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CALIBRATION OF A MULTIVARIATE PARMA MODEL FOR THE OTTAWA RIVER SYSTEM

Report prepared for
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1 INTRODUCTION

The present report summarizes various aspects of the development and calibration of a multivariate PARMA model for the Ottawa River system, or, more precisely, for five regions in that system. The overall objective of the study is to develop a generator of daily simultaneous flows at 30 sites. The generation of a large number of multi-site flow sequences for input to current management models permits to study the reliability of the system, both in terms of hydropower production and in terms of adequacy of the hydraulic installations. The main interest in the current project is the reliability assessment of existing constructions such as dams, spillways, dikes, etc., vis-à-vis extreme floods. Floods occur over a relatively short period of time, in Quebec usually in the spring as a result of snow melt. This is why it is necessary to consider time-steps as small as one day. Since the PARMA type model is unsuitable for generating daily flows and also is practically limited by the number of sites that can be handled, it has to be combined with various disaggregation models. In the present report, only one component of the flow generator is considered, namely a 5-region, weekly PARMA model. Generated weekly regional flows will be disaggregated spatially to each site in the region and to a daily time step, but that part of the generator is not described here. The delineation of 27 gauged sites in the Ottawa River system into five regions has been carefully done with the emphasis on maximizing the statistically similarity of sites within regions. This work is described in Mathier et al. (1995). Although no particular attention was paid to the geographical location of the sites, the five regions turned out to be geographically contiguous. They will in the following be referred to as North West (NW), North East (NE), East (E), Central (C), and South (S) regions.

The distributions of aggregated weekly flows in the five regions have been carefully examined and transformed to normality (J. Grygier, personal communication). The results presented in the following deal only with the transformed data space. Although a good model performance in the transformed data space does not guarantee an equally good performance in real space, it is generally acknowledged that for a model to perform well in real space, it must do well in the transformed space. Hence, the objective of this part of the project will be to identify the most adequate model for the transformed data.

The models considered here are the class of multivariate PARMA(p,q) models (periodic autoregressive moving average). The data are assumed normalized and standardized to zero mean and unit variance, but even after removal of the periodic mean and variance, the data series may still exhibit periodicity in the week-to-week correlations. Therefore it is necessary to consider a model with time-varying parameters, such as the multivariate PARMA models, whose general form is:

$$\mathbf{x}_{v,\tau} = \sum_{i=1}^{p} \mathbf{\Phi}_{i,\tau} \mathbf{x}_{v,\tau-i} + \boldsymbol{\varepsilon}_{v,\tau} - \sum_{j=1}^{q} \mathbf{\Theta}_{j,\tau} \boldsymbol{\varepsilon}_{v,\tau-j} \tag{1}$$

The model relates the present flows at n sites (elements of vector **x**) to the p previous flows and to the q previous innovations. The model and its special cases will be described in detail later. In its general form, the parameter matrices $\mathbf{\Phi}_{i,\tau}$ and $\mathbf{\Theta}_{i,\tau}$ are allowed to be full. A substantial simplification can be obtained by assuming that these matrices are diagonal (Salas et al., 1980). This uncouples the equations and permits to model each (aggregated) site independently. The spatial dependence is introduced by generating innovation vectors with correlated elements. This type of model, commonly denoted contemporaneous, permits in principle to preserve explicitly the spatial correlation of flows at lag 0, whereas there is no explicit provision for preserving correlations at higher lags. However, contemporaneous models have been used in several studies and are generally found to yield good results.

The periodicity of the parameters and statistics related to them introduces some difficulties in identifying the appropriate model order. Classical identification techniques for stationary Box-Jenkins ARMA models are not directly applicable to seasonal models. The autocorrelation function and the partial autocorrelation function, which are the usual tools for identifying the orders of stationary models, are meaningless when seasonality in the model parameters is present. One can gain some insight by looking at the correlations between periods, but in the case of weekly flows, an exhaustive analysis would be very tedious. Moreover, one cannot expect to arrive at a unique conclusion as to which values of p and q should be used, since generally the correlation pattern depend on the period. The approach taken here is the trial-and-error method. Some a priori chosen models are fitted to the observed data and their performances are evaluated and compared. With the limited data available for the Ottawa River, it is suggested that models beyond PARMA(2,2) should not be considered. The PARMA(2,2) model defines a class consisting of PARMA(p,q) models with $\max\{p,q\} \le 2$. This class comprises among others the popular PARMA(1,0) and PARMA(1,1) models, as well as the PARMA(2,1) model. The PARMA(1,1) model is generally found to perform better than the PARMA(2,0) model which is the reason why the latter is not used in this study. PARMA(2,1) models are generally preferable to PARMA(1,2) model, and only the former is considered here. Hence, four PARMA models constitute the group of candidates to be examined in this study.

The first part of this report describes properties of the univariate model. Chapter 2 presents the criteria on which we based the model selection. Chapter 3 describes the three different estimation methods that were considered in this study and a comparison between them. Chapter 4 contains a brief description of the program CSU5 which eventually was used to calibrate the PARMA models. In Chapter 5, we present the results of the calibration of the univariate models.

In the second part of the report, the spatial dimension of the multivariate model is considered. The method of moments was used to estimate the cross-covariance matrices of residuals. This new method for calibrating higher-order contemporaneous models is described in detail in Chapter 6. In Chapter 7, we present the results of the spatial estimation. A few concluding remarks are given in Chapter 8.

2 MODEL SELECTION CRITERIA

In the following, we consider the case of univariate PARMA models and the problem of determining the appropriate model order. As mentioned in the introduction, most of the classical methods for model identification do not apply to periodic models, so one usually has to base the choice of model order on a trial-and-error search. The parameters of each considered model is estimated, and one may compute various statistical properties of interest and perform a global comparison of the involved models using the historical data series as reference. Some of the properties that one would usually examine are the periodic means, variances, and period-to-period correlations (periodic autocorrelation). In the transformed data space, the periodic means can usually be reproduced exactly (identical to the historical)[1], whereas the periodic variances and autocorrelation may be more or less close to the historical values used to calibrate the model. Commonly, model properties are obtained by generating long series of flow data. Especially if data have been re-transformed to real space, this is the most straightforward method for deriving the statistical properties of the model. However, for model development, in particular the selection of model order, it may suffice to examine the statistical properties in the transformed data space. Generally, one cannot expect a model to perform well in real space, if it fails to perform well in the transformed space (Stedinger, 1981). In this study, an analytical technique, based on the periodic Yule-Walker equations, are used to compute the periodic variance and autocorrelation in the transformed data space. This technique is described below.

The univariate PARMA(2,2) model relates the present flow to preceding flows and innovations by the following functional relationship

$$x_{v,\tau} = \sum_{i=1}^{2} \phi_{i,\tau} x_{v,\tau-i} + \varepsilon_{v,\tau} - \sum_{j=1}^{2} \theta_{j,\tau} \varepsilon_{v,\tau-j} \qquad (2)$$

where $x_{v,\tau}$ represents the normalized and standardized flow in year v, period τ, $\phi_{i,\tau}$ are autoregressive parameters depending on the specific period of the year, and $\theta_{i,\tau}$ are moving average parameters, also depending on the period. It is assumed that there are ω periods in the year. Due to the normalization of the flows, the innovations, $\varepsilon_{v,\tau}$, are normally distributed. Moreover, since the data are assumed standardized to zero mean and unit variance, the innovations also have zero mean. The variance of the innovations is denoted

[1]Here the term "reproduced exactly" does not imply that each generated series has the same mean as the historical series, but rather that the expected value of the mean is identical to the historical mean. Same comment applies to the variance and correlations.

g_τ and generally depends on the period. Due to the limited amount of data available for calibration, it may be appropriate to consider also models of lower order. Popular submodels are the PARMA(1,0) and PARMA(1,1), which in many cases provide a satisfactory description of the correlation structure of observed flows. However, when the considered time scale is very short (as for example in the case of weekly flows), the seasonal autocorrelation structure may exhibit irregularities which cannot be adequately captured by low-order PARMA models. This is why we adopt the PARMA(2,2) as a general class of models that comprises itself and any submodel, i.e. any PARMA(p,q) model with $\max\{p,q\} \le 2$.

The values of the parameters of a PARMA(p,q) model uniquely define the covariance structure of the model through the so-called periodic Yule-Walker equations which for the PARMA(2,2) model read (Appendix A):

$$\begin{aligned} m_\tau(0) = {} & \phi_{1,\tau} m_\tau(1) + \phi_{2,\tau} m_\tau(2) + g_\tau - \theta_{1,\tau} g_{\tau-1}\left[\phi_{1,\tau} - \theta_{1,\tau}\right] \\ & - \theta_{2,\tau} g_{\tau-2}\left[\phi_{1,\tau}\phi_{1,\tau-1} - \phi_{1,\tau}\theta_{1,\tau-1} + \phi_{2,\tau} - \theta_{2,\tau}\right] \end{aligned} \tag{3a}$$

$$m_\tau(1) = \phi_{1,\tau} m_{\tau-1}(0) + \phi_{2,\tau} m_{\tau-1}(1) - \theta_{1,\tau} g_{\tau-1} - \theta_{2,\tau} g_{\tau-2}\left[\phi_{1,\tau-1} - \theta_{1,\tau-1}\right] \tag{3b}$$

$$m_\tau(2) = \phi_{1,\tau} m_{\tau-1}(1) + \phi_{2,\tau} m_{\tau-2}(0) - \theta_{2,\tau} g_{\tau-2} \tag{3c}$$

$$m_\tau(k) = \phi_{1,\tau} m_{\tau-1}(k-1) + \phi_{2,\tau} m_{\tau-2}(k-2) \qquad k > 2 \tag{3d}$$

where the periodic autocovariance function is defined as $m_\tau(i) = E\left[X_\tau\, X_{\tau-i}\right]$. The above equations are valid for any submodel of the PARMA(2,2) by setting particular parameters equal to zero. For instance, the periodic Yule-Walker equations corresponding to a PARMA(2,1) are obtained by setting $\theta_{2,\tau}$ equal to zero. The periodic Yule-Walker equations serve an important purpose by allowing a fast and straightforward calculation of the periodic variance and the periodic autocorrelation corresponding to a given estimated model. Equation (3c) can be used to remove $m_\tau(2)$ from (3a). After some manipulations, the first two periodic Yule-Walker equations can be written

$$\begin{aligned} & -\hat{\phi}_{2,\tau}^2 m_{\tau-2}(0) + m_\tau(0) - \hat{\phi}_{1,\tau}\hat{\phi}_{2,\tau} m_{\tau-1}(1) - \hat{\phi}_{1,\tau} m_\tau(1) \\ & \quad = \hat{g}_\tau - \hat{\theta}_{1,\tau}\hat{g}_{\tau-1}\left[\hat{\phi}_{1,\tau} - \hat{\theta}_{1,\tau}\right] - \hat{\theta}_{2,\tau}\hat{g}_{\tau-2}\left[\hat{\phi}_{1,\tau}\hat{\phi}_{1,\tau-1} - \hat{\phi}_{1,\tau}\hat{\theta}_{1,\tau-1} + 2\hat{\phi}_{2,\tau} - \hat{\theta}_{2,\tau}\right] \end{aligned} \tag{4a}$$

$$-\hat{\phi}_{1,\tau} m_{\tau-1}(0) - \hat{\phi}_{2,\tau} m_{\tau-1}(1) + m_\tau(1) = -\hat{\theta}_{1,\tau}\hat{g}_{\tau-1} - \hat{\theta}_{2,\tau}\hat{g}_{\tau-2}\left[\hat{\phi}_{1,\tau-1} - \hat{\theta}_{1,\tau-1}\right] \tag{4b}$$

where "^" is used to designate particular estimates. It is seen that the above equations constitute a linear system of 2ω equations with 2ω unknowns, namely $m_\tau(0)$ and $m_\tau(1)$ for $\tau = 1,2,..,\omega$. The system is readily solved for $m_\tau(0)$ and $m_\tau(1)$, and the complete periodic autocorrelation function can then be constructed from (3c-d). It is useful to emphasize that the implementation of (4a-b) also permits to evaluate the properties of any submodel of PARMA(2,2) by equating particular parameters to zero.

