paper the proposed bivariate index-flood model is based on (4). The  
 8 resolution of equation (4), using copula and margin expressions, leads to several solutions called  
 9 combinations. These combinations constitute the corresponding quantile curve.

10 The usual univariate quantiles are special cases of the bivariate quantile curve given in (2)  
 11 or (4). Indeed, Figure 1 illustrates that the univariate quantiles represent the extreme points of the  
 12 proper part of the bivariate quantile curve.

13 The following notations are employed throughout the paper and are illustrated in Figure 1:

14  $QC_p$  is the bivariate quantile curve associated to a risk  $p$  of the non-exceeding event on variables  
 15  $X$  and  $Y$  which corresponds to  $QC_{X,Y}(p, \epsilon - -)$  of equation (2) (we may denote it by  
 16  $QC_{X,Y}(p)$  if it is necessary to make the emphasis on the variables);

17  $Q_{x,y}(p)$  represents a point (a combination) of the curve  $QC_p$ ;

18  $QC_x(p)$  and  $QC_y(p)$  are the coordinates of the point  $Q_{x,y}(p)$ , that is

$$19 \quad Q_{x,y}(p) = (QC_x(p), QC_y(p)).$$



1 The univariate quantiles are denoted as  $QD_x(p)$  and  $QD_y(p)$  when directly evaluated and  
2  $QL_x(p)$  and  $QL_y(p)$  when deduced as extreme values from the bivariate quantile curve. A  
3 complete list of the notations used in the paper is presented at the end of the document.

4  
5 We close this section by summarizing some key facts that are related to the notion of  
6 bivariate quantile in local FA (for more details see Chebana and Ouarda, 2008b):

- 7 1. The quantile curves, for practical reasons, are composed of two parts: naïve part and proper  
8 part (central part). The naïve part is composed of two segments starting at the end of each  
9 extremity of the proper part. These segments are parallel to the axis. The points that define  
10 the extremities correspond to the maximum value for each of the variables  $x$  and  $y$  in the  
11 empirical version of the quantile curve or in the case where the marginal distributions are  
12 bounded (right or left according to the considered event). In the case of quantiles  
13 corresponding to the parent distribution when the margins are not bounded, there are two  
14 options to identify the extremities of the proper part. It is possible to take the extremities to  
15 be related to those of the empirical version. It is also possible to select the extremities to be  
16 as close as needed to the asymptotes. The first option is useful for the comparison of the  
17 empirical and true quantiles. For simplicity, Figure 1 presents the case where the margins  
18 are right-bounded.
- 19 2. The marginal quantiles correspond to the extreme scenarios of the proper part related to the  
20 event.
- 21 3. For a given sample, univariate estimation results should be used cautiously since the  
22 combination of the univariate quantile values of each variable does not correspond to the  
23 desired risk and hence may lead to wrong conclusions.

1 4. Some events, relating both variables, cannot be expressed in the univariate context.  
2 5. The number of bivariate quantile scenarios in the proper part decreases, when the risk  $p$   
3 increases, and hence the proper part of the quantile curve becomes shorter.  
4 These key statements are illustrated in Figure 1. In the remainder of the paper, if it is not  
5 specified, the quantile curve refers to the proper part of the full curve.

6 Generally, explicit or analytical expressions of bivariate quantiles are not available.  
7 Hence, bivariate quantiles are obtained numerically by resolving equations (2) or (4). Some  
8 difficulties may arise when solving these equations, especially for values of  $p$  that are very close  
9 to 1 and/or for complex distributions (margins and copula). The procedure employed to obtain  
10 multivariate quantile curves is parametric. That is the joint distribution  $F$  is shown to belong to a  
11 class of parametric distributions with unknown parameters to be estimated. The class of  
12 parametric distributions is identified using goodness-of-fit tests. The parametric estimation  
13 approach is commonly used in hydrologic FA. In the univariate context some nonparametric  
14 approaches have been employed in hydrologic FA (see e.g., Adamowski and Feluch, 1990;  
15 Ouarda et al., 2001) but these methods are of limited use for hydraulic design of major structures  
16 as indicated by Singh and Strupczewski (2002).

17

### 18 **3. Multivariate index-flood model**

19 The following procedure represents a complete multivariate version of regional FA. It  
20 includes the two main steps: delineation of a homogeneous region and estimation of the extreme  
21 event. The step dealing with the delineation of a region is treated in Chebana and Ouarda (2007)  
22 who proposed multivariate discordancy  $D$  and homogeneity  $H$  statistical tests based on  
23 multivariate  $L$ -moments. The statistic  $H$  is also employed as a heterogeneity measure for a given

1 region. The development of the estimation step is the object of the present paper. It consists in  
2 extending the index-flood model to the multivariate framework. For notation clarity and  
3 computation simplicity, the procedure is presented for the bivariate setting. Nevertheless, the  
4 procedure can be conceptually extended to higher dimensions. However, some theoretical and  
5 practical elements need to be developed such as copula modeling, parameter estimation and  
6 computational aspects. These elements are discussed at the end of the present section.

7         Given a set of  $N$  sites with record length  $n_i$  at site  $i, i=1, \dots, N$ , the problem is to estimate,  
8 at the target-site  $\ell$ , the quantile of interest corresponding to a given risk  $p, 0 < p < 1$  (or  
9 equivalently a return period  $T$ ). The data are of the form  $(x_{ij}, y_{ij})$  for  $j = 1, \dots, n_i$  and  $i = 1, \dots, N$   
10 where  $x$  and  $y$  represent realisations of the considered variables. Let  $qC_p$  be the regional growth  
11 curve which represents a quantile curve common to every site in the region. It can be seen as a  
12 “regional quantile curve” and can be obtained on the basis of the standardized data of the whole  
13 region.

14

15         The procedure is described as follows:

16 1. Identify the set of sites (region) to be used in the estimation as follows (Chebana and Ouarda,  
17 2007):

18 1.1. Apply the multivariate discordancy test  $D$  to identify discordant sites to be removed from  
19 the region,

20 1.2. Check the homogeneity of the remaining sites by applying the multivariate homogeneity  
21 test  $H$ . Assume these sites are indexed from 1 to  $N'$  (with  $N' \leq N$ ).

22 2. For each site  $i, i = 1, \dots, N'$ :

23 2.1. Assess the location parameters  $\hat{\mu}_{i,X}$  and  $\hat{\mu}_{i,Y}$ ,

- 1        2.2. Standardize the sample  $(x_{ij}, y_{ij})$  to be  $(x'_{ij} = x_{ij}/\hat{\mu}_{i,X}, y'_{ij} = y_{ij}/\hat{\mu}_{i,Y})$ ,
- 2        3. Select a family of regional multivariate distributions to fit the standardized data of the whole
- 3        region  $(x'_{ij}, y'_{ij})$  for  $j = 1, \dots, n_i$  and  $i = 1, \dots, N'$ : this includes the marginal distributions as
- 4        well as a copula. In the present context, assume that the regional distribution depends upon  $s$
- 5        parameters denoted  $\theta_1, \dots, \theta_s$ .
- 6        4. Estimate the parameters of the distribution obtained in step 3:
- 7        4.1. Obtain an estimator  $\hat{\theta}_k^{(i)}$  of the  $k^{\text{th}}$  parameter from the standardized data of the  $i^{\text{th}}$  site,  $k =$
- 8         $1, \dots, s$  and  $i = 1, \dots, N'$ . The maximum likelihood method or the  $L$ -moments-based
- 9        method can be used for the estimation.
- 10       4.2. Obtain the weighted regional parameter estimators:

$$11 \quad \hat{\theta}_k^{(R)} = \frac{\sum_{i=1}^{N'} n_i \hat{\theta}_k^{(i)}}{\sum_{i=1}^{N'} n_i}, \quad k = 1, \dots, s \quad (5)$$

- 12       5. Evaluate, for a given value of  $p$ , different combinations of the estimated growth curve
- 13        $\hat{q}_{x,y}(p)$  from (5) using the fitted multivariate distribution with the corresponding weighted
- 14       regional parameters  $\hat{\theta}_k^{(R)}$  with  $k = 1, \dots, s$ .
- 15       6. Multiply component-wise each growth curve combination by the location parameter of the
- 16       target-site  $\ell$ ,  $\hat{\mu}_{\ell,X}$  and  $\hat{\mu}_{\ell,Y}$ :

$$17 \quad (\hat{Q}_{x,y}(p))_{\ell} = \begin{pmatrix} \hat{\mu}_{\ell,X} \\ \hat{\mu}_{\ell,Y} \end{pmatrix} \hat{q}_{x,y}(p) \quad 0 < p < 1 \quad (6)$$

- 18       Hence the obtained result in (6) is an estimate of the local quantile curve corresponding to the
- 19       target-site.

1 Note that  $\hat{\mu}_{\ell,X}$  and  $\hat{\mu}_{\ell,Y}$ , representing the indices of the target-site  $\ell$ , are generally  
 2 assumed to be location parameters. Particular values of these indices can be the sample median or  
 3 the sample mean. Furthermore, for an ungauged site, they can be obtained from its  
 4 meteorological and physiographical features, for instance, through a linear model. Since the  
 5 classical index-flood model is based on the non-exceedance probability  $F(x) = P(X \leq x)$ , we  
 6 have considered, in this procedure, the probability of the event  $\{X \leq x, Y \leq y\}$ . However, if  
 7 another event is of interest, appropriate changes can easily be brought to the proposed model.

8 To deal with step 3 in the described procedure, goodness-of-fit tests are required for  
 9 copula as well as for marginal distributions. Such tests are well-known in the literature for  
 10 univariate distributions. For instance, the empirical cumulative distribution function given by  
 11 Cunnane (1978) can be used. Recently some statistical tests (numerical or graphical) have been  
 12 developed to test copula's goodness-of-fit (see, e.g. Fermanian, 2005 or Genest et al., 2009).