The above procedure provides an effective means to compare different model options and to guide in the selection of the most adequate. For example, one may wish to compare a PARMA(1,1), a PARMA(2,1), and a PARMA(2,2) for modeling weekly data at a given location. The first step is to obtain parameter estimates for each model considered, for example by one of the methods described in the next section. Then the periodic variances and autocovariances are computed from (4a-b) and plotted along with the corresponding historical values and a graphical comparison can be made.

The above procedure constitutes the main basis for our model selection. There are other properties that could be examined as well, including stationarity conditions and whiteness of the residual series.

3 ESTIMATION METHODS

3.1 Introduction

Prior to undertaking the task of estimating and identifying the appropriate models for the five regions in the Ottawa River, a preliminary analysis of different estimation techniques was made. This analysis led to the conclusion that the method of least squares, as implemented in the software CSU5 developed by the research group of J. Salas at Colorado State University, provided results that were acceptable to all participants in the project, and the CSU5 was consequently selected and used for model calibration. However, for completeness, the three estimation alternatives we examined are briefly described here.

3.2 Estimation by the method of least squares

The LS estimators are the set of parameters that minimizes the sum of squared residuals, i.e. (assuming a PARMA(p,q)-model)

$$\{\phi_{i,\tau},\theta_{j,\tau}\} \quad \text{for which} \quad \sum_{v=1}^{N}\sum_{\tau=1}^{\omega}\left[x_{v,\tau}-\sum_{i=1}^{p}\phi_{i,\tau}x_{v,\tau-i}+\sum_{j=1}^{q}\theta_{j,\tau}\varepsilon_{v,\tau-j}\right]^2 \quad \text{is minimum}$$

The estimate of the residual variance, g_τ, is obtained directly from the series of residuals. Usually one omits the first p data from the summation, because the p preceding values are unknown (Box and Jenkins' back forecast method does not apply to periodic series). The first q innovations are commonly set to zero. The LS method is fairly straightforward, but very computer intensive. The LS method as implemented in CSU5 is described in more detail in the next section.

3.3 Estimation by the method of moments

The moment estimators of the parameters of a PARMA(p,q) model are the solution to the first $p+q+1$ periodic Yule-Walker equations in which the periodic autocovariance function is computed from the data. Hence, if a moment solution exists, the corresponding model preserves exactly the variance plus the lagged correlations up to order $p+q$. For low-order PARMA-models, a moment solution is fairly straightforward (Salas et al., 1982), but if the order of the moving average component of the model exceeds one, then an analytical solution is not available, and even with numerical means, it seems impossible to obtain a solution (Bartolini and Salas, 1993). Therefore, in our study moment solutions were restricted to PARMA(1,0), PARMA(1,1), and PARMA(2,1) models. Unfortunately, in the

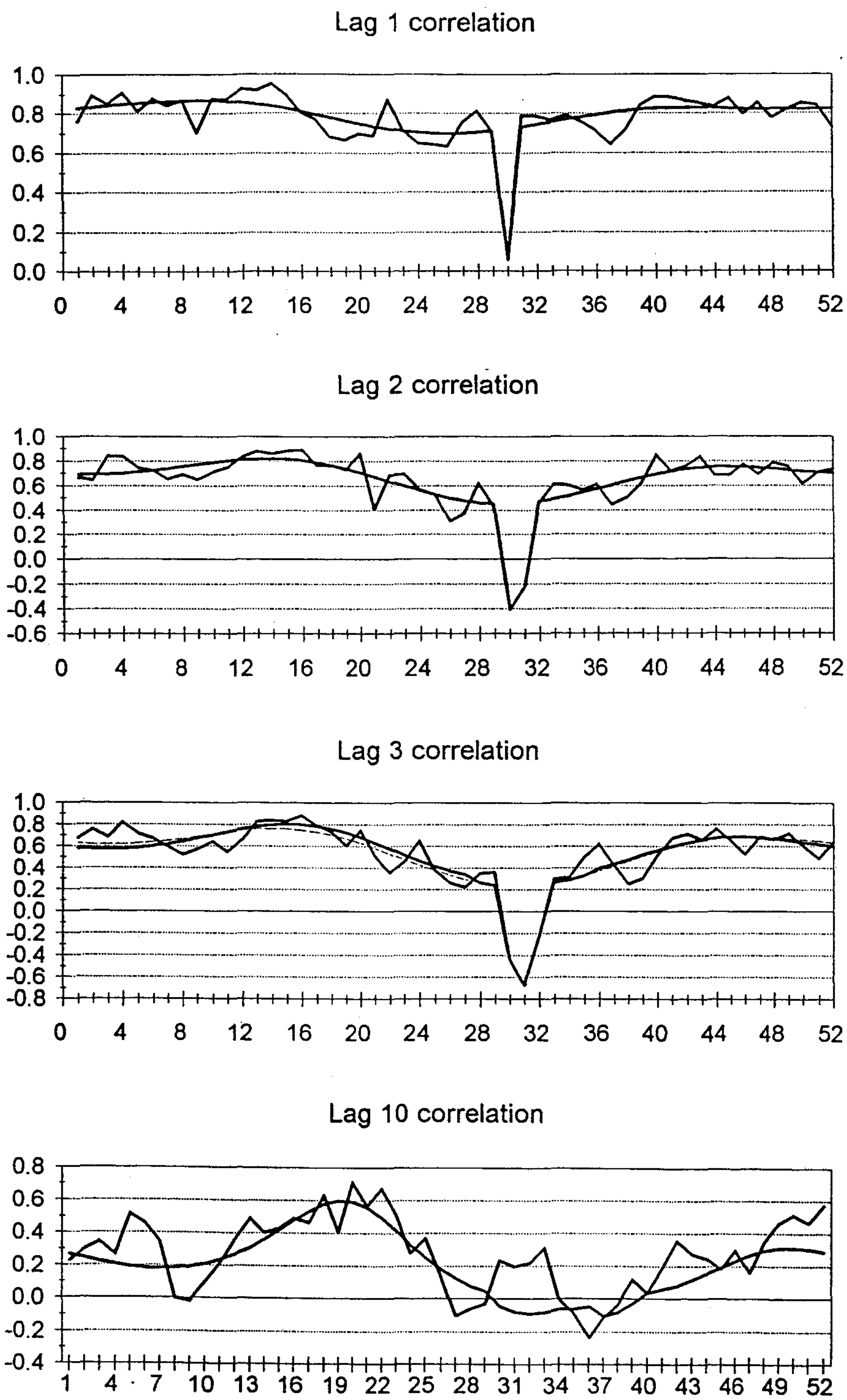


Figure 3.1 Periodic covariance function of a moment estimated PARMA(1,1) model for the Central region

case of PARMA(1,1) and PARMA(2,1) models a moment solution may not exist. When calibrating the models, some of the residual variances may turn out negative, indicating that there is no feasible solution to the periodic Yule-Walker equations. When applying the method of moments to the Ottawa River data, it was found necessary to smooth the periodic autocorrelation in order to obtain feasible solutions. A Fourier analysis showed that two harmonics were generally enough to describe adequately the periodic autocorrelations, except for the spring flood period. Hence, in our approach we first replaced the data during the flood period with typical values prior to and after the flood period. The coefficients of the first two harmonics were determined. Then the irregular correlations in the flood season were inserted in the smoothed periodic autocorrelation function, and the resulting function was used to calibrate the PARMA-model.

The above approach was applied to the data from the five regions in the Ottawa River. Results from fitting a PARMA(1,1) to the Central region is shown in Figure 3.1. The variance, and the lags 1 and 2 of the transformed data are reproduced exactly (i.e. identical to the historical, transformed series.) It is particularly interesting to study the performance of higher order lags which are not explicitly preserved. It is seen that up to at least lag 10, the PARMA(1,1)-model provides an excellent description of the correlation structure.

One obvious advantage of smoothing the periodic autocorrelation functions is that the number of independent parameters can be substantially reduced. For example, a PARMA(1,1)-model contains 3ω-parameters, which, in the case of weekly data, means 156 parameters. Since the fitting of parameters to smoothed periodic autocorrelations functions results in smoothed parameters, one can reduce the number of "independent" parameters to two times the number of harmonics considered plus the number of original correlations used during the flood period (in addition to the means, variances, and transformation parameters).

3.4 Estimation using the MATLAB/ARMAX function

A third method for estimating the parameters of PARMA(2,2) models was suggested and developed by Mrs. L. Fagherazzi, Hydro-Quebec. It is based on the ARMAX subroutine in MATLAB's System Identification Toolbox. The key idea of this approach is to estimate the parameters of each season independently of the other seasons, ignoring the functional relationship between the moving average parameters of adjacent periods, but allowing for a fast and simple estimation. The first step is to sequentially extract one column from the data matrix given by:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1\omega} \\ x_{21} & x_{22} & x_{23} & & x_{2\omega} \\ \vdots & & & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{n\omega} \end{matrix}$$

The two preceding columns are used as exogenous input series (some adjustment is needed for the first two periods), and the series of flows in a particular period, with length equal to the number of years available for the study, is modeled by a moving average process of order 2. The estimation is based on the method of least squares.

It was found that the estimates of the autoregressive parameters is equivalent to those one would obtain by fitting a traditional PAR(2) model with either the method of moments or the method of least squares (autoregressive parameters generally do not pose any estimation problems). The meaning of the moving average parameters is somewhat obscure, since they do not relate directly to the original time series, but rather to the series of one period's flow over n years in which there is no significant autocorrelation

3.5 Comparison of estimation methods

In order to choose the most satisfactory estimation method, a simulation study was carried out, in which the three methods described above were used to calibrate the data series for the South region. The performance of each method was evaluated by generating, for each of the three estimation methods and for various model orders, 1000 series of 23 years of weekly data (corresponding to the length of the historical series) and by computing various statistics in the real data space. These statistics comprised periodic means, variances, lag 1 week-to-week correlations, annual mean, annual maximum, and the distribution of the annual maximum. It was found that in certain periods, the re-transformation had a significant impact on the preservation of statistics. It is well-known that a good preservation of correlations in the transformed data space does not guarantee a good preservation of correlations in real space. However, in principle it should be possible to preserve the periodic real means exactly, but some departure was observed in the beginning of the year. This can only be ascribed to the transformation. Also the standard deviation was in some cases way off (Figure 3.2). Figure 3.3 shows the average lag 1 week-to-week correlation for a PARMA(1,1)/LS, a PARMA(1,1)/MOM, and a PARMA(2,2)/MATLAB model. Not

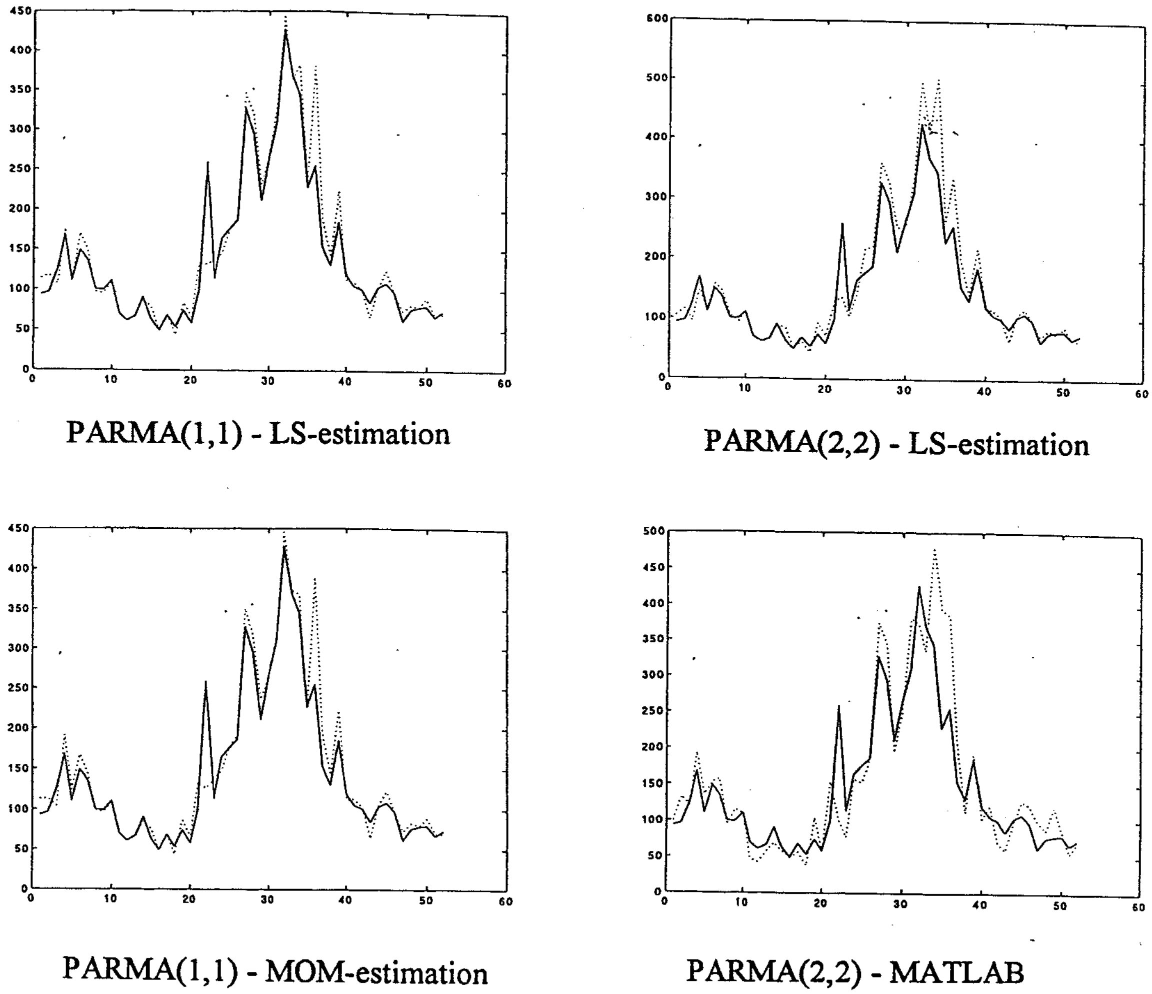
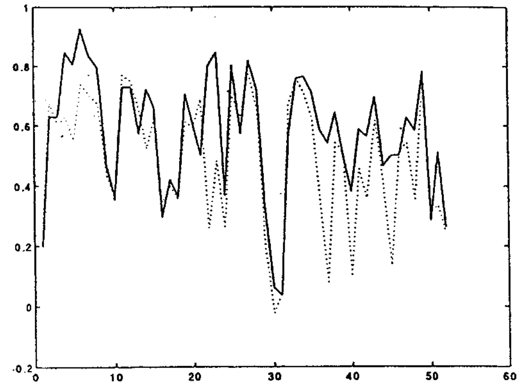
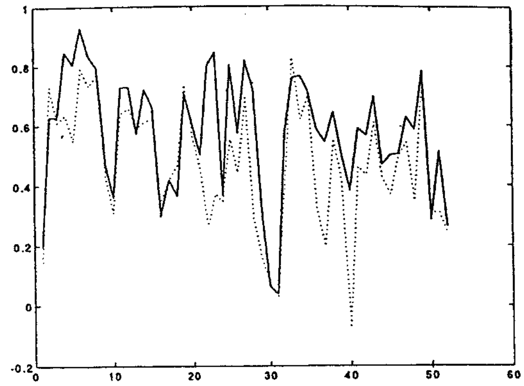


Figure 3.2 Periodic standard deviation of simulated flows for the South region. (dotted lines represent simulated values)

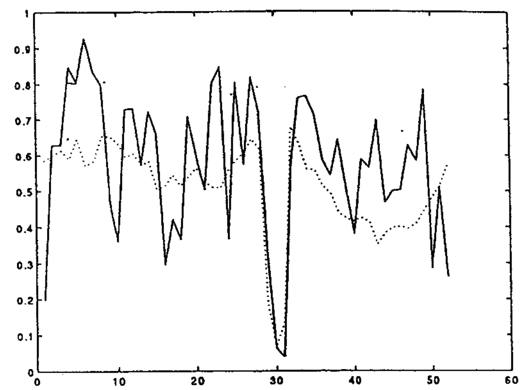
surprisingly, it is seen that the MATLAB-method results in a poor preservation of the correlation structure. Because of the smoothing of correlations in the case of estimation by the method of moments, the generated mean correlations are also fairly smooth as opposed to the model based on LS-estimation, in which the periodic real correlations fluctuate, sometimes very close to, but other times quite away from the historical value. The choice between the LS-method and the MOM-method is not evident. After a careful study of the different statistical characteristics of series generated with the three methods, it was decided to adopt the LS-method for estimating the parameters of the five regional series in the Ottawa River.



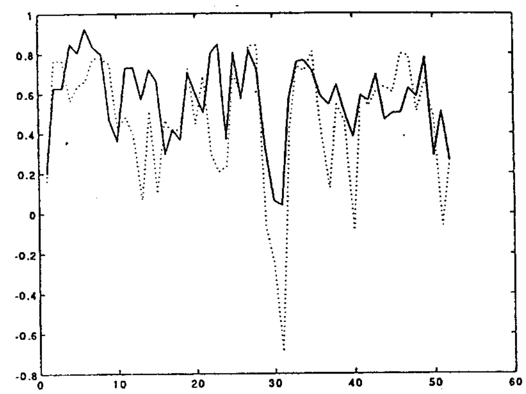
PARMA(1,1) - LS-estimation



PARMA(2,2) - LS-estimation



PARMA(1,1) - MOM-estimation



PARMA(2,2) - MATLAB

Figure 3.3 Lag 1 week-to-week correlation of simulated flows for the South region (dotted lines represent simulated values)

4 DESCRIPTION OF CSU5

CSU5 is a computer software developed at Colorado State University by J. Salas and others for calibration of PARMA models. The program permits to estimate the parameters of several types of periodic models, including the PARMA(p,q). The program was originally written for application to monthly flows, so the code had to be revised for the current project, in which weekly data are used.

Parameters are estimated by the LS method, briefly described in the preceding chapter. A method of moment solution is used as starting point in the search for a minimum. Initially, the program attempts to estimate a PARMA(1,1)-model by the method of moments. If a solution does not exist, a PARMA(1,0) moment solution is used as starting values. With an initial estimation of the parameters, the Powell method of direct search is invoked to find the set of parameters that minimizes the sum of squared residuals. The p flows and q innovations preceding the first data (year 1, period 1) is set to zero, and the objective function to be minimized is therefore a sum of $n\omega$ terms, where n is the number of years. The search terminates when a user-specified accuracy has been attained. The variance of the residual series is used as estimator of g_τ.

The program provides a variety of outputs such as periodic means and variances of the input series and of the residual series, periodic autocorrelation structure, and statistics of the aggregated annual flows. The program also tests for whiteness of residuals and for stationarity of the solution. The program, however, does not provide output of the periodic autocovariance function corresponding to the solution. Using MATLAB, we implemented a routine for calculating this important property following the procedure described in Chapter 2.

When estimating the parameters of PARMA(2,2)-models, the program turned out to be quite sensitive to the particular computer on which it was run, and also to the Fortran compiler used to compile the source code. A series of test runs based on the same monthly data set were made on different computers and with different compilers in order to examine the differences in the solutions. Although the estimated parameters differed substantially, the periodic autocovariance structures of the estimated models were quite similar, and it was consequently found impossible to conclude that any particular computer/compiler combination was significantly superior to the others. Since no particular preference could be attributed to a specific computer, it was decided to use a PC, on which the manipulation of data is more tractable than on a mainframe. The instability in the solution can be explained by the high dimensions of the PARMA(2,2)-model. In fact, the objective function supposedly is very flat around the minimum, indicating that a large number of solutions may yield virtually identical results in terms of periodic autocovariance structure. The use of high-order models represents one particular point of view in modeling, namely that the large number of parameters and the corresponding instability in the solution is unimportant as long as the covariance structure of the model reasonably well describes the observed historical correlations.

5 RESULTS OF SINGLE-SITE ESTIMATION

5.1 Introduction

As mentioned in the introduction, our approach to estimating the parameters of the univariate PARMA-models is the trial-and-error method. Four models, namely the PARMA(1,0), PARMA(1,1), PARMA(2,1), and PARMA(2,2), were considered. The main emphasis was put on finding the model whose temporal covariance structure was closest to the historical (transformed data). Figure 5.1 shows the observed lag 1 to 4 week-to-week correlation of weekly flows at the Central in the Ottawa River system. Note that around the spring flood period, there is a significant drop in the correlation. The same drop is seen in the lag 2, 3, and 4 week-to-week correlation and can easily be explained. In Quebec, there is usually one big spring flood each year occurring around April-May and extending over a period of a few days. The fact that the spring flood extends over a period of the same order of magnitude as the time scale of the considered flows (weekly) and that the flood season (i.e. the period in which the spring flood is likely to occur) on the other hand extends over 4-5 weeks give rises to the negative lagged correlations. If the flood does not occur in the first half of the flood season, then it will occur in the second half, and vice-versa. Hence, there will be a tendency that in the flood season of anyone year, one week has a large flow while the others have small flows compared to their means, i.e. a negative correlation.

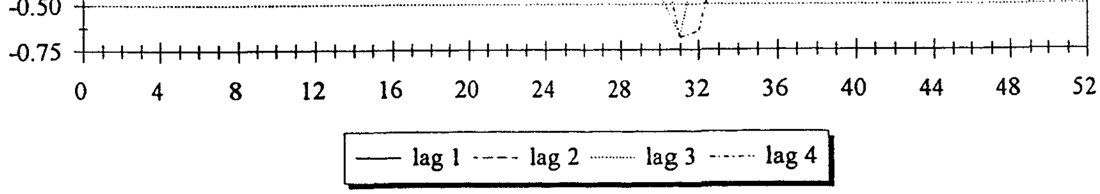


Figure 5.1 Observed lag 1 to 4 week-to-week correlation in the Central region.