13 We end this section by stating the required elements to define the above procedure in a  
 14 multivariate setting. Let  $(X_1, \dots, X_d)$  be a random vector defined on  $R^d$ ,  $d \geq 1$ , with joint  
 15 distribution  $F$  and marginal distributions  $F_{X_1}, \dots, F_{X_d}$ . On the basis of Sklar's theorem, there exists  
 16 a copula  $C$  such that  $F(x_1, \dots, x_d) = C(F_{X_1}(x_1), \dots, F_{X_d}(x_d))$  for real  $x_1, \dots, x_d$ .

17 Assume that we are interested in the event  $\{X_1 \leq x_1, \dots, X_d \leq x_d\}$ . Then the corresponding  
 18 multivariate quantile is given by:

$$19 \quad Q_{X_1, \dots, X_d}(p) = \left\{ (x_1, \dots, x_d) \in R^d \text{ such that } x_j = F_{X_j}^{-1}(u_j); u_j \in [0, 1], j = 1, \dots, d : C(u_1, \dots, u_d) = p \right\}$$

20 For  $d = 3$  the multivariate quantile represents a surface in a three-dimensional space. For the  
 21 target-site  $\ell$ , equation (6) of the index-flood model becomes:

$$1 \quad \left( \hat{Q}_{x_1, \dots, x_d}(p) \right)_\ell = \begin{pmatrix} \hat{\mu}_{\ell, X_1} \\ \vdots \\ \hat{\mu}_{\ell, X_d} \end{pmatrix} \hat{q}_{x_1, \dots, x_d}(p) \quad 0 < p < 1,$$

2           Therefore, all the theoretical elements required to define the procedure in a  $d$ -dimensional  
3 space are available. However, in practice, some difficulties arise. A key point is related to the  
4 effective modeling of the multivariate copula. Even though some well-known classes of  
5 Archimedean copulas and extreme value copulas are available in the multivariate setting, they are  
6 not convenient to model when the dependence structure is complex. Fitting other kinds of copulas  
7 is a topic of continuous development. The number of parameters to be estimated (related to each  
8 marginal distribution and to the copula) grows quickly with the dimension  $d$ . The numerical  
9 difficulties encountered in the bivariate setting become even more important.

10

#### 11 **4. Performance evaluation using simulation**

12           In order to evaluate the performance of the proposed model, a simulation study is carried  
13 out. Before starting the simulation procedure, it is required to define the regions to be simulated  
14 and convenient evaluation criteria. Note that these evaluation criteria are also employed in  
15 Chebana and Ouarda (2008b) and are adapted in the present work to the regional context. Recall  
16 that the variables  $X$  and  $Y$  are selected to be respectively the flood volume and flood peak (Figure  
17 2).

18

##### 19 **4.1 Simulated regions**

20           As it was already underlined, to apply the index-flood model, the region should be  
21 homogeneous. However, to be more realistic, the region can also be “possibly homogeneous”  
22 rather than “exactly homogeneous” (see Hosking and Wallis, 1997). In that case, the value of the

1 corresponding heterogeneity statistical measure  $H$  should be between 1 and 2. Since, the present  
 2 study is a continuation of the work by Chebana and Ouarda (2007), the same regional distribution  
 3 can be considered, namely, a bivariate distribution with Gumbel margins and Gumbel logistic  
 4 copula given respectively by:

$$5 \quad F_X(x) = \exp\left\{-\exp\left(-\frac{x-\beta_X}{\alpha_X}\right)\right\}, \quad x \text{ real, } \alpha_X > 0 \text{ and } \beta_X \text{ real} \quad (7)$$

$$6 \quad C_\gamma(u,v) = \exp\left\{-\left[(-\log u)^\gamma + (-\log v)^\gamma\right]^{1/\gamma}\right\}, \quad \gamma \geq 1 \text{ and } 0 \leq u, v \leq 1 \quad (8)$$

7 By replacing each  $x$  by  $y$  in (7), we obtain the expression of the marginal distribution of the  
 8 variable  $Y$ . The case where  $\gamma=1$  in (8) corresponds to complete independence of the two  
 9 variables.

10 The corresponding parameters of the bivariate distribution when the region is  
 11 homogeneous are:

$$12 \quad \alpha_X = 300.22, \beta_X = 1239.80, \alpha_Y = 15.85, \beta_Y = 51.85 \text{ and } \gamma = 1.414 \quad (9)$$

13 This value of  $\gamma$  is equivalent to the correlation coefficient  $\rho = 0.5$  and the Kendall's tau  
 14 coefficient where  $\tau = 0.3$   $\tau = 4E[F(X,Y)] - 1$ . Indeed, the parameter  $\gamma$  is related to  
 15 respectively  $\rho$  and  $\tau$  according to the following expressions (see Gumbel and Mustafi, 1967 and  
 16 Genest and Rivest, 1993):

$$17 \quad \gamma = \frac{1}{\sqrt{1-\rho}}, \quad 0 \leq \rho < 1 \quad (10)$$

$$18 \quad \gamma = 1 + \frac{\tau}{1-\tau} \quad (11)$$

19 The parameter values in (9) are those of a real data treated in Yue and Rasmussen (2002) and  
 20 concern the Skootamatta River in Ontario, Canada.

21 Three kinds of regions are considered as follows:

- 1 - The first region is *homogeneous (Homog)*: all sites have the same distribution with the  
2 same parameters given above.
- 3 - The second one is a *30% completely heterogeneous region (HetCo30)*: the scale and  
4 dependence parameters ( $\alpha_x, \alpha_y$  and  $\gamma$ ) increase linearly from the first to the last site in  
5 the 30% range centered around the homogeneous region parameters, e.g., for a given  
6 value of  $\alpha_x$  the variation is in the range  $[\alpha_x(1-0.3/2), \alpha_x(1+0.3/2)]$ .
- 7 - The third one is a *50% heterogeneous region on the marginal parameters (HetMa50)*: it is  
8 the same as the above region but the dependence parameter  $\gamma$  is fixed and the variation is  
9 on the marginal parameters  $\alpha_x$  and  $\alpha_y$ .

10 The representative simulated regions are composed of a number of sites  $N = 15$  and each  
11 site contains  $n = n_i = 30$  observations where  $\gamma = 1.414$  (in the remainder of the paper other values  
12 of  $\gamma$  are also considered). For the regions *HetCo30* and *HetMa50*, the corresponding mean  
13 values of the heterogeneity statistical measure  $H$  are respectively 1.30 and 1.36. Hence they can  
14 be considered effectively as “possibly homogeneous” regions. These values are obtained  
15 following the procedure defined in Chebana and Ouarda (2007). Note that the location  
16 parameters  $\beta_x$  and  $\beta_y$  are considered to be fixed in the generated regions at the values given in  
17 (9). These parameters have no effect on the heterogeneity measure  $H$  since we are interested in  
18 the variability aspect (see also Hosking and Wallis, 1993 or Chebana and Ouarda, 2007).

19 According to the linear variation of the parameters in the regions *HetCo30* and *HetMa50*,  
20 the corresponding sites are ranked from 1 to  $N$ . That is, for instance, the smallest parameter  
21 values and the largest ones are associated respectively to the first site and to the last site whereas  
22 the parameters in (9) correspond to the middle site.



1 For comparison purposes, and to study the effect of various factors on the estimation  
2 results, other regions than the representative ones are generated:

3 i. Regions *Homog* and *HetMa50* in the independence case ( $\gamma = 1$  in the Gumbel logistic  
4 copula (8) with  $n_i = n = 30$  and  $N = 15$  for each region.

5 ii. Regions *Homog*, *HetCo30* and *HetMa50* in the dependence case ( $\gamma = 1.414$ ) where  $n =$   
6 30 and 60 with  $N = 15$  for each region.

7 iii. Regions *Homog*, *HetCo30* and *HetMa50* in the dependence case ( $\gamma = 1.414$ ) where  $N =$   
8 10, 15, 20, 50 and 100 with  $n = 30$  for each region.

9 iv. Regions *HetCo60* and *HetCo80* (similar to *HetCo30* with 60% and 80% instead of 30%)  
10 in the dependence case where  $\gamma = 3.162$  (equivalent to  $\rho = 0.9$ ) with  $n = 30$  and  $N = 15$   
11 for each region.

12  
13 The values of  $n$  and  $N$  are selected on the basis of situations commonly encountered in RFA (e.g.  
14 Hosking and Wallis, 1997 or Chebana and Ouarda, 2007). Note that the region *HetCo30* cannot  
15 be considered in the independent case (i) where the parameter  $\gamma$  is fixed at  $\gamma = 1$ . The regions in  
16 (iv) are heterogeneous with heterogeneity mean measures  $H = 2.48$  and  $5.29$  respectively. They  
17 are considered to show the effect of the heterogeneity of the region on the estimation  
18 performances. The value of  $\gamma = 3.162$  is considered to ensure that  $\gamma \geq 1$  for each site in the  
19 region. There are some intersections between the above considered cases, for instance, both cases  
20 (ii) and (iii) contain the region *HetMa50* with  $n = 30$ ,  $N = 15$  and  $\gamma = 1.414$ . These repetitions  
21 are kept for the coherence of the result presentation.

22 Ghoudi et al. (1998) developed an algorithm for the generation of samples of a bivariate  
23 variable  $(X, Y)$  according to the extreme value copula. This algorithm is used in the present case

1 since the Gumbel logistic copula (8) is also an extreme value copula. The algorithm is  
 2 summarized in Chebana and Ouarda (2007).

3

#### 4 **4.2 Performance evaluation criteria**

5 In the present multivariate context, the bivariate quantile is a curve. Hence, the estimation  
 6 result is a curve instead of a real value as in the univariate framework. Consequently, the usual  
 7 performance evaluation criteria are not adapted and should be defined differently. To evaluate the  
 8 performance of the method, given the true quantile curve, an estimation of the corresponding  
 9 quantile curve is obtained. Then, the evaluation consists in the assessment of the distance  
 10 between the true and estimated curves.

11

12 In the present context the quantile curve is a function. Consequently, we can adopt the  
 13 notations:  $(x, g(x))$  for the regional growth curve and  $(x, G(x))$  for the at-site quantile curve.  
 14 These notations will ensure the clarity of the definition of the evaluation criteria. Let  $M$  be the  
 15 number of simulation repetitions, and let  $\hat{g}_p^{[m]}(x)$  and  $\hat{G}_p^{[m],i}(x)$  be the ordinates, respectively, of  
 16 the  $m$ th repetition of the estimated regional growth curve and site- $i$  quantile estimate for non-  
 17 exceedence probability  $p$  ( $0 < p < 1$ ). Then, the corresponding *coordinate-wise* relative errors are  
 18 given by:

$$19 \quad r_p^{[m],i}(x) = \frac{\hat{g}_p^{[m]}(x) - g_p^i(x)}{g_p^i(x)} \quad \text{and} \quad R_p^{[m],i}(x) = \frac{\hat{G}_p^{[m],i}(x) - G_p^i(x)}{G_p^i(x)} \quad (12)$$

20 for fixed  $x$  component along the proper part of the curve where  $g_p^i(x)$  and  $G_p^i(x)$  are  
 21 respectively the true at-site growth curve and quantile curve ordinates. Note that the differences  
 22 in the numerator represent vertical distances between points of the underlying curves. As

1 indicated in Section 2.2, the points that define the extremities of the proper part correspond to the  
 2 maximum value for each of the variables  $x$  and  $y$  in the estimated quantile curve. Since the  
 3 Gumbel distribution is not right-bounded, in order to obtain the values in (12), we select the  
 4 extremities of the true curve according to those of the estimated curve.

5  
 6 In the coordinate-wise relative errors (12), there are three index dimensions: the index  $i$  is  
 7 related to sites, the index  $m$  is related to the simulation replications and the last index  $x$  is related  
 8 to the quantile combinations. Therefore, it is necessary to summarize the relative errors (12) to be  
 9 interpretable. To avoid repetition, we focus on the relative errors related to the quantile and those  
 10 of the growth curve can be obtained in a similar manner.

11 To summarize errors (12) with respect to  $x$ , we consider distances or norms in functional  
 12 spaces. In such spaces, some possible criteria are known as the  $L^r$  distances with  $r \geq 1$ . They are  
 13 defined between functions  $f_1$  and  $f_2$  on a given space  $S$  with a positive measure  $\lambda$  as

14  $\|f_1 - f_2\|_r = \left( \int_S |f_1 - f_2|^r d\lambda \right)^{1/r}$  (see, e.g., Jones, 1993, Chapter 10). The particular cases  $L^1$ ,  $L^2$  and

15  $L^\infty$  are the most commonly used. Note that the  $L^1$  distance is more intuitive and more  
 16 representative than  $L^2$  and  $L^\infty$ , but is more complex to handle in theoretical proofs because of the  
 17 presence of the absolute value. Furthermore, when using the  $L^1$ , the bias can not be evaluated  
 18 since, as a metric, it is always positive. For this reason, and to keep the same commonly  
 19 employed performance criteria in frequency analysis, we proceed as follows. Let  $L_p^i$  be the length  
 20 of the proper part of the true quantile curve  $QC_p^i$ , then the relative integrated error is :

21 
$$RIE_i^{*[m]}(p) = \frac{1}{L_p^i} \int_{QC_p^i} R_p^{[m],i}(x) dx, \quad i = 1, \dots, N, \quad 0 < p < 1, \quad m = 1, \dots, M \quad (13)$$

1 Note that the integral  $RIE_i^{*[m]}(p)$  is similar to  $L^l$  but it is not a distance in the formal sense, since  
 2 it may have negative values. To be differentiated from  $L^l$ , the “pseudo-distance” associated to  
 3  $RIE_i^{*[m]}(p)$  is denoted by  $L^{l*}$ .  $RIE_i^{*[m]}(p)$  allows to assess the regional bias. However, it is not  
 4 appropriate for the variance evaluation since it may have some null values whereas the estimation  
 5 is poor. For this reason, the root-mean-square errors are evaluated on the basis of the  $L^l$  distance.  
 6 In the present context, the corresponding  $L^l$  distance is given by:

$$7 \quad RIE_i^{[m]}(p) = \frac{1}{L_p^i} \int_{QC_p^i} |R_p^{[m]i}(x)| dx, \quad i = 1, \dots, N, \quad 0 < p < 1, \quad m = 1, \dots, M \quad (14)$$

8  
 9 In order to evaluate the estimation error for a site  $i$ , on the basis of  $RIE_i^{*[m]}(p)$  and  
 10  $RIE_i^{[m]}(p)$ , the bias and root-mean square errors are given respectively by:

$$11 \quad B_i(p) = 100 \frac{1}{M} \sum_{m=1}^M RIE_i^{*[m]}(p) \quad \text{and} \quad R_i(p) = 100 \sqrt{\frac{1}{M} \sum_{m=1}^M (RIE_i^{[m]}(p))^2} \quad (15)$$

12 To summarize these criteria over the sites of the region, it is possible to average them to  
 13 obtain the regional bias, the absolute regional bias and the regional quadratic error respectively  
 14 given by:

$$15 \quad RB^R(p) = \frac{1}{N} \sum_{i=1}^N B_i(p), \quad ARB^R(p) = \frac{1}{N} \sum_{i=1}^N |B_i(p)| \quad \text{and} \quad RRMSE^R(p) = \frac{1}{N} \sum_{i=1}^N R_i(p) \quad (16)$$

16  
 17 The role of each one of these criteria is explained, for instance, in Hosking and Wallis (1997), in  
 18 the univariate setting. The  $RB^R$  measures the tendency of quantile estimates to be uniformly too  
 19 high or too low across the whole region; the  $ARB^R$  measures the tendency of quantile estimates to  
 20 be consistently high at some sites and low at others; and the  $RRMSE^R$  measures the overall  
 21 deviation of estimated quantiles from true quantiles.

22

### 1 4.3 Simulation procedure

2           Once the regions and the evaluation criteria are identified, the simulation procedure can  
3 be defined. It is mainly based on the general procedure given in Section 3. In the simulation  
4 procedure, there is no need to apply the discordancy test. The distribution is known a-priori.  
5 Hence step 3 in Section 3 is omitted. The repetition and evaluation steps are only for the  
6 simulation procedure and do not concern the general procedure. The simulation procedure  
7 consists of the following steps:

8 1. Generate a region as described in Section 4.2 with data denoted as  
9  $(x_{ij}, y_{ij}), j = 1, \dots, n_i$  and  $i = 1, \dots, N$ .

10 2. For each site  $i, i = 1, \dots, N$ :

11 2.1. Evaluate the sample mean on both variables:  $\hat{\mu}_{i,X}$  and  $\hat{\mu}_{i,Y}$ ,

12 2.2. Standardize the sample to obtain  $(x'_{ij} = x_{ij}/\hat{\mu}_{i,X}, y'_{ij} = y_{ij}/\hat{\mu}_{i,Y}), j = 1, \dots, n_i$ ,

13 2.3. Estimate the parameters of the standardized sample related to the bivariate distribution of  
14 the generated region:

15 - For the marginal Gumbel distribution, the estimators  $\hat{\alpha}_X^{(i)}, \hat{\alpha}_Y^{(i)}, \hat{\beta}_X^{(i)}$  and  $\hat{\beta}_Y^{(i)}$  are  
16 obtained using the  $L$ -moment method given by Hosking and Wallis (1997).

17 - The parameter  $\gamma$  of copula is estimated by  $\hat{\gamma}^{(i)}$  using expression (11).

18 3. Obtain the regional parameters using the weighted mean given by (5).

19 4. Obtain the combinations of the bivariate regional growth curve  $qC_p$  using (7) and (8)  
20 replaced in (4), on the basis of the standardized data, for a fixed value of the risk  $p$  (here we  
21 take  $p = 0.9, 0.99$  and  $0.995$ ).

- 1 5. Apply the index-flood model (6) using the growth curves obtained in step 4 and location  
2 estimators obtained from step 2.1.
- 3 6. Repeat steps 1 to 5  $M$  times where  $M$  is a large number, say  $M = 2000$ .
- 4 7. Evaluate for each site the true quantile curves  $QC_p$  using the parameters of the parent  
5 distribution according to the type of region (e.g. *Homog*, *HetMa50* and *HetCo30*).
- 6 8. Evaluate the true bivariate growth curves  $qC_p$  using the parameters of the parent distribution  
7 for each site as follows:
  - 8 8.1. Find the population means  $\mu_x$  and  $\mu_y$  from the distribution parameters on each variable.  
9 In the present case, for the Gumbel distribution, we have :  $\mu_x = \beta_x + 0.5772\alpha_x$  and  
10  $\mu_y = \beta_y + 0.5772\alpha_y$ ,
  - 11 8.2. Divide each component of the true quantiles  $QC_p$  (from step 7) by the corresponding  
12 population mean (from step 8.1).
- 13 9. Evaluate the performance criteria described in equations (13) and (14), then (15) and finally  
14 (16).

15  
16 Step 8 is introduced to evaluate the performance of the estimation of the growth curve. In  
17 addition, note that step 8 produces a true growth curve for each site which is required for the  
18 simulation of heterogeneous regions. However, these growth curves are identical for  
19 homogeneous regions as it is assumed by the index-flood model. This is similar to the univariate  
20 setting employed, for instance, by Hosking and Wallis (1997).