As the emphasis of the present study is on the generation of extreme flows, particular attention was paid to a fair modeling of correlations during the flood season. In fact, the negative correlations appearing in the first lagged week-to-week correlations should be reasonably well reproduced in order to generate realistic flood scenarios. This may imply severe requirements to the flow generation model. For example, the PARMA(1,0) was found unable to reproduce the observed correlations in a satisfactory way, and results for that model are not presented.

Some general remarks on the results from the CSU5 program are appropriate here. It was generally impossible to obtain a feasible PARMA(1,1) moment solution as starting point for the least square search algorithm, so a PARMA(1,0) was used instead. Estimation of PARMA(2,1) and PARMA(2,2) parameters on a PC typically took two to three hours depending on the number of years available for the site. The hypothesis of normality of residuals was always rejected. This must be ascribed to the data transformations which do not always result in normally distributed input data. Likewise, the Anderson tests of uncorrelated residuals were also rejected. There is no exact test for whiteness when the correlations structure of the data is periodic, and when applied to weekly data, the Anderson test is too powerful for practical application (Tao and Delleur, 1976, p. 1548). We decided to ignore the problem of some autocorrelation in the residual series, and also the problem of non-normality. The latter issue might be of some concern, but as it is related to the transformation of flows, it falls outside the scope of the present work.

The number of years available for the analysis is shown in Table 5.1. The fact that the series are relatively short explains much of the fluctuations in the observed periodic statistics.

Table 5.1 Length of data series

NO	1960-1989	30 years
NE	1961-1989	29 years
E	1968-1989	22 years
C	1965-1991	27 years
S	1967-1989	23 years

Results of the fitting of three PARMA-models, PARMA(1,1), PARMA(2,1), and PARMA(2,2) with the method of least squares (program CSU5) are presented in Appendix C. The periodic autocorrelation structure of each model has been computed according to the algorithm described in Chapter 2 (the MATLAB code is presented in Appendix B). A series of figures were prepared, each corresponding to a given lagged periodic week-to-week correlation. More specifically, we considered the variance, and the lagged correlations of order 1, 2, 3, 4, 5, and 10. When comparing the three PARMA models most attention was given to the variance, followed by the lag 1 week-to-week correlation, followed by the lag 2, and so forth. Furthermore, we paid special attention to the statistical properties during the flood season, since badly represented statistics during this period may result in flood scenarios that deviate significantly from the historical (in an average sense).

5.2 North East region

From the figures in Appendix C1, it is seen that the PARMA(1,1) preserves the variance of the transformed, standardized flows very well. In the first few periods it deviates from 1, probably due to the initialization of the first residual in the LS-estimation algorithm. The variance of both PARMA(2,1) and PARMA(2,2) deviates significantly from 1. Deviations of 20-25% are observed in some periods. In the critical flood season (week 26-35) the PARMA(2,2) seems to deviate more than the PARMA(2,1) from 1. For correlations up to lag 4, PARMA(2,1) and PARMA(2,2) do well, whereas the PARMA(1,1) does not satisfactorily described the critical correlations during the flood season. The PARMA(2,1) seems to do slightly better than the PARMA(2,2) in describing the lagged correlation. The PARMA(2,1)-model was therefore selected as the most adequate model for the North East region.

5.3 North West region

For the North West region, the variance is best reproduced by the PARMA(1,1)-model. The PARMA(2,2) generally does a better job than the PARMA(2,1) in reproducing the periodic variance, and notably in the critical flood season. As for the lagged correlations, the PARMA(1,1) fails to reproduce satisfactorily the correlations during the flood periods, whereas the other two models essentially do equally well. In the light of these observations, it was decided to choose the PARMA(2,2)-model for the North West region.

5.4 East region

Again the PARMA(1,1) preserves the variance almost exactly. The PARMA(2,1) and PARMA(2,2) fluctuates quite much and seem to be equally good (or bad). The same thing can be said about the lagged correlations, where there is little difference between the two. The PARMA(1,1) also here fails to reproduce the negative correlations during the flood season. Since the two higher order models yield quite similar results, the PARMA(2,1)-model is selected in accordance with the principle of parameter parsimony.

5.5 Central region

For the Central region, the variance of the PARMA(2,2) deviates up to 70% from the historical, transformed variance during several weeks around and following the flood season, and it was therefore excluded. Both PARMA(1,1) and PARMA(2,1) do a good job in describing the variance and the lagged correlations, although for the last property, the PARMA(2,1) seems slightly superior. It is therefore selected as the appropriate model.

5.6 South region

Historical statistics for the South region are characterized by large fluctuations. The PARMA(2,2) yields a relatively poor representation of the variance of the transformed process, the PARMA(1,1) does excellently, and the PARMA(2,1) is somewhere in between. As for the lagged correlations, the PARMA(2,2) seems worst, the PARMA(1,1) seems better, and the PARMA(2,1) seems best, although the difference between PARMA(1,1) and PARMA(2,1) is small. As a whole, it seems that the PARMA(1,1) provides a reasonable description of the data for the South region.

5.7 Summary of results

The estimated model parameters for the five sites are listed in Table 5.2

Table 5.2 Estimated PARMA(p,q)-parameters for the five regions in the Ottawa River.

	North East Region				North West Region				
τ	$\phi_{1,\tau}$	$\phi_{2,\tau}$	$\theta_{1,\tau}$	g_τ	$\phi_{1,\tau}$	$\phi_{2,\tau}$	$\theta_{1,\tau}$	$\theta_{2,\tau}$	g_τ
1	1.2740	-0.1320	0.6600	0.3540	1.8640	-0.9000	1.5360	-0.1810	0.2256
2	1.9950	-0.9790	1.0100	0.1475	0.3230	0.5210	-0.4530	0.4190	0.3457
3	0.1680	0.6240	-0.8470	0.2520	1.2140	-0.2940	0.4090	0.1570	0.3238
4	1.8280	-0.7440	1.3200	0.1739	0.0940	0.6710	-0.6090	-0.1640	0.1781
5	0.6260	0.2450	0.1710	0.2256	1.4960	-0.6230	0.7760	0.0500	0.3588
6	-1.8100	2.2770	-2.7980	0.2043	0.4040	0.2170	-0.5730	-0.3430	0.3047
7	1.2780	-0.4300	0.2020	0.1568	0.7360	0.1790	-0.2830	0.2680	0.2025
8	1.0630	-0.1330	0.0420	0.1204	0.6230	0.2120	0.5130	-0.3210	0.1714
9	1.0300	-0.1830	0.4010	0.3341	0.3000	0.6150	-0.3790	0.4140	0.2266
10	1.2480	-0.2840	0.4730	0.2611	1.6960	-0.7430	1.2070	-0.5270	0.2218
11	1.5960	-0.5560	0.8130	0.1665	1.6550	-0.7700	1.3580	-0.3450	0.4020
12	1.3440	-0.3430	0.6890	0.1467	0.4470	0.4970	-0.3800	0.4110	0.1289
13	0.6070	0.1860	0.1190	0.3994	0.2540	0.6530	-0.6570	0.2800	0.2581
14	2.2870	-1.0200	1.4860	0.1592	2.4320	-1.3050	1.7710	0.1060	0.3192
15	1.4650	-0.5870	1.0900	0.3588	1.6670	-0.6260	1.2890	0.1580	0.3058
16	0.7100	0.1780	0.2290	0.3636	0.5130	0.4510	0.4100	0.1180	0.3636
17	1.2230	-0.3680	0.2030	0.3422	2.1960	-1.5610	2.1810	-1.7820	0.5155
18	1.2420	-0.4630	0.6290	0.5055	1.8700	-0.6370	1.7940	-0.8600	0.3869
19	-0.4920	0.8380	-1.2470	0.4610	0.8890	0.0260	1.0620	-0.1400	0.4225
20	-0.7500	0.9730	-1.5340	0.4706	0.7130	0.1930	0.6660	-0.2180	0.4900
21	2.8560	-1.5760	2.5200	0.4369	-0.3020	0.3960	-0.8130	-0.3670	0.6241
22	1.1670	-0.2220	0.9310	0.3516	0.4860	0.5640	-0.0820	0.7760	0.5329
23	1.0630	-0.0630	1.0440	0.4147	1.0400	0.1030	0.6300	0.3080	0.5213
24	0.4550	0.1310	0.1100	0.7430	0.3100	0.0390	-0.3790	-0.1180	0.6889
25	-2.0930	1.4150	-2.4260	0.6806	-0.0930	0.5700	-0.6070	-0.0730	0.5184
26	0.3690	0.1930	-0.1720	0.6675	-1.2690	1.9660	-1.9950	1.2490	0.4277
27	1.6820	-0.4610	0.8050	0.2034	0.9060	0.0010	0.0460	0.1750	0.3192
28	0.2730	0.5430	-0.7460	0.1576	-0.0100	0.5310	-0.9370	-0.3710	0.2228
29	2.1870	-1.2460	1.1980	0.2520	1.0970	-0.3210	0.3110	0.1510	0.5155
30	-1.8380	1.9690	-2.8480	0.4556	0.8900	-0.6260	0.0850	0.3740	0.7090
31	0.3810	-0.4470	-0.5960	0.4665	-1.9970	-0.8520	-2.5590	-2.6720	0.3624
32	0.7640	-0.4980	-0.1420	0.3770	0.9630	-0.2540	-0.2170	0.1880	0.1673
33	0.6940	0.0970	-0.2920	0.2430	1.0860	-0.1770	-0.1860	0.2800	0.2510
34	2.1950	-1.1550	1.3220	0.2162	0.5920	0.1520	-0.4960	0.0220	0.2905
35	-0.6910	1.3030	-1.6810	0.2714	1.5970	-0.7490	0.5740	0.0820	0.3446
36	1.2670	-0.4220	0.4230	0.3295	-0.7490	0.8250	-1.7700	-0.5980	0.4160
37	-0.7520	1.2650	-1.5910	0.3147	1.9560	-0.6290	1.1510	0.6210	0.3036
38	1.8850	-1.1360	0.8410	0.3697	0.0830	0.5060	-0.8270	-0.2840	0.2777
39	0.7310	0.0320	-0.2360	0.2852	1.3710	-0.4650	0.5900	0.1760	0.3648
40	1.1720	-0.4980	0.1110	0.4290	2.0230	-0.7540	1.2370	0.5340	0.1122
41	1.0850	-0.1440	0.4610	0.3672	1.8880	-1.0070	1.6320	-0.1790	0.2200
42	0.3940	0.3680	-0.5310	0.2323	0.4730	0.4100	-0.3850	0.8870	0.2510
43	1.7550	-0.8370	1.3040	0.4369	0.7280	0.0600	0.0760	-0.0370	0.4134
44	0.5250	0.1870	-0.3840	0.3080	1.0320	-0.0530	0.1650	0.8060	0.2256
45	-1.3910	1.6280	-2.7000	0.2228	1.0290	-0.1620	0.2980	0.2390	0.4638
46	1.0560	-0.2510	0.0720	0.2116	1.8290	-0.7480	0.9720	0.5090	0.2938
47	1.7610	-0.7590	0.9110	0.2162	1.0790	-0.2000	0.8310	-0.2370	0.3025
48	2.3950	-1.3500	1.3790	0.1640	1.1760	-0.5610	0.0630	-0.8810	0.2460
49	-0.5270	1.2410	-1.6940	0.1901	-0.5920	1.1760	-0.8060	-0.3160	0.3576
50	0.3700	0.4400	-0.5780	0.2570	0.7830	0.0980	0.4830	0.2980	0.4761
51	1.1310	-0.3880	0.2340	0.4382	-0.7210	1.2550	-1.4660	0.9920	0.4747
52	-0.6300	0.7810	-1.4610	0.5476	2.0580	-0.7120	1.3170	0.3370	0.2209