21 After the bivariate procedure, the univariate estimation procedure is also applied for  
22 comparison purposes. Note that some elements of the present simulation procedure are inspired  
23 by Hosking and Wallis (1997) and by Chebana and Ouarda (2007).

## 1 **5. Results**

2           The application of the described simulation procedure leads to the results presented  
3 herein. This section is divided into three parts: in the first one we present the preliminary  
4 simulations, in the second we present and analyse the main results and in the last part we study  
5 the effect of some factors on the performance of the proposed model.

6

### 7 **5.1 Preliminary simulations**

8           Before analysing the results, three preliminary simulations are produced to explain some  
9 of the previously introduced notions. The first one corresponds to one repetition ( $M = 1$ ) of the  
10 simulation procedure on the region *HetMa50* with  $p = 0.9$  where  $n = 30$  and  $N = 15$ . Figure 3a  
11 shows the true and estimated quantile curves of the first, the middle and the last sites in the  
12 region. Table 1 presents estimation relative errors related to the first, the middle and the last sites  
13 in the same simulated region. Relative errors for bivariate quantiles, as curves, are evaluated  
14 using (13). The univariate quantiles are evaluated directly or as extreme scenarios of the proper  
15 part of the bivariate quantile curves. The univariate estimations are evaluated on the basis of the  
16 usual relative errors (e.g. Hosking and Wallis, 1997). Based on criteria (13), from Table 1, we  
17 observe that the middle site, defined with parameters (9), of the homogeneous region, is the one  
18 best estimated. The estimation of the first site is acceptable whereas the estimation of the last site  
19 is the worst. The values in Table 1 reflect the evaluation criteria being used (13) for the bivariate  
20 quantile estimation. For instance, it is clear from Figure 3a that the bivariate quantile curve of the  
21 last site is underestimated with a high negative value (-20.94 %), the univariate quantiles are both  
22 underestimated and the relative error related to  $Y$  is negative with high magnitude. In addition, for  
23 each site, similarity is observed between the estimation errors of the univariate quantiles

1 evaluated both directly and as extreme scenarios even when only one sample is considered. Note  
2 that similar errors do not imply similar estimated values. Table 2 presents the true values of the  
3 univariate quantiles evaluated directly and as extreme scenarios of the bivariate curve. Table 2  
4 provides also an indication about the relative difference between the two estimates. These relative  
5 differences are very low (less than 0.5%). Therefore, one can consider that values obtained by the  
6 two different methods are almost the same.

7         The second preliminary simulation results are presented in Table 3 which summarises the  
8 values of the criteria related to the same previously generated region for the bivariate as well as  
9 the univariate estimation. For the bivariate case we opted to present all possible combinations of  
10 criteria ( $RB^R$ ,  $ARB^R$  and  $RRMSE^R$ ) and “norms” ( $L^{I*}$ ,  $L^I$  and  $L^2$ ). Note that the values in Table 3  
11 represent the whole region while values in Table 1 represent only particular sites. Bivariate  
12 results show, as expected, that it is appropriate to use  $L^{I*}$  for the  $RB^R$  and  $ARB^R$  evaluation and  $L^I$   
13 for the  $RRMSE^R$  evaluation. The values associated to  $L^2$  are given only as an indication. Hence,  
14 and in order to save space in the remainder of the paper, they are omitted. Furthermore, the  
15 regional averages of the univariate quantile errors estimated directly or as extreme scenarios are  
16 similar. Figures 3b,c show the true and estimated quantile curves, respectively, for all sites of the  
17 generated region. It can be seen that the true quantile curves are well ordered whereas the  
18 estimated ones intersect. This is because in the first case the parameters are known and ordered  
19 whereas in the second case the parameters are estimated and do not necessarily keep their order.  
20 Nevertheless, the whole view of the region, composed by the two groups of curves (true and  
21 estimated), shows a good agreement as a region and not for each site alone. The curves in Figures  
22 3b,c are also in agreement with the values corresponding to  $L^{I*}$  in Table 3. The agreement here  
23 concerns the whole region (all sites together) rather than site by site.



1           As it was previously indicated, univariate quantiles can be either estimated directly using  
2 the index-flood model or deduced from the bivariate estimation as extreme scenarios (extreme  
3 points of the quantile curve). Hence, in this third preliminary simulation, it is valuable to compare  
4 univariate quantiles obtained from both estimations. Table 4 shows, on the basis of  $M = 2000$   
5 generated *HetMa50* regions with  $p = 0.9, 0.99$  and  $0.995$ , relative errors corresponding to  
6 univariate quantile estimation. These results combined to those in Table 2 indicate that, for a  
7 given variable  $X$  or  $Y$ , directly evaluated quantile estimates are very similar to those obtained as  
8 extreme points of the bivariate quantile curves for all values of  $p$  and for each criterion. This  
9 result remains valid for the outputs of the main simulations. Consequently, results related to the  
10 univariate quantiles when evaluated as extreme points are omitted from the next simulation  
11 results. This result shows that values provided by multivariate FA are very similar to those  
12 obtained by the univariate FA and also with an equivalent accuracy.

13

## 14 **5.2 Main results**

15           The main simulation results are presented in Tables 5a,b for all the considered regions.  
16 Even though, the focus is on quantile curve estimation, the evaluation of growth curve estimation  
17  $qC_p$  is also reported in order to explain the quantile results. From both tables, three apparent  
18 elements can be observed:

- 19 - In general, the values of the performance criteria increase with respect to the risk  $p$  in both  
20 univariate and bivariate settings. This behaviour, well known in the univariate FA, is not  
21 systematic in the bivariate estimation, especially in terms of  $RRMSE^R$ . The usual explanation  
22 is that generally in FA, a quantile associated to a risk  $p$  is more accurately estimated than  
23 another one associated to a risk  $p'$  if  $p < p'$  (when  $p$  and  $p'$  are close to 1). The reason is that

1 for a small risk, the corresponding quantile is close to the central body of the distribution, and  
2 hence, an important part of the data contributes to its estimation. However, in the bivariate  
3 setting, the situation is not similar. Indeed, in the multivariate context, the central part of a  
4 distribution contains little probability mass compared to the univariate setting. This is very  
5 obvious in higher dimensions; see Scott (1992) for more details and examples.

6 - The relative bias  $RB^R$  is very small in all regions and for all values of  $p$ . However, bivariate  
7  $RB^R$ 's are larger than those of each one of the univariate but without exceeding 1.17%. The  
8  $RB^R$  low values are due to the symmetry regarding the parameters of the simulated regions.

9 - The growth curve  $qC_p$  results, for each value of  $p$ , are very similar for both variables  
10 separately (univariate) and also jointly (bivariate), especially in terms of  $RRMSE^R$ . However,  
11 this is not the case for quantile  $QC_p$  estimation where differences are noticeable between  
12 bivariate and univariate results. This can be explained by the errors induced from the  
13 estimation of the index  $\mu$ . That is, if one variable has a high error in its index  $\mu$ , then, when  
14 multiplying it by the growth curve  $qC_p$ , the final estimation result is affected accordingly.

15 Note that the uncertainty related to the mean has more effect on the variability of the  
16 quantiles (through the  $RRMSE^R$ ) than on the bias ( $RB^R$ ) because of error compensation in the  
17  $RB^R$ .

18 In the homogeneous regions (Table 5a), the variability expressed in terms of the  $RRMSE^R$   
19 in the growth curve estimation  $qC_p$  is small compared to that of  $QC_p$  for a fixed  $p$ . Hence, in  
20 homogeneous regions, the variability in  $QC_p$  estimation originates essentially from the  
21 estimation of the index  $\mu$  in equations (1) and (6). However, with respect to  $p$ , the variability in  
22  $qC_p$  estimation increases faster than the variability in  $QC_p$ . This result may be explained by the

1 fact that the mean has more influence on the central part of the distribution than on the tail.  
2 Hence, the contribution of the index variability decreases for large values of  $p$ .

3 Table 5b presents results of *Homog* and *HetMa50* when the variables  $X$  and  $Y$  are assumed  
4 to be independent, that is  $\gamma = 1$  in copula (8). From Tables 5a,b, the comparison of the dependent  
5 and independent cases reveals two principal elements. First, there is no significant difference in  
6 the univariate results: The results of univariate estimation quantiles remain almost the same in  
7 both dependent and independent cases. The reason is that, intuitively, the marginal distributions  
8 are not affected by the copula, and mathematically, the copula has always the same values in the  
9 extreme points, that is  $C(u, 1) = u$  and  $C(1, v) = v$  for all  $u, v$  in  $[0, 1]$  (see e.g. Nelsen, 2006).  
10 Second, in bivariate quantile estimation, the criteria values are slightly smaller in the independent  
11 case than in the dependent one. This may be justified by the presence of an extra parameter to be  
12 estimated in the dependent case (the parameter  $\gamma$ ). This parameter in the independent case is not  
13 estimated and is fixed at  $\gamma = 1$ . We conclude that univariate estimation ignores the dependence  
14 structure of the event.