Table 5.2.*(cont.)*

	East Region				Central Region			
τ	$\phi_{1,\tau}$	$\phi_{2,\tau}$	$\theta_{1,\tau}$	g_τ	$\phi_{1,\tau}$	$\phi_{2,\tau}$	$\theta_{1,\tau}$	g_τ
1	-1.9750	2.1270	-3.0740	0.1163	0.5000	0.3470	-0.1400	0.4330
2	0.6650	0.1160	-0.3930	0.2107	2.4540	-1.3140	1.5630	0.1260
3	4.4630	-2.8420	3.6180	0.0918	0.1610	0.6950	-0.5550	0.2247
4	1.4070	-0.4400	1.7340	0.0900	0.5520	0.3590	-0.2150	0.1665
5	0.4990	0.3200	-1.2600	0.0497	1.5160	-0.6310	0.7850	0.3283
6	0.2780	0.5660	-2.4530	0.0488	0.4430	0.3390	-0.4900	0.2209
7	0.8710	0.0540	0.7700	0.0876	0.7420	0.0120	-0.4020	0.2652
8	1.1740	-0.3240	0.2140	0.1731	0.9490	-0.0670	-0.0950	0.2470
9	0.8890	0.0300	0.3750	0.2581	-0.5130	1.1300	-1.1950	0.4970
10	2.1450	-1.0870	1.1580	0.1849	-0.5800	1.0500	-1.4370	0.2294
11	1.4560	-0.5140	0.7030	0.2007	0.2920	0.4990	-0.7190	0.1927
12	1.4490	-0.4320	0.7050	0.1764	2.3620	-1.2790	1.5980	0.0986
13	2.9620	-1.7580	2.6810	0.1714	1.8950	-0.9100	1.2010	0.1414
14	0.6360	0.2970	-0.5390	0.1267	-0.2170	1.0850	-1.3180	0.0784
15	0.9410	-0.0440	0.5470	0.0955	1.5110	-0.5660	1.0970	0.1858
16	2.0060	-1.0040	1.1750	0.0930	1.1940	-0.1670	1.4140	0.1011
17	0.1870	0.5880	-0.6000	0.1376	0.6060	0.2870	0.5810	0.3192
18	1.4540	-0.5150	0.6250	0.1884	0.6860	0.2420	0.6030	0.3600
19	1.1300	-0.2250	0.3060	0.1731	0.3130	0.5170	-0.0010	0.4147
20	2.3280	-1.2420	1.6000	0.2343	0.6000	0.4590	0.5030	0.2088
21	2.7280	-1.5640	1.9080	0.2652	0.6390	-0.0530	-0.4200	0.4886
22	2.1520	-1.1630	1.1570	0.2052	0.8290	0.1330	0.0910	0.2218
23	-0.4440	1.1570	-1.4780	0.1945	0.2370	0.4630	-0.3670	0.4610
24	0.2740	0.5120	-0.7550	0.2480	2.6180	-1.3040	2.2120	0.5155
25	-0.3830	1.0740	-1.4400	0.2079	-0.0680	0.5380	-0.7070	0.5402
26	-1.9280	2.1240	-3.0710	0.4045	0.9360	-0.2790	0.2250	0.5730
27	0.4590	0.1020	-0.6940	0.2642	-0.0680	0.5430	-0.8320	0.4122
28	0.1910	0.5810	-0.8370	0.1498	-0.0420	0.6060	-0.9570	0.3318
29	-0.2730	0.7260	-1.6330	0.5685	-0.8650	1.1980	-1.8200	0.4343
30	-0.4810	0.0940	-1.5310	0.2162	1.6920	-1.6320	1.0990	0.5170
31	1.1360	-0.6510	1.1810	0.1697	0.8900	-0.1990	0.2980	0.2460
32	0.7310	-0.1160	-0.6960	0.3352	1.8890	-0.9950	0.8690	0.2992
33	1.8900	-0.9240	0.9350	0.2061	0.7180	0.0520	0.2680	0.4212
34	0.5980	0.1690	-0.3970	0.3844	-0.4560	0.8500	-1.3030	0.3147
35	1.7940	-0.7420	0.8730	0.2621	2.1100	-1.1340	1.4110	0.3411
36	1.5600	-0.6910	0.5920	0.1962	-0.1350	0.7040	-0.9200	0.4462
37	2.3640	-1.3000	1.5360	0.1608	-0.2890	0.6450	-1.1110	0.4970
38	1.4680	-0.5420	0.8670	0.2352	1.4100	-0.3820	0.9010	0.4238
39	-0.2020	0.7060	-1.4990	0.1772	-0.1100	0.6590	-1.1220	0.2190
40	0.6410	0.1990	-0.4180	0.3329	1.1350	-0.1400	0.5630	0.1798
41	2.8380	-1.6420	2.0100	0.1739	0.6070	0.1620	-0.7600	0.1608
42	3.0950	-2.2010	2.6300	0.2401	0.1420	0.6020	-1.1580	0.1980
43	1.1230	-0.1100	0.1980	0.2025	1.0570	-0.0790	0.6830	0.2125
44	-0.4660	1.2140	-1.2820	0.3469	1.0140	-0.2000	0.0100	0.2970
45	0.8170	1.2050	-1.8580	0.3399	-0.2190	0.8660	-1.3040	0.1832
46	1.4630	-0.5730	0.7610	0.3147	1.5460	-0.5920	1.2010	0.3114
47	1.5270	-0.3680	0.4920	0.0918	1.0710	-0.1550	0.2960	0.2621
48	-0.2240	1.1730	-1.2030	0.1529	0.7100	0.1660	0.2710	0.3434
49	-1.0500	1.8210	-1.9740	0.1190	0.2280	0.4830	-0.6110	0.3272
50	1.5130	-0.7440	0.7730	0.3795	-1.1070	1.5130	-2.2340	0.1648
51	2.1140	-0.9100	1.3840	0.2125	0.9030	-0.0630	0.0280	0.2894
52	-1.1180	1.5980	-1.9900	0.2992	0.8010	0.0590	0.4150	0.4160

Table 5.2 *(cont.)*

	South Region		
τ	$\phi_{1,\tau}$	$\theta_{1,\tau}$	g_τ
1	1.1110	1.2380	0.5198
2	0.8170	0.7010	0.6740
3	0.7140	0.4110	0.7868
4	1.4450	1.2860	0.4872
5	1.1360	1.1620	0.3660
6	0.8740	0.1750	0.3352
7	0.7080	-0.0600	0.4692
8	0.7770	0.1060	0.4556
9	0.8050	0.7070	0.6593
10	0.9790	1.0340	0.6577
11	1.1580	0.6950	0.4007
12	0.9120	0.4740	0.4212
13	1.0920	1.1400	0.3215
14	0.8590	0.6750	0.4844
15	0.8920	0.3430	0.4449
16	0.5570	0.1240	0.7482
17	1.4960	1.2400	0.3709
18	0.8700	0.5250	0.4858
19	0.9780	0.5210	0.4083
20	0.6570	0.3950	0.7140
21	0.6370	-0.0610	0.5213
22	0.6530	0.2920	0.7586
23	0.7940	0.8830	0.8263
24	-0.6750	-1.2580	0.6320
25	0.3980	-0.4220	0.5285
26	-0.1520	-0.9990	0.6006
27	0.0380	-0.9440	0.4238
28	0.4300	0.2850	0.8855
29	-1.3940	-1.9750	0.4720
30	0.8810	0.5730	0.5491
31	0.5490	0.4920	0.8668
32	1.4410	1.0130	0.5550
33	0.7750	0.2540	0.5883
34	0.8260	0.4300	0.6194
35	0.4030	-0.3610	0.5746
36	0.2760	0.0500	0.9235
37	2.1880	2.2490	0.7174
38	0.5440	0.0760	0.7621
39	0.7670	0.5300	0.8154
40	-0.1640	0.0200	1.0302
41	-0.6480	-0.9100	0.8336
42	1.9550	2.0170	0.6593
43	0.6060	0.4160	0.8446
44	1.4030	1.6030	0.6496
45	1.0910	0.9820	0.5868
46	1.0090	0.8660	0.5776
47	0.4930	0.2570	0.8668
48	1.5160	1.4880	0.6839
49	0.7850	0.5150	0.7500
50	0.7490	0.8150	0.8464
51	1.1180	0.7850	0.7379
52	1.1890	1.1370	0.6257

6 ESTIMATION OF THE CROSS-COVARIANCE OF RESIDUALS

6.1 Introduction

In order to generate flow sequences that are coherent in space, a multivariate model must be formulated. Due to the complexity of parameter estimation, the parameter matrices given in (1) are rarely considered full. An exception is the multivariate PAR(p)-model of Matalas (1967), which, however, is deemed inadequate for the present study. A common approach is to consider the parameter matrices $\mathbf{\Phi}_{i,\tau}$ and $\mathbf{\Theta}_{i,\tau}$ diagonal. This procedure is denoted contemporaneous modeling, because only the lag-0 cross-correlation of (transformed) flows can be explicitly modeled through the spatial covariance of the residuals. By considering the parameter matrices diagonal, the different sites involved in the analysis are decoupled and can be studied separately. Hence, when the parameters at each site have been estimated, only the spatial correlations of residuals remain to be determined.

In this section, we address the question of how to estimate the spatial correlation of residuals. There are essentially two possible avenues: the method of maximum likelihood and the method of moments. The former is by far the most used for more complicated seasonal models. It consists in deriving the residual series of each site and then to compute the correlation matrix for each season from these series. The ML method is thus easy to implement, and always yields results, but it tends to underestimate the true correlations. Stedinger et al. (1985) derived the moment estimator of the covariance matrix in the case of a contemporaneous stationary ARMA(1,1) model, and specifically conclude that for multivariate annual ARMA(1,1) models, the estimator of the residual covariance matrix that reproduces the observed correlation generally is superior to the maximum likelihood estimator.

Haltiner and Salas (1988) extended Stedinger results to the contemporaneous PARMA(1,1) case. Their derivation of the estimator of $\mathbf{G}_\tau$ (periodic cross-covariance matrix of the residuals) is instructive and is briefly reviewed here. By squaring the left and the right hand sides of (1), with $p = q = 1$, one obtains

$$\begin{aligned}
E\left[\mathbf{x}_\tau\mathbf{x}_\tau^T\right] &= E\left[\left(\hat{\mathbf{\Phi}}_\tau\mathbf{x}_{\tau-1}+\boldsymbol{\varepsilon}_\tau-\hat{\mathbf{\Theta}}_\tau\boldsymbol{\varepsilon}_{\tau-1}\right)\left(\hat{\mathbf{\Phi}}_\tau\mathbf{x}_{\tau-1}+\boldsymbol{\varepsilon}_\tau-\hat{\mathbf{\Theta}}_\tau\boldsymbol{\varepsilon}_{\tau-1}\right)^T\right] \\
&= \hat{\mathbf{\Phi}}_\tau E\left[\mathbf{x}_{\tau-1}\mathbf{x}_{\tau-1}^T\right]\hat{\mathbf{\Phi}}_\tau^T+\hat{\mathbf{\Phi}}_\tau E\left[\mathbf{x}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T\right]-\hat{\mathbf{\Phi}}_\tau E\left[\mathbf{x}_{\tau-1}\boldsymbol{\varepsilon}_{\tau-1}^T\right]\hat{\mathbf{\Theta}}_\tau^T \\
&\quad +E\left[\boldsymbol{\varepsilon}_\tau\mathbf{x}_{\tau-1}^T\right]\hat{\mathbf{\Phi}}_\tau^T+E\left[\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_\tau^T\right]+E\left[\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_{\tau-1}^T\right]\hat{\mathbf{\Theta}}_\tau^T \\
&\quad -\hat{\mathbf{\Theta}}_\tau E\left[\boldsymbol{\varepsilon}_{\tau-1}\mathbf{x}_{\tau-1}^T\right]\hat{\mathbf{\Phi}}_\tau^T-\hat{\mathbf{\Theta}}_\tau E\left[\boldsymbol{\varepsilon}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T\right]+\hat{\mathbf{\Theta}}_\tau E\left[\boldsymbol{\varepsilon}_{\tau-1}\boldsymbol{\varepsilon}_{\tau-1}^T\right]\hat{\mathbf{\Theta}}_\tau^T
\end{aligned} \tag{5}$$

where the year index has been omitted for notational convenience. $\hat{\mathbf{\Phi}}_\tau$ and $\hat{\mathbf{\Theta}}_\tau$ are assumed known. Noting that $E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T]=\mathbf{G}_\tau$, $E[\mathbf{x}_{\tau-1}\boldsymbol{\varepsilon}_\tau^T]=0$, and $E[\boldsymbol{\varepsilon}_\tau\boldsymbol{\varepsilon}_{\tau-1}^T]=0$, the above relation can be written:

$$\mathbf{G}_\tau=\hat{\mathbf{M}}_\tau(0)-\hat{\mathbf{\Phi}}_\tau\hat{\mathbf{M}}_{\tau-1}(0)\hat{\mathbf{\Phi}}_\tau^T+\hat{\mathbf{\Phi}}_\tau\mathbf{G}_{\tau-1}\hat{\mathbf{\Theta}}_\tau^T+\hat{\mathbf{\Theta}}_\tau\mathbf{G}_{\tau-1}\hat{\mathbf{\Phi}}_\tau^T-\hat{\mathbf{\Theta}}_\tau\mathbf{G}_{\tau-1}\hat{\mathbf{\Theta}}_\tau^T \tag{6}$$

in which only the $\mathbf{G}_\tau$-matrices are unknown. Note that $\mathbf{G}_\tau$ depends on $\mathbf{G}_{\tau-1}$ etc. If a feasible solution to (6) exists, then the model will exactly reproduce the variance of the (transformed) flows at each site. However, there is a potential risk in combining, for example, LS-estimators of the autoregressive and moving average parameters with moment estimators of the residual variances. The combined solution may not be coherent and could at some sites lead to periodic autocorrelations that deviate more from the observed correlations than the pure LS-estimation. It then becomes particularly important to compute each sites periodic autocorrelation as described previously. If the solution to (6) is unsatisfactory, one can choose to preserve only the cross-covariances between sites exactly, i.e. the off-diagonal elements of $\hat{\mathbf{M}}_\tau(0)$. The procedure for doing this will be described later. In the following, we consider the problem of estimating $\mathbf{G}_\tau$ by the method of moments when the individual site models are PARMA(2,2), or, in general, any submodel hereof.

The derivation of (6) is fairly straightforward, because $\mathbf{G}_\tau$ can be easily expressed (although implicitly) as a function of the properties to be preserved, $\hat{\mathbf{M}}_\tau(0)$. In the case of the contemporaneous PARMA(2,2) model, the derivation of a $\mathbf{G}_\tau$-estimator is less evident. Note that if the left and right hand site of (1), with $p=q=2$ are squared, then lagged cross-correlations will appear. However, contemporaneous PARMA models do not permit to preserve explicitly lagged cross-correlations. Preservation of the symmetric $\hat{\mathbf{M}}_\tau(0)$ matrices imposes $\omega m(m+1)/2$ constraints which is exactly the number of degrees of freedom in the ω $\mathbf{G}_\tau$ matrices. Therefore, Haltiner and Salas' results for the contemporaneous PARMA(1,1) model are not generally applicable to models of higher order. In the present study, we have developed a moment estimation method for any PARMA(p,q) model with $\max\{p,q\} \le 2$.

6.2 Moment estimation of spatial correlations

To estimate $\mathbf{G}_\tau$, we shall make use of the periodic multivariate Yule-Walker equations. The following definition is important:

$$E\left[\mathbf{x}_\tau \mathbf{x}_{\tau-1}^T\right] = \mathbf{M}_\tau(1) \quad \text{and hence} \quad E\left[\mathbf{x}_{\tau-1} \mathbf{x}_\tau^T\right] = \mathbf{M}_\tau^T(1) \tag{7}$$

where T indicates a transposed vector or matrix. Note that $\mathbf{M}_\tau(1)$ will in general not be symmetric. With these definitions, one can deduce the first three multivariate periodic Yule-Walker equations (Appendix A):

$$\begin{aligned}\mathbf{M}_\tau(0) &= E\left[\mathbf{x}_\tau \mathbf{x}_\tau^T\right] \\ &= \mathbf{M}_\tau(1)\,\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_\tau(2)\,\mathbf{\Phi}_{2,\tau}^T + \mathbf{G}_\tau - \left[\mathbf{\Phi}_{1,\tau} - \mathbf{\Theta}_{1,\tau}\right]\mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T \\ &\quad - \left[\mathbf{\Phi}_{1,\tau}\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Phi}_{1,\tau}\mathbf{\Theta}_{1,\tau-1} + \mathbf{\Phi}_{2,\tau} - \mathbf{\Theta}_{2,\tau}\right]\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T\end{aligned} \tag{8a}$$

$$\begin{aligned}\mathbf{M}_\tau^T(1) &= E\left[\mathbf{x}_{\tau-1}\mathbf{x}_\tau^T\right] = \mathbf{M}_{\tau-1}(0)\,\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{\tau-1}(1)\,\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-1}\mathbf{\Theta}_{1,\tau}^T \\ &\quad - \left[\mathbf{\Phi}_{1,\tau-1} - \mathbf{\Theta}_{1,\tau-1}\right]\mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T\end{aligned} \tag{8b}$$

$$\mathbf{M}_\tau^T(2) = E\left[\mathbf{x}_{\tau-2}\mathbf{x}_\tau^T\right] = \mathbf{M}_{\tau-1}^T(1)\,\mathbf{\Phi}_{1,\tau}^T + \mathbf{M}_{\tau-2}(0)\,\mathbf{\Phi}_{2,\tau}^T - \mathbf{G}_{\tau-2}\mathbf{\Theta}_{2,\tau}^T \tag{8c}$$

A careful inspection of these equations shows that the relations corresponding to the diagonal elements are simply the univariate cases given in (3a-c). On the other hand, the off-diagonal elements of $\mathbf{M}_\tau(0)$, $\mathbf{M}_\tau(1)$, and $\mathbf{M}_\tau(2)$, as expressed in the above equations, are defined in terms of off-diagonal elements of themselves and off-diagonal elements of $\mathbf{G}_\tau$, but do not involve any terms from the diagonal of these matrices. This important observation implies that the variance terms of $\mathbf{G}_\tau$, i.e. the diagonal elements can be estimated one at the time, and the covariances, i.e. off-diagonal elements can be estimated independently of the variances. Moreover, when estimating the correlations of residuals, one needs only consider two sites at the time, since other sites do not affect the particular element in $\mathbf{G}_\tau$ that corresponds to the two sites. Hence, in the following we develop the estimation procedure for two sites.

First, equation (8c) is used to eliminate $\mathbf{M}_\tau(2)$ from (8a). Then the equations corresponding to the off-diagonal elements of $\mathbf{M}_\tau(0)$ and $\mathbf{M}_\tau(1)$ are written out. Considering first the entry (1,2), we obtain from (8a-b)

$$\begin{aligned}-\Big[\Big(\hat{\phi}^{(1)}_{1,\tau}\hat{\phi}^{(1)}_{1,\tau-1}-\hat{\phi}^{(1)}_{1,\tau}\hat{\theta}^{(1)}_{1,\tau-1}+\hat{\phi}^{(1)}_{2,\tau}-\hat{\theta}^{(1)}_{2,\tau}\Big)\hat{\theta}^{(2)}_{2,\tau}+\hat{\theta}^{(1)}_{2,\tau}\hat{\phi}^{(2)}_{2,\tau}\Big]\ G^{(12)}_{\tau-2}-\Big(\hat{\phi}^{(1)}_{1,\tau}-\hat{\theta}^{(1)}_{1,\tau}\Big)\hat{\theta}^{(2)}_{1,\tau}\ G^{(12)}_{\tau-1} \\ +G^{(12)}_{\tau}+\hat{\phi}^{(1)}_{1,\tau}\hat{\phi}^{(2)}_{2,\tau}\ M^{(12)}_{\tau-1}(1)+\hat{\phi}^{(2)}_{1,\tau}\ M^{(12)}_{\tau}(1) \quad = \quad \hat{M}^{(12)}_{\tau}(0)-\hat{\phi}^{(1)}_{2,\tau}\hat{\phi}^{(2)}_{2,\tau}\ \hat{M}^{(12)}_{\tau-2}(0)\end{aligned} \tag{9a}$$

$$\hat{\theta}^{(2)}_{2,\tau}\Big(\hat{\phi}^{(1)}_{1,\tau-1}-\hat{\theta}^{(1)}_{1,\tau-1}\Big)\ G^{(12)}_{\tau-2}+\hat{\theta}^{(2)}_{1,\tau}\ G^{(12)}_{\tau-1}-\hat{\phi}^{(2)}_{2,\tau}\ M^{(12)}_{\tau-1}(1)+M^{(21)}_{\tau}(1) \quad = \quad \hat{\phi}^{(2)}_{1,\tau}\ \hat{M}^{(12)}_{\tau-1}(0) \tag{9b}$$

The '^' is used here to distinguish known terms, either estimated parameters or moments calculated from the data, from the terms which are to be estimated at this step. Hence, there is an important difference between, for example, $\hat{M}^{(12)}_\tau(0)$ and $M^{(12)}_\tau(1)$. The former is estimated from the data and preserved by the model; the latter is a model property, not necessarily equal to what one would get by estimating it from the data. $G^{(12)}_\tau$ is the covariance between the residuals at time τ, and $\hat{M}^{(12)}_\tau(0)=\hat{M}^{(21)}_\tau(0)$ is the cross-covariance of lag zero, estimated from the data. Since the data are assumed standardized, the cross-covariance is equal to the cross-correlation. The lag 1 cross-covariance is defined as $M^{(12)}_\tau(1)=E\big[x^{(1)}_\tau x^{(2)}_{\tau-1}\big]$. It is important to note that in general $M^{(12)}_\tau(1)\neq M^{(21)}_\tau(1)$. The above expressions have been obtained from (8a-b) by considering the element (1,2). Two other sets of equations can be obtained by considering the elements (2,1) in (8a-b), which corresponds to switching the site indices in (9a-b). Note that the element $G^{(12)}_\tau$ remains unaltered, since $G^{(12)}_\tau=G^{(21)}_\tau$. Hence, we essentially have 4ω linear equations with 3ω unknowns, namely $G^{(12)}_\tau$, $M^{(12)}_\tau(1)$, and $M^{(21)}_\tau(1)$ for $\tau=1,..,\omega$. In fact, one of the sets of equations is superfluous, and can be omitted. Eventually, it can be used to check the solution. Hence, we consider the system of linear equations consisting of (8a-b) and (8b) with reversed site indices. These 3ω equations can without any major difficulty be solved for $G^{(12)}_\tau$, $M^{(12)}_\tau(1)$, and $M^{(21)}_\tau(1)$.