15 Figure 4 shows the quantile estimation performance with respect to the 15 sites within the  
16 same simulated regions *HetMa50* presented in Table 5a. The  $B_i$  and  $R_i$ , defined in (15), represent  
17 respectively the *RB* and the *RRMSE* for a site  $i$ . Similar results are obtained for the other types of  
18 regions and hence they are omitted. From Figure 4 the behaviours of both the *RB* and the  
19 *RRMSE*, obtained from the bivariate or univariate models are similar. It is observed that for each  
20 value of  $p$ , the bias is positive for the first half of the sites (from the 1<sup>st</sup> to the 8<sup>th</sup> site) and  
21 negative for the other half (from the 9<sup>th</sup> to the 15<sup>th</sup> site). This fact is observed in the univariate  
22 setting as well as in the bivariate one. It may be explained as follows: In the first half, the true  
23 quantile  $Q$  is smaller than the average quantile over the region  $\bar{Q}$ ,  $Q \leq \bar{Q}$ , since small values of

1 the scale parameter  $\alpha$  reduce quantile values in regions like *HetMa50*. Furthermore, the average  
2 quantile value should be very close to the estimated one  $\hat{Q}$  ( $\bar{Q} \approx \hat{Q}$ ). Therefore, the quantile  
3 relative error is positive since it is greater than the negligible difference error between the  
4 estimate and the average quantiles ( $\hat{Q} - Q \geq \hat{Q} - \bar{Q} \approx 0$ ). The other half of the sites, where the  
5 bias is negative, can be treated similarly. Furthermore, we note that the *RRMSE* is small for sites  
6 with parameters close to those of the homogeneous region given in (9) and it increases according  
7 to the deviation of the site parameters from the central one. This is more apparent for high values  
8 of  $p$ . This behaviour of the RB and *RRMSE* with respect to site number is also observed in the  
9 univariate index-flood model (Hosking and Wallis, 1997).

10

### 11 **5.3 Effect of various factors on the estimation results**

12 The proposed estimation model (6) may be affected by several factors. In this section, we  
13 present a short study dealing with the impact of the record length  $n$ , the region size  $N$  as well as  
14 the degree of region heterogeneity. Table 6 presents estimation results for the *Homog*, *HetCo30*  
15 and *HetMa50* regions when the variables are dependent where  $N = 15$  and  $n = 30, 60$  and  $100$ . To  
16 facilitate comparisons, results for  $n = 30$  are taken from Table 5a. We observe that when  $n$   
17 increases, the main improvement is related to the  $RRMSE^R$  for each value of  $p$ . However, the  $RB^R$   
18 and the  $ARB^R$  remain almost constant. Note that the values of the  $RB^R$  and the  $ARB^R$  are very low  
19 and their variations can be considered as proportionally similar to those of the  $RRMSE^R$ . On the  
20 one hand, the improvement of the  $RRMSE^R$ , with respect to  $n$ , is related to the heterogeneity  
21 degree of the region. That is, the improvement decreases slightly from *Homog* to *HetMa50*. On  
22 the other hand, the improvement for the bivariate estimation is slightly more important than for  
23 the univariate estimation in all considered regions.

1 Similarly to Table 6, Table 7 presents estimation results of 0.99-quantiles for *Homog*,  
2 *HetCo30* and *HetMa50* regions when the variables are dependent where  $n = 30$  and  $N = 10, 15,$   
3  $20, 50$  and  $100$ . In order to simplify the comparison, results for  $N = 15$  are taken from Table 5a.  
4 We observe, for a given type of region, a very slight improvement (less than 1%) of the  $RRMSE^R$   
5 in both univariate and bivariate estimations whereas the  $RB^R$  and the  $ARB^R$  are almost constant  
6 but with a variability proportionally similar to the variability of the  $RRMSE^R$ . This behaviour with  
7 respect to  $N$  is similar for the three region types although the values are different. The results  
8 corresponding to the 0.9- and 0.995-quantiles lead to similar conclusions and hence are not  
9 presented.

10 An important assumption of RFA (and the Index-flood model) is the homogeneity of the  
11 region which is checked in the delineation step. To study the effect of the heterogeneity degree of  
12 a region, we consider five regions with different heterogeneity degrees: one homogeneous region,  
13 two possibly homogeneous regions and two heterogeneous regions. The corresponding results are  
14 presented in Table 8. They indicate that, for each fixed value of  $p$  in the univariate as well as the  
15 bivariate settings, the quantile and the growth curve estimation errors increase with respect to the  
16 heterogeneity degree expressed through the mean values of  $H$ . It can be concluded that the  
17 heterogeneity degree has a negative effect on the performance of the estimation procedure.

18 We conclude from Tables 6 and 7 that the impact of  $n$  and  $N$  is not significant on regional  
19 quantile estimation. Note that, in the hydrological context, the variations of  $n$  and  $N$  are generally  
20 small. However, as concluded in Chebana and Ouarda (2007), the effect of  $n$  and  $N$  is very  
21 important in the delineation step. Hence, the impact of the record length  $n$  and the region size  $N$  is  
22 indirect on the estimation step through the homogeneous region selection in the delineation step.  
23 Even though the estimation is not greatly affected by increasing values of  $N$ , there is still  
24 significant gain for carrying out the regionalisation (transfer of information from other sites in the

1 region). Indeed, when  $N = 1$ , if the target-site contains enough data, then quantile estimation can  
2 be obtained directly by local FA. The regional methodology is of interest when the target-site is  
3 ungauged or partially ungauged so that the local estimation is not possible or not efficient. The  
4 homogeneity or the possible homogeneity of regions are important conditions to the good  
5 performance of the procedure. The above conclusions are similar to those obtained by Hosking  
6 and Wallis (1997) for the univariate model.

7 In the previous results, univariate and bivariate estimates were shown to be different in  
8 terms of values but similar in terms of behaviour. The explanation lies partially in the difference  
9 in the criteria employed to evaluate the performances of each model. Furthermore, the main  
10 differences between univariate and bivariate models are conceptual. The univariate estimation  
11 results are presented only to be compared to the extreme points of the bivariate quantiles.

12

## 13 **6. Conclusions and future work**

14 In the present paper we proposed an extension of the index-flood model to the  
15 multivariate context. The proposed estimation procedure with the multivariate discordancy and  
16 homogeneity tests constitute a complete multivariate RFA procedure. Even though the procedure  
17 is shown to be valid in the multivariate setting, the present paper focuses on the bivariate case.  
18 The proposed model is based on copulas and on a bivariate quantile version. The bivariate  
19 quantile version employed is a curve composed by several statistically similar combinations,  
20 since they lead to the same risk. The univariate estimated quantiles, correctly combined, are  
21 particular cases corresponding to the extreme scenarios of the bivariate quantile curve. According  
22 to the available resources and the nature of the project, one or more convenient scenarios may be

1 selected. Hence, the bivariate setting offers more flexibility to designers than the univariate  
2 framework.

3  
4 Simulation results of the bivariate version of the index-flood model are similar to those of  
5 the univariate model in terms of behaviour of the corresponding performance criteria. The results  
6 of univariate FA are provided (as extreme points) by the multivariate FA, and with an equivalent  
7 accuracy. The proposed model performs better when the region is close to homogeneity. Its  
8 performance is not significantly affected by small variations of the record length of sites or the  
9 region size. However, the whole regionalization procedure is affected by these factors through the  
10 delineation step. On the other hand, the univariate estimation results remain almost unchanged if  
11 the variables are dependent or independent. Hence, the univariate quantile estimates do not take  
12 into account the dependence structure of the variables characterizing the event.

13 In the present study several elements of multivariate RFA are treated. Nevertheless, the  
14 following issues, among others, should have the merit to be developed in future efforts:

- 15 - Adaptation of the model for the estimation of other events of interest such as the  
16 simultaneous exceedence event expressed as  $\{X \geq x, Y \geq y\}$ ,
- 17 - Estimation of the multivariate index-flood  $\mu$  for ungauged sites using their  
18 physiographical characteristics,
- 19 - Definition of sharp criteria to measure the model performances. Indeed, if other  
20 phenomena and other types of copula are considered, then “distances” will be between  
21 sets instead of functions, since generally a quantile curve is a “set of points” and not  
22 necessarily a function (which is a particular case),

- 1 - Development of confidence *bands* associated to the regional estimates of the quantile  
2 curves. This is of interest to evaluate the amount of variation in the curve estimation.
- 3 - A thorough sensitivity study of the impact of different factors that may affect the  
4 performances of the model, separately or combined. Such factors include: the estimation  
5 method of the distribution parameters, the fitted regional distribution including copula, as  
6 well as the effect of a misspecification of the bivariate distribution.

7

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12 helped improve the quality of the paper.

13



## 1 Notation list:

$\alpha_X, \beta_X, \alpha_Y, \beta_Y$	Parameters of the marginal Gumbel distributions
$\gamma$	Dependence parameter in the Gumbel logistic copula
$\rho$	Correlation coefficient
$\tau$	Kendall tau coefficient
$F, F_X$ and $F_Y$	Joint distribution and marginal distributions for the random variables $X$ and $Y$
$C_\gamma(\cdot, \cdot)$	Gumbel logistic copula with parameter $\gamma$
$D$	Bivariate discordancy test
$H$	Bivariate homogeneity test
$n$ or $n_i$	Site record length (of site $i$ )
$N$	Number of sites in a region
$M$	Number of replicates for the simulations
$QC_p$	Bivariate quantile curve associated to a risk $p$ of the non-exceeding event
$Q_{x,y}(p)$	A point (a combination) of the curve $QC_p$
$QC_x(p)$ and $QC_y(p)$	Coordinates of the point $Q_{x,y}(p)$ , that is $Q_{x,y}(p) = (QC_x(p), QC_y(p))$
$QD_X(p)$ and $QD_Y(p)$	Univariate quantiles when directly evaluated
$QL_X(p)$ and $QL_Y(p)$	Univariate quantiles when deduced as extreme values from the bivariate quantile curve
$R_p^{[m],i}(\cdot)$	Coordinate-wise relative errors of the proper part of the quantile curve of site $i$ corresponding to the replication $m$ and for a risk $p$
$L_p^i$	Length of the proper part of the true quantile curve $QC_p^i$ of site $i$ and for a risk $p$
$RIE_i^{*[m]}(p)$	Relative integrated error related to $R_p^{[m],i}(\cdot)$
$RIE_i^{[m]}(p)$	Relative integrated error related to $ R_p^{[m],i}(\cdot) $
$B_i(p)$	Bias for a site $i$ evaluated on the basis of $RIE_i^{*[m]}(p)$
$R_i(p)$	Root-mean square error for a site $i$ evaluated on the basis of $RIE_i^{[m]}(p)$
$RB^R(p)$	Regional bias evaluated as a mean over the region of $B_i(p)$
$ARB^R(p)$	Regional absolute bias evaluated as a mean over the region of $ B_i(p) $
$RRMSE^R(p)$	Regional quadratic error evaluated as a mean over the region of $R_i(p)$
The notations employed for the quantile curve are valid for the growth curve quantile by replacing $Q$ by $q$	

2

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8

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Table 1: Relative errors (%) corresponding to the 1<sup>st</sup> site, the 8<sup>th</sup> site and the last site of the 0.9-quantile estimates. The generated region is *HetMa50* with  $n = 30$  and  $N = 15$ . The univariate quantiles are evaluated directly and as extreme points of the bivariate quantile curves. The relative errors of the bivariate quantiles are evaluated using (13)

	1 <sup>st</sup> site	8 <sup>th</sup> site	15 <sup>th</sup> site
$RIE^*_i(p)$ for $QC_p$	3.78	2.09	-20.94
Relative error for $QL_X$	1.22	0.33	-6.23
Relative error for $QD_X$	1.24	0.42	-6.09
Relative error for $QL_Y$	1.41	2.14	-15.90
Relative error for $QD_Y$	1.44	2.24	-15.77



Table 2: True values of the univariate quantiles evaluated directly and as extreme points of the bivariate quantile curve using the parameters of site 8.