If the variance of the flows at each site is not preserved exactly by the individual univariate models, then $\hat{M}^{(12)}_\tau(0)$ on the right side of the above equations must be adjusted with a factor $\sqrt{M^{11}_\tau(0)\,M^{22}_\tau(0)}$ where the terms in the square root are the variances produced by the individual models. This adjustment is necessary in order to correctly reproduce the correlation of flows.

The complex structure of the individual site models may impose such constraints on each series that exact preservation of the spatial cross-correlation of flows is not feasible. For example, it may appear that some of the estimated correlations between residuals are greater than one or less than minus one. This, of course, is meaningless, and some adjustment of the estimated $\mathbf{G}_\tau$-matrices is needed. In general, the requirement to the $\mathbf{G}_\tau$ matrices is that they be positive semidefinite. Hence, if a given $\mathbf{G}_\tau$ matrix is negative definite, an adjustment must be made to make it positive semidefinite. Generally, this will imply that the correlation of flows will no longer be exactly preserved. The adjustment should have as little influence on the $\mathbf{G}_\tau$ matrices as possible. There seems to be no standard method for adjusting symmetric, negative definite matrices so as to make them positive semidefinite. If a $\mathbf{G}_\tau$ matrix is negative definite (i.e. have negative eigenvalues) then one could proceed as follows:

1) Decompose the $\mathbf{G}$ matrix in eigenvectors and eigenvalues, $\mathbf{P}$ and $\mathbf{\Lambda}$, where the columns of $\mathbf{P}$ contain the eigenvectors of $\mathbf{G}$, and $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues on the diagonal. Hence, $\mathbf{G} = \mathbf{P\Lambda P'}$.
2) Set the negative eigenvalue in $\mathbf{\Lambda}$ equal to zero. This defines a new matrix $\mathbf{\Lambda^*}$.
3) Compute the matrix $\mathbf{G^*} = \mathbf{P\Lambda^* P'}$, which is positive semi-definite.
4) In order to preserve the variance terms of the original $\mathbf{G}$ matrix (i.e. the diagonal elements), perform the following computation

   $$\mathbf{G}_{\mathrm{adj}} = \mathbf{U\,G^*U}$$

   where

   $$\mathbf{U} = \begin{bmatrix} \sqrt{G^{(11)}/G^{*(11)}} & 0 & & 0 \\ 0 & \sqrt{G^{(22)}/G^{*(22)}} & & 0 \\ & & \ddots & \\ 0 & 0 & & \sqrt{G^{(mm)}/G^{*(mm)}} \end{bmatrix}$$

In order to check to what degree the cross-covariance of the transformed flows are reproduced by the obtained estimates of $G_\tau^{(12)}$, equations (8a-b) (with $\mathbf{M}_\tau(2)$ eliminated from (8a) by means of (8c)) are reformulated as a system of 4ω linear equations in $M_\tau^{(12)}(0)$, $M_\tau^{(21)}(0)$, $M_\tau^{(12)}(1)$, and $M_\tau^{(21)}(1)$. These equations become:

$$-\hat{\phi}^{(1)}_{2,\tau}\hat{\phi}^{(2)}_{2,\tau}\ M^{(12)}_{\tau-2}(0)+M^{(12)}_{\tau}(0)-\hat{\phi}^{(1)}_{1,\tau}\hat{\phi}^{(2)}_{2,\tau}\ M^{(12)}_{\tau-1}(1)-\hat{\phi}^{(2)}_{1,\tau}\ M^{(12)}_{\tau}(1)$$
$$=-\hat{\theta}^{(1)}_{2,\tau}\hat{G}^{(12)}_{\tau-2}\hat{\phi}^{(2)}_{2,\tau}+\hat{G}^{(12)}_{\tau}-\left(\hat{\phi}^{(1)}_{1,\tau}-\hat{\theta}^{(1)}_{1,\tau}\right)\hat{G}^{(12)}_{\tau-1}\hat{\theta}^{(2)}_{1,\tau} \tag{10a}$$
$$-\left(\hat{\phi}^{(1)}_{1,\tau}\hat{\phi}^{(1)}_{1,\tau-1}-\hat{\phi}^{(1)}_{1,\tau}\hat{\theta}^{(1)}_{1,\tau-1}+\hat{\phi}^{(1)}_{2,\tau}-\hat{\theta}^{(1)}_{2,\tau}\right)\hat{G}^{(12)}_{\tau-2}\hat{\theta}^{(2)}_{2,\tau}$$

$$\hat{\phi}^{(2)}_{1,\tau}\ M^{(12)}_{\tau-1}(0)+\hat{\phi}^{(2)}_{2,\tau}\ M^{(12)}_{\tau-1}(1)-M^{(21)}_{\tau}(1)=\hat{G}^{(12)}_{\tau-1}\hat{\theta}^{(2)}_{1,\tau}+\left(\hat{\phi}^{(1)}_{1,\tau-1}-\hat{\theta}^{(1)}_{1,\tau-1}\right)\hat{G}^{(12)}_{\tau-2}\hat{\theta}^{(2)}_{2,\tau} \tag{10b}$$

plus the same two sets of equations with reversed site indices. Strictly, only three equations are needed, since $M^{(12)}_{\tau}(0)=M^{(21)}_{\tau}(0)$. However, the above formulation provides a test of coherence. In a first step, one should verify that $M^{(12)}_{\tau}(0)$ is identical to $M^{(21)}_{\tau}(0)$. If not, this could indicate a programming error or lack of precision in the computation. The model cross-covariance $M^{(12)}_{\tau}(0)$ can be compared with the observed cross-covariance of the transformed data $\hat{M}^{(12)}_{\tau}(0)$.

Lagged cross-correlations are not preserved explicitly. The extent to which an estimated model produces lagged cross-correlations that resembles the observed can be examined by comparing $\mathbf{M}_{\tau}(1)$ and $\mathbf{M}_{\tau}(2)$ of the model with the corresponding observed lagged cross-covariance matrices. Note that $\mathbf{M}_{\tau}(1)$ is obtained as a biproduct in the estimation of $\mathbf{G}_{\tau}$. With known $\mathbf{M}_{\tau}(0)$ and $\mathbf{M}_{\tau}(1)$, $\mathbf{M}_{\tau}(2)$ is readily obtained from (8c). One can generalize equation (3d) to the multivariate case:

$$\mathbf{M}_{\tau}(k)=E\left[\mathbf{x}_{\tau}\mathbf{x}^{T}_{\tau-k}\right]=\mathbf{\Phi}_{1,\tau}\ \mathbf{M}_{\tau-1}(k-1)+\mathbf{\Phi}_{2,\tau}\ \mathbf{M}_{\tau-2}(k-2) \qquad k>2 \tag{11}$$

but usually only the first or second lagged correlations need to be examined.

7 RESULTS OF MULTI-SITE ESTIMATION

The method of moments was used to estimate the off-diagonal elements of the periodic cross-covariance matrices of residuals. The MATLAB computer codes used to compute these properties are presented in Appendix B. The $\mathbf{G}_\tau$-matrices have dimension 5x5, but only 10 elements in each are unknown, corresponding to the number of combinations of two sites out of five. Estimated autoregressive and moving average parameters of each pair of sites, as well as the observed periodic cross-correlation of transformed flows, were entered into a program, which as output yielded the periodic cross-covariance of residuals needed to reproduce the cross-correlations of flows. This resulted in ten vectors of periodic cross-covariances of residuals, representing an initial estimate of the ten off-diagonal elements of the $\mathbf{G}_\tau$-matrices. As noted in the previous section, there is no guarantee that this is a feasible solution. The $\mathbf{G}_\tau$-matrices must be positive semidefinite, but there is no provision for this in the method of moments. In fact, the problem pertaining to negative definite matrices turned out to be more severe than expected. Only one of the 52 matrices were positive definite, 21 matrices had one negative eigenvalue, 28 had two negative eigenvalues, and two had three negative eigenvalues. All matrices with negative eigenvalues were modified with the technique described in the previous section. This had a significant effect on some of the elements of the matrices. The matrix $\mathbf{G}_1$ had three negative eigenvalues, $\mathbf{G}_2$ had two negative eigenvalues, and $\mathbf{G}_3$ had one negative eigenvalue. The first three G-matrices were changed as follows[2]:

$$G_1 = \begin{bmatrix} 0.35 & 0.49 & 0.70 & 0.53 & -0.43 \\ & 0.23 & 0.88 & 0.63 & -0.45 \\ & & 0.12 & 0.79 & -0.25 \\ & & & 0.43 & -0.35 \\ & & & & 0.52 \end{bmatrix} \rightarrow G_1^* = \begin{bmatrix} 0.35 & 0.28 & 0.20 & 0.38 & -0.32 \\ & 0.23 & 0.16 & 0.31 & -0.24 \\ & & 0.12 & 0.22 & -0.15 \\ & & & 0.43 & -0.29 \\ & & & & 0.52 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.15 & 0.11 & -0.14 & 0.00 & 0.17 \\ & 0.35 & -0.23 & 0.24 & 0.12 \\ & & 0.21 & -0.23 & 0.45 \\ & & & 0.13 & 0.13 \\ & & & & 0.67 \end{bmatrix} \rightarrow G_2^* = \begin{bmatrix} 0.15 & 0.10 & -0.04 & 0.04 & 0.10 \\ & 0.35 & -0.15 & 0.20 & 0.10 \\ & & 0.21 & -0.08 & 0.25 \\ & & & 0.13 & 0.06 \\ & & & & 0.67 \end{bmatrix}$$

2) The elements of the matrices correspond to the regions in the following order: North East, North West, East, Central, and South.