		Direct	As extreme point	Relative difference*
$p = 0.9$	$X$	2016.7	2021.4	0.23 %
	$Y$	92.8	93.1	0.32 %
$p = 0.99$	$X$	2828.0	2832.3	0.15 %
	$Y$	135.7	135.9	0.15 %
$p = 0.995$	$X$	3068.2	3081.8	0.44 %
	$Y$	148.4	149.1	0.47 %

\* In the relative difference we assume that the direct quantile value is the reference value, hence relative difference = (As extreme point-Direct)/Direct

Table 3: Relative errors (%) corresponding to the 0.9-quantile estimates of the generated region. The generated region is *HetMa50* with  $n = 30$  and  $N = 15$ . The univariate quantiles are evaluated directly and as extreme points of the bivariate quantile curves. The relative errors of the bivariate quantiles are evaluated using (16)

	Bivariate			Univariate evaluated			
	$L^{1*}$	$L^1$	$L^2$	as extreme points		Directly	
				$QL_X$	$QL_Y$	$QD_X$	$QD_Y$
$RB^R$	0.20	9.55	11.24	-0.06	0.14	0.02	0.24
$ARB^R$	9.03	9.55	11.24	4.61	6.31	4.61	6.33
$RRMSE^R$	9.03	9.55	11.24	4.61	6.31	4.61	6.33

Table 4: Relative errors (%) of univariate quantiles evaluated directly and as extreme points of the bivariate quantile curve. The corresponding region is *HetMa50* with  $n = 30$  and  $N = 15$ .

	<b><math>p=0.9</math></b>				<b><math>p=0.99</math></b>				<b><math>p=0.995</math></b>			
	$QL_X$	$QL_Y$	$QD_X$	$QD_Y$	$QL_X$	$QL_Y$	$QD_X$	$QD_Y$	$QL_X$	$QL_Y$	$QD_X$	$QD_Y$
$RB^R$	-0.03	0.04	0.01	0.09	0.32	0.37	0.35	0.41	0.17	0.30	0.23	0.37
$ARB^R$	3.12	3.50	3.08	3.47	5.50	5.87	5.48	5.86	5.98	6.45	5.93	6.41
$RRMSE^R$	6.20	7.33	6.18	7.32	8.03	9.12	8.02	9.12	8.42	9.53	8.40	9.51

Table 5a: Estimation results (%) for the considered regions when the variables are dependent ( $N=15, n=30$ )

		Quantile estimation			Growth curve estimation				
		<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$qD_X$	$qD_Y$		
<b>Homog</b> ( $H$ mean $\approx 0$ )	<b><math>P = 0.9</math></b>	$RB^R$	-0.22	-0.13	-0.08	-0.23	-0.07	-0.04	
		$ARB^R$	0.22	0.13	0.12	0.23	0.07	0.04	
		$RRMSE^R$	9.18	5.14	6.21	0.98	0.93	1.08	
	<b><math>P=0.99</math></b>	$RB^R$	-0.07	-0.04	-0.07	-0.22	-0.06	-0.08	
		$ARB^R$	0.13	0.09	0.11	0.22	0.06	0.08	
		$RRMSE^R$	9.93	5.37	6.50	1.67	1.71	1.84	
	<b><math>p=0.995</math></b>	$RB^R$	-0.48	-0.24	-0.16	-0.45	-0.22	-0.14	
		$ARB^R$	0.48	0.24	0.17	0.45	0.22	0.14	
		$RRMSE^R$	9.10	5.42	6.43	2.06	1.79	1.91	
	<b>HetCo30</b> ( $H$ mean = 1.30)	<b><math>p = 0.9</math></b>	$RB^R$	-0.10	-0.08	-0.02	-0.16	-0.02	0.01
			$ARB^R$	3.18	1.85	2.09	1.89	1.85	2.07
			$RRMSE^R$	9.75	5.52	6.62	2.18	2.16	2.43
<b><math>p=0.99</math></b>		$RB^R$	0.21	0.01	0.04	-0.16	0.01	0.06	
		$ARB^R$	5.86	3.25	3.45	3.48	3.25	3.50	
		$RRMSE^R$	11.56	6.47	7.53	3.96	3.82	4.09	
<b><math>p = 0.995</math></b>		$RB^R$	-0.16	-0.04	-0.01	-0.54	-0.02	-0.00	
		$ARB^R$	5.89	3.57	3.81	3.83	3.54	3.77	
		$RRMSE^R$	11.06	6.71	7.74	4.34	4.11	4.43	
<b>HetMa50</b> ( $H$ mean = 1.36)		<b><math>p = 0.9</math></b>	$RB^R$	0.34	0.01	0.09	-0.05	0.06	0.12
			$ARB^R$	6.51	3.08	3.47	3.73	3.09	3.46
			$RRMSE^R$	11.52	6.18	7.32	3.90	3.30	3.72
	<b><math>p = 0.99</math></b>	$RB^R$	1.17	0.35	0.41	0.00	0.33	0.40	
		$ARB^R$	10.58	5.48	5.86	6.30	5.45	5.87	
		$RRMSE^R$	14.76	8.02	9.12	6.63	5.87	6.31	
	<b><math>p = 0.995</math></b>	$RB^R$	0.77	0.23	0.37	-0.52	0.25	0.38	
		$ARB^R$	10.69	5.93	6.41	6.70	5.93	6.33	
		$RRMSE^R$	14.42	8.40	9.51	7.04	6.35	6.79	

\*The  $RB^R$  and  $ARB^R$  are evaluated using  $RIE_i^{*[m]}(p)$  and the  $RRMSE^R$  is evaluated using  $RIE_i^{[m]}(p)$

Table 5b: Estimation results (%) for the considered regions when the variables are independent ( $N=15, n=30$ )

		Quantile estimation			Growth curve estimation				
		<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$qD_X$	$qD_Y$		
<b>Homog</b>	<b><math>p = 0.9</math></b>	$RB^R$	-0.07	-0.04	-0.06	-0.11	-0.08	-0.06	
		$ARB^R$	0.18	0.10	0.15	0.11	0.08	0.06	
		$RRMSE^R$	7.86	5.10	6.18	1.01	0.96	1.05	
	<b><math>p = 0.99</math></b>	$RB^R$	-0.12	-0.13	-0.04	-0.13	-0.09	-0.07	
		$ARB^R$	0.15	0.13	0.11	0.13	0.09	0.07	
		$RRMSE^R$	8.25	5.39	6.46	1.61	1.69	1.81	
	<b><math>p = 0.995</math></b>	$RB^R$	-0.34	-0.16	-0.17	-0.26	-0.15	-0.17	
		$ARB^R$	0.34	0.17	0.18	0.26	0.15	0.17	
		$RRMSE^R$	7.21	5.41	6.49	1.81	1.84	1.93	
	<b>HetMa50</b>	<b><math>p = 0.9</math></b>	$RB^R$	0.53	0.09	0.11	0.03	0.05	0.11
			$ARB^R$	7.06	3.06	3.47	4.22	3.09	3.46
			$RRMSE^R$	10.90	6.14	7.28	4.39	3.32	3.71
<b><math>p = 0.99</math></b>		$RB^R$	1.09	0.26	0.41	0.02	0.30	0.39	
		$ARB^R$	10.60	5.43	5.91	6.43	5.45	5.87	
		$RRMSE^R$	13.82	8.02	9.14	6.71	5.86	6.31	
<b><math>p = 0.995</math></b>		$RB^R$	0.87	0.31	0.38	-0.53	0.32	0.37	
		$ARB^R$	10.30	5.91	6.32	6.89	5.93	6.33	
		$RRMSE^R$	13.11	8.40	9.48	7.20	6.37	6.79	

\*The  $RB^R$  and  $ARB^R$  are evaluated using  $RIE_i^{[m]}(p)$  and the  $RRMSE^R$  is evaluated using  $RIE_i^{[m]}(p)$

Table 6: Quantile estimation results (%) for regions with different record lengths ( $n= 30, 60$  and  $100$  with  $N = 15$ )

	$n = 30$			$n = 60$			$n = 100$		
	$Biv. *$	$QD_X$	$QD_Y$	$Biv. *$	$QD_X$	$QD_Y$	$Biv. *$	$QD_X$	$QD_Y$
<b>Homog</b>					$p = 0.9$				
$RB^R$	-0.22	-0.13	-0.08	0.00	0.01	-0.01	0.00	-0.02	-0.01
$ARB^R$	0.22	0.13	0.12	0.11	0.04	0.08	0.08	0.04	0.06
$RRMSE^R$	9.18	5.14	6.21	6.67	3.64	4.39	5.26	2.83	3.43
					$p=0.99$				
$RB^R$	-0.07	-0.04	-0.07	-0.05	-0.02	-0.07	-0.04	-0.07	-0.01
$ARB^R$	0.13	0.09	0.11	0.09	0.05	0.09	0.07	0.07	0.03
$RRMSE^R$	9.93	5.37	6.50	7.13	3.76	4.56	5.57	2.93	3.52
					$p=0.995$				
$RB^R$	-0.48	-0.24	-0.16	-0.04	-0.03	0.00	-0.15	-0.08	-0.08
$ARB^R$	0.48	0.24	0.17	0.12	0.06	0.08	0.19	0.10	0.10
$RRMSE^R$	9.10	5.42	6.43	6.83	3.81	4.64	5.42	2.98	3.56
<b>HetCo30</b>					$p = 0.9$				
$RB^R$	-0.10	-0.08	-0.02	0.07	0.03	0.01	0.12	0.03	0.04
$ARB^R$	3.18	1.85	2.09	3.27	1.87	2.04	3.28	1.82	2.03
$RRMSE^R$	9.75	5.52	6.62	7.43	4.19	4.98	6.34	3.47	4.12
					$p=0.99$				
$RB^R$	0.21	0.01	0.04	0.39	0.13	0.09	0.46	0.10	0.19
$ARB^R$	5.86	3.25	3.45	6.09	3.24	3.47	6.26	3.26	3.55
$RRMSE^R$	11.56	6.47	7.53	9.71	5.24	5.97	8.69	4.59	5.23
					$p = 0.995$				
$RB^R$	-0.16	-0.04	-0.01	0.30	0.15	0.16	0.27	0.14	0.13
$ARB^R$	5.89	3.57	3.81	6.13	3.53	3.76	6.26	3.51	3.77
$RRMSE^R$	11.06	6.71	7.74	9.34	5.45	6.17	8.47	4.81	5.43
<b>HetMa50</b>					$p = 0.9$				
$RB^R$	0.34	0.01	0.09	0.51	0.08	0.14	0.55	0.12	0.14
$ARB^R$	6.51	3.08	3.47	6.67	3.09	3.46	6.75	3.07	3.46
$RRMSE^R$	11.52	6.18	7.32	9.72	4.97	5.83	8.84	4.38	5.10
					$p = 0.99$				
$RB^R$	1.17	0.35	0.41	1.19	0.31	0.42	1.26	0.35	0.45
$ARB^R$	10.58	5.48	5.86	10.91	5.47	5.87	11.04	5.44	5.86
$RRMSE^R$	14.76	8.02	9.12	13.34	6.97	7.75	12.71	6.45	7.16
					$p = 0.995$				
$RB^R$	0.77	0.23	0.37	1.10	0.40	0.50	1.07	0.44	0.45
$ARB^R$	10.69	5.93	6.41	10.83	5.93	6.33	10.96	5.93	6.37
$RRMSE^R$	14.42	8.40	9.51	13.08	7.39	8.19	12.46	6.91	7.60

\*The  $RB^R$  and  $ARB^R$  are evaluated using  $RIE_i^{*[m]}(p)$  and the  $RRMSE^R$  is evaluated using  $RIE_i^{[m]}(p)$

Table 7: Estimation results (%) of 0.99-quantiles for regions with different sizes ( $N = 10, 15, 20, 50$  and  $100$  with  $n = 30$ )

	<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$QD_X$	$QD_Y$	<i>Biv.*</i>	$QD_X$	$QD_Y$
	<b><math>N = 10</math></b>			<b><math>N = 15</math></b>			<b><math>N = 20</math></b>			<b><math>N = 50</math></b>			<b><math>N = 100</math></b>		
<i>n = 30, Homog and p = 0.99</i>															
$RB^R$	-0.08	-0.10	-0.04	-0.07	-0.04	-0.07	-0.18	-0.08	-0.14	-0.24	-0.16	-0.11	-0.20	-0.11	-0.11
$ARB^R$	0.17	0.12	0.10	0.13	0.09	0.11	0.20	0.12	0.15	0.27	0.17	0.13	0.24	0.12	0.14
$RRMSE^R$	10.21	5.59	6.65	9.93	5.37	6.50	9.75	5.25	6.35	9.49	5.09	6.18	9.42	5.04	6.14
<i>n = 30, HetCo30 and p = 0.99</i>															
$RB^R$	0.21	-0.01	0.06	0.21	0.01	0.04	0.30	0.08	0.08	0.17	0.01	0.03	0.11	-0.01	-0.00
$ARB^R$	6.05	3.35	3.63	5.86	3.25	3.45	5.85	3.21	3.46	5.64	3.09	3.31	5.64	3.07	3.32
$RRMSE^R$	12.06	6.77	7.81	11.56	6.47	7.53	11.54	6.39	7.49	11.22	6.16	7.22	11.16	6.10	7.17
<i>n = 30, HetMa50 and p = 0.99</i>															
$RB^R$	1.01	0.23	0.36	1.17	0.35	0.41	1.10	0.35	0.39	0.94	0.21	0.32	0.89	0.24	0.26
$ARB^R$	10.81	5.62	6.05	10.58	5.48	5.86	10.36	5.32	5.74	10.16	5.20	5.59	10.06	5.13	5.53
$RRMSE^R$	15.17	8.32	9.45	14.76	8.02	9.12	14.57	7.89	9.01	14.16	7.61	8.71	14.04	7.53	8.61

\*The  $RB^R$  and  $ARB^R$  are evaluated using  $RIE_i^{*[m]}(p)$  and the  $RRMSE^R$  is evaluated using  $RIE_i^{[m]}(p)$

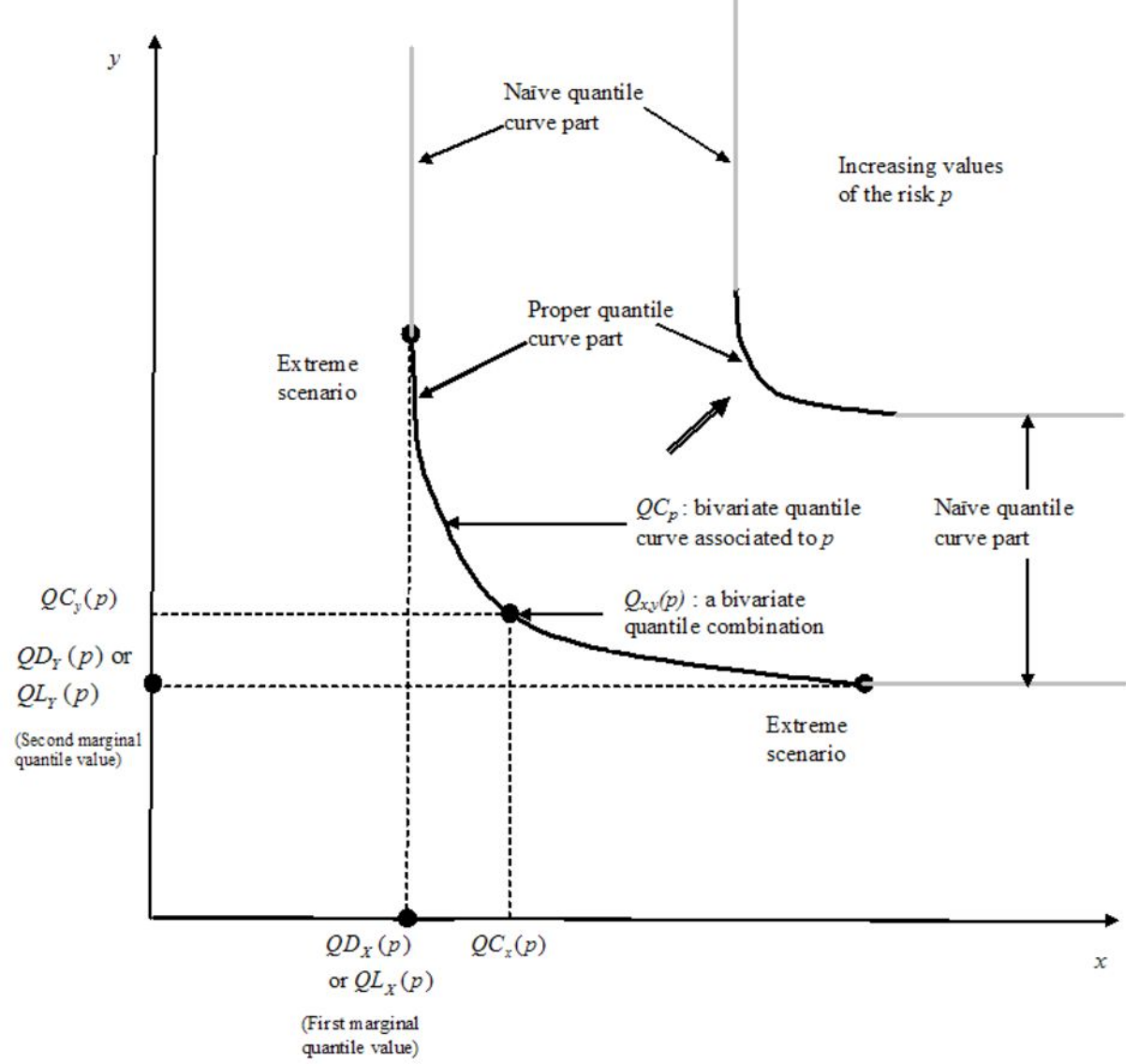
Table 8: Quantile estimation results (%) for regions with different heterogeneity degrees for  $p = 0.99$  with  $n= 30$  and  $N = 15$ .

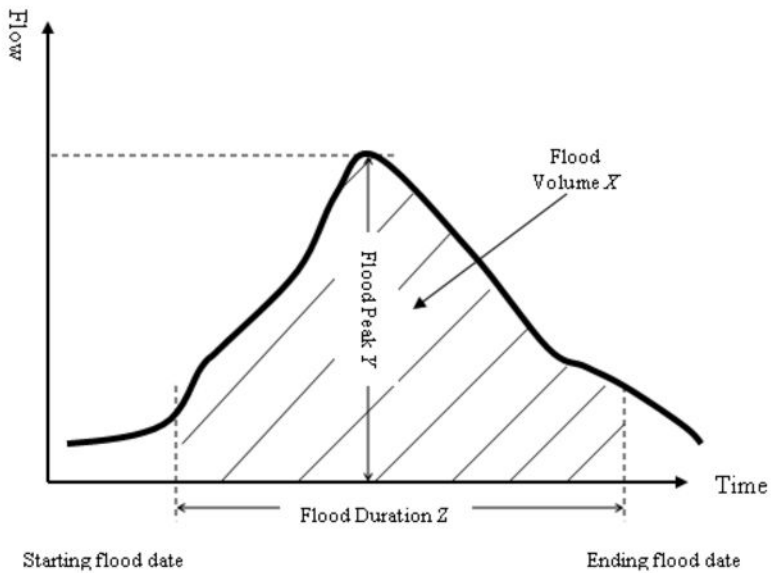
		Quantile estimation			Growth curve estimation		
		<i>Biv.</i> *	$QD_X$	$QD_Y$	<i>Biv.</i> *	$qD_X$	$qD_Y$
<b>Homog</b> ( $H$ mean $\approx 0$ )	$RB^R$	-0.07	-0.04	-0.07	-0.22	-0.06	-0.08
	$ARB^R$	0.13	0.09	0.11	0.22	0.06	0.08
	$RRMSE^R$	9.93	5.37	6.50	1.67	1.71	1.84
<b>HetCo30</b> ( $H$ mean = 1.30)	$RB^R$	0.21	0.01	0.04	-0.16	0.01	0.06
	$ARB^R$	5.86	3.25	3.45	3.48	3.25	3.50
	$RRMSE^R$	11.56	6.47	7.53	3.96	3.82	4.09
<b>HetMa50</b> ( $H$ mean = 1.36)	$RB^R$	1.17	0.35	0.41	0.00	0.33	0.40
	$ARB^R$	10.58	5.48	5.86	6.30	5.45	5.87
	$RRMSE^R$	14.76	8.02	9.12	6.63	5.87	6.31
<b>HetCo60**</b> ( $H$ mean = 3.23)	$RB^R$	1.45	0.44	0.56	0.24	0.47	0.58
	$ARB^R$	11.42	6.54	7.05	6.31	6.56	7.07
	$RRMSE^R$	15.78	8.88	10.01	6.66	6.93	7.47
<b>HetCo80**</b> ( $H$ mean = 5.29)	$RB^R$	2.65	0.96	1.11	0.43	0.91	1.07
	$ARB^R$	15.39	8.87	9.54	8.45	8.84	9.55
	$RRMSE^R$	19.08	10.83	12.04	8.73	9.13	9.86

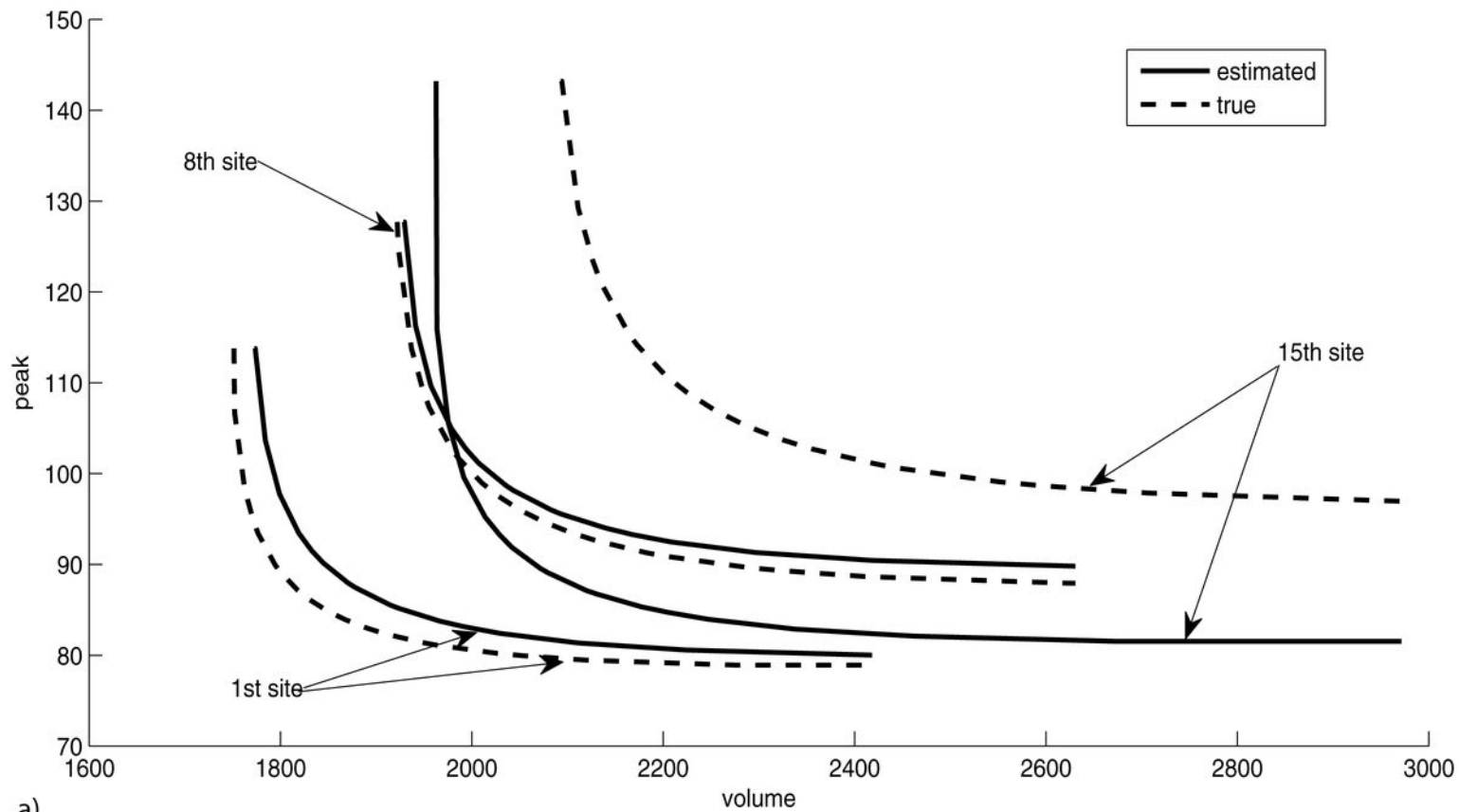
\*The  $RB^R$  and  $ARB^R$  are evaluated using  $RIE_i^{[m]}(p)$  and the  $RRMSE^R$  is evaluated using  $RIE_i^{[m]}(p)$

\*\*The dependence parameter for these regions is  $\gamma = 3.162$

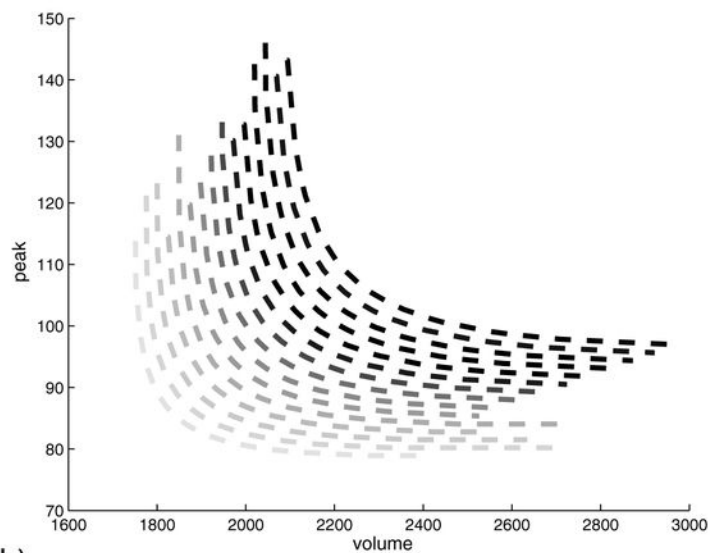




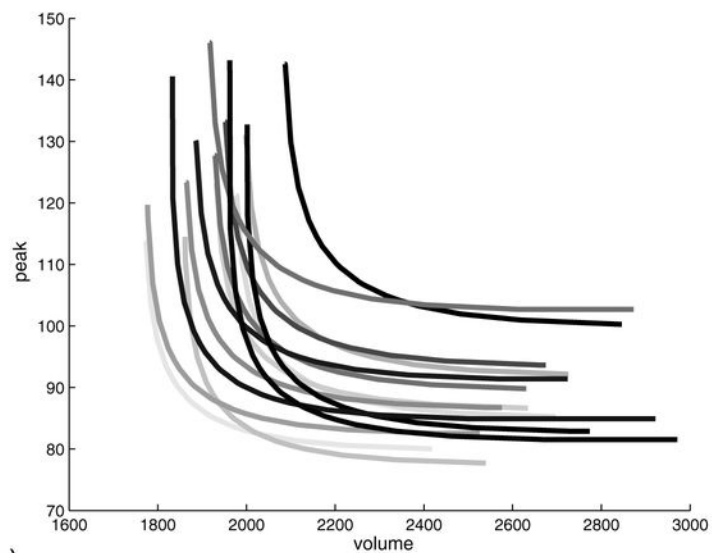




a)



b)



c)