Table 7.1 Elements of the periodic covariance matrices of residuals

τ	g11	g12	g13	g14	g15	g22	g23	g24
1	0.3540	0.2819	0.1983	0.3844	-0.3204	0.2256	0.1604	0.3104
2	0.1475	0.1047	-0.0418	0.0357	0.1041	0.3457	-0.1538	0.2035
3	0.2520	0.2643	-0.0939	0.1330	-0.1907	0.3238	-0.1158	0.1726
4	0.1739	0.1243	-0.0828	0.1308	-0.1998	0.1781	-0.1066	0.1279
5	0.2256	0.1637	0.0511	0.1373	-0.0928	0.3588	0.0997	0.2250
6	0.2043	0.0737	0.0322	-0.0602	0.1127	0.3047	-0.0843	0.1700
7	0.1568	0.1043	0.0818	0.0313	-0.0207	0.2025	-0.0189	0.1924
8	0.1204	0.0891	0.0087	0.1350	0.0634	0.1714	-0.1255	0.1965
9	0.3341	0.1797	0.1806	0.3385	-0.0497	0.2266	0.0253	0.2070
10	0.2611	0.1483	0.1206	0.1560	0.3679	0.2218	0.0120	0.0579
11	0.1665	0.0998	-0.1170	-0.0437	-0.1495	0.4020	-0.0716	0.0803
12	0.1467	0.0800	-0.0575	0.1137	0.1331	0.1289	0.0257	0.0593
13	0.3994	0.1710	0.0253	0.1468	0.2901	0.2581	-0.0384	0.0827
14	0.1592	0.0777	-0.0231	-0.0044	0.2548	0.3192	0.1274	0.0933
15	0.3588	0.2497	0.0655	0.1050	-0.0952	0.3058	-0.0151	-0.0056
16	0.3636	0.2921	0.0097	0.0231	0.2039	0.3636	0.0452	0.0399
17	0.3422	0.2851	0.0499	0.1618	0.0004	0.5155	-0.0470	0.0020
18	0.5055	-0.0077	0.0851	0.2432	0.1203	0.3869	0.0124	-0.0937
19	0.4610	-0.0392	-0.0423	0.0047	-0.0215	0.4225	0.0047	-0.0961
20	0.4706	0.0868	0.1353	0.1472	0.1871	0.4900	0.2834	0.0475
21	0.4369	0.5121	0.2590	0.4168	0.3277	0.6241	0.2573	0.4833
22	0.3516	0.2948	0.1994	0.2632	0.4194	0.5329	0.1558	0.2256
23	0.4147	0.4542	0.2495	0.3121	0.4338	0.5213	0.3004	0.3131
24	0.7430	0.3105	0.1525	0.2697	0.2692	0.6889	0.2722	0.3792
25	0.6806	0.4882	0.3356	0.5324	0.4302	0.5184	0.2969	0.4530
26	0.6675	0.5072	0.3969	0.5377	0.3914	0.4277	0.2611	0.3925
27	0.2034	0.2490	0.2102	0.2779	-0.1121	0.3192	0.2755	0.3224
28	0.1576	0.1197	0.1118	0.1740	-0.3173	0.2228	0.1213	0.1870
29	0.2520	0.2706	0.2833	0.1998	-0.0884	0.5155	0.3249	0.2423
30	0.4556	0.5538	0.3039	0.4557	-0.3235	0.7090	0.3522	0.5259
31	0.4665	0.3651	0.2642	0.1795	0.0084	0.3624	0.2444	0.2513
32	0.3770	0.1795	0.0452	0.3284	0.3886	0.1673	-0.0756	0.1871
33	0.2430	0.1785	0.1085	0.3050	0.2000	0.2510	0.2056	0.2699
34	0.2162	0.1662	0.2619	0.1525	0.2040	0.2905	0.1460	0.1854
35	0.2714	0.2403	0.0766	0.1565	-0.0142	0.3446	0.2329	0.2954
36	0.3295	0.3052	0.1473	0.3783	0.0429	0.4160	0.2357	0.3863
37	0.3147	0.2520	0.2015	0.1428	0.2755	0.3036	0.2057	0.3109
38	0.3697	0.1511	0.1803	0.1494	-0.0672	0.2777	0.1142	0.0803
39	0.2852	0.0789	0.0879	0.1997	-0.0186	0.3648	-0.1003	0.1103
40	0.4290	0.2127	0.1176	0.1307	0.1578	0.1122	0.0935	0.0586
41	0.3672	0.2834	0.1876	0.2311	-0.1522	0.2200	0.1351	0.1827
42	0.2323	0.2307	0.2010	0.1273	0.1208	0.2510	0.2374	0.1788
43	0.4369	0.4056	0.1447	0.1675	-0.4186	0.4134	0.2055	0.2066
44	0.3080	0.1563	0.2114	0.0811	-0.0284	0.2256	0.2293	0.0137
45	0.2228	0.2628	0.1255	0.0141	0.0084	0.4638	0.3139	0.1815
46	0.2116	0.2288	0.1804	0.2136	0.0681	0.2938	0.1906	0.2303
47	0.2162	0.1340	0.0707	0.1538	-0.2314	0.3025	0.1665	0.2743
48	0.1640	0.1064	0.0895	0.2331	0.1378	0.2460	-0.0773	0.1629
49	0.1901	0.1640	-0.0246	-0.0104	0.0435	0.3576	0.0918	0.2334
50	0.2570	0.2490	0.2624	0.1251	-0.2266	0.4761	0.3775	0.2640
51	0.4382	0.2652	0.0296	0.3061	-0.1207	0.4747	0.1537	0.2712
52	0.5476	0.1644	0.1804	0.0530	-0.3323	0.2209	-0.1398	0.2026

Table 7.1 *(cont.)*

τ	g25	g33	g34	g35	g44	g45	g55
1	-0.2385	0.1163	0.2243	-0.1451	0.4330	-0.2880	0.5198
2	0.0958	0.2107	-0.0837	0.2521	0.1260	0.0558	0.6740
3	-0.1443	0.0918	-0.1419	-0.0540	0.2247	0.0686	0.7868
4	-0.0551	0.0900	-0.1179	-0.0174	0.1665	-0.0195	0.4872
5	-0.0592	0.0497	0.1252	0.0374	0.3283	0.0713	0.3660
6	-0.0403	0.0488	-0.1032	-0.0059	0.2209	0.0441	0.3352
7	-0.0068	0.0876	-0.0881	-0.0060	0.2652	0.0715	0.4692
8	0.0124	0.1731	-0.1038	0.1036	0.2470	0.0934	0.4556
9	-0.0604	0.2581	0.0983	-0.2111	0.4970	-0.1181	0.6593
10	0.3144	0.1849	0.2018	0.2055	0.2294	0.2834	0.6577
11	0.1832	0.2007	-0.0413	0.0489	0.1927	0.1081	0.4007
12	0.0576	0.1764	-0.0776	0.0896	0.0986	0.0768	0.4212
13	0.1256	0.1714	-0.1084	0.1097	0.1414	0.0571	0.3215
14	0.2417	0.1267	0.0979	0.0599	0.0784	0.0684	0.4844
15	-0.0683	0.0955	0.1113	0.0553	0.1858	-0.0846	0.4449
16	0.1411	0.0930	0.0953	0.0859	0.1011	0.0864	0.7482
17	0.1456	0.1376	0.2000	0.0260	0.3192	-0.0026	0.3709
18	0.1302	0.1884	0.2329	0.0704	0.3600	0.1108	0.4858
19	0.2606	0.1731	0.2466	0.2038	0.4147	0.2509	0.4083
20	0.0979	0.2343	0.0180	0.1411	0.2088	0.3106	0.7140
21	0.4068	0.2652	0.2130	0.2242	0.4886	0.1565	0.5213
22	0.3821	0.2052	0.1967	0.3896	0.2218	0.3939	0.7586
23	0.5210	0.1945	0.2057	0.3751	0.4610	0.5041	0.8263
24	0.5494	0.2480	0.3557	0.3820	0.5155	0.5429	0.6320
25	0.4290	0.2079	0.2392	0.3158	0.5402	0.2610	0.5285
26	0.3702	0.4045	0.4669	0.3412	0.5730	0.4687	0.6006
27	-0.0850	0.2642	0.2488	-0.1131	0.4122	-0.1767	0.4238
28	-0.3993	0.1498	0.2112	-0.2945	0.3318	-0.4104	0.8855
29	0.0144	0.5685	0.2202	0.1847	0.4343	0.0879	0.4720
30	-0.4482	0.2162	0.3250	-0.1890	0.5170	-0.1985	0.5491
31	0.2253	0.1697	0.1596	0.1069	0.2460	0.3536	0.8668
32	0.1900	0.3352	-0.0151	0.2225	0.2992	0.3252	0.5550
33	0.1226	0.2061	0.1613	0.1671	0.4212	0.1483	0.5883
34	0.0347	0.3844	0.2496	0.3407	0.3147	0.1171	0.6194
35	0.0079	0.2621	0.2840	0.1095	0.3411	0.0761	0.5746
36	-0.0327	0.1962	0.2070	0.1880	0.4462	0.0737	0.9235
37	0.1668	0.1608	0.2027	0.2275	0.4970	0.1751	0.7174
38	0.2454	0.2352	0.2717	0.2563	0.4238	0.3310	0.7621
39	-0.5292	0.1772	-0.0444	0.1470	0.2190	-0.0662	0.8154
40	0.0859	0.3329	0.1291	0.4196	0.1798	0.4007	1.0302
41	-0.1480	0.1739	0.0835	0.1674	0.1608	-0.2044	0.8336
42	0.1376	0.2401	0.1987	0.1599	0.1980	0.0672	0.6593
43	-0.3117	0.2025	0.1907	-0.0603	0.2125	-0.2018	0.8446
44	0.0036	0.3469	0.1439	0.2155	0.2970	0.0793	0.6496
45	0.1234	0.3399	0.1636	0.3401	0.1832	0.0696	0.5868
46	0.0587	0.3147	0.3061	0.3568	0.3114	0.2986	0.5776
47	-0.1176	0.0918	0.1496	-0.0542	0.2621	-0.2083	0.8668
48	0.0440	0.1529	0.1155	0.1287	0.3434	0.1132	0.6839
49	0.0102	0.1190	0.1839	0.1843	0.3272	0.1679	0.7500
50	0.0227	0.3795	0.1743	0.0324	0.1648	-0.0457	0.8464
51	0.0030	0.2125	0.1352	0.1507	0.2894	0.0575	0.7379
52	-0.2546	0.2992	-0.1284	0.1419	0.4160	0.0423	0.6257

$$G_3 = \begin{bmatrix} 0.25 & 0.32 & -0.34 & 0.10 & -0.22 \\ & 0.32 & -0.19 & 0.29 & -0.14 \\ & & 0.09 & -0.66 & -0.15 \\ & & & 0.22 & 0.07 \\ & & & & 0.79 \end{bmatrix} \rightarrow G_3^{*} = \begin{bmatrix} 0.25 & 0.26 & -0.09 & 0.13 & -0.19 \\ & 0.32 & -0.12 & 0.17 & -0.14 \\ & & 0.09 & -0.14 & -0.05 \\ & & & 0.22 & 0.07 \\ & & & & 0.79 \end{bmatrix}$$

It is seen that part of the problem is that some of the covariances in the initial matrices are "too" high as compared with the corresponding variances. For example, the moment solution requires that the correlation coefficient between the residuals in region 1 and 3 at time $\tau = 1$ be $0.70 / \sqrt{0.35 \cdot 0.12} = 3.4$. Since correlation coefficients are restricted to the interval $[-1;1]$, the element at row 1, column 3 must be reduced with at least a factor 3.4. One cannot generally conclude that the number of negative eigenvalues determines the "amount" of correction needed, since also the value of the eigenvalues are important. The number of negative eigenvalues, which in our algorithm is set to zero, determines the rank of the adjusted G-matrices. For example, the matrix G_1, which has three negative eigenvalue, is modified to a matrix with rank 2. This means that the generated residual vectors at $\tau = 1$ only have two degrees of freedom, or, in other words, two elements of the residual vectors uniquely determine the three others. Note that the variance terms of the matrices (diagonal elements) remain unchanged. They are identical to the values listed in Table 5.2.

The complete set of adjusted G-matrices is given in Table 7.1. An important step in the analysis is to evaluate the consequences of the adjustments made on the initial moment estimates. The theoretical procedure described in the previous section was implemented (see Appendix B) and invoked in order to compute the cross-covariance of generated flows. The results are shown in Figure 7.1. The following conclusions can be drawn from the figures:

1. The correlations between the transformed flows in certain regions are substantially underestimated in the beginning of the (hydrological) year. This is especially true for the combinations involving the East region. It should be noted, however, that the flows in this period are relatively low, thus reducing the practical impact of this underestimation.

2. The cross-correlations are generally well preserved during the critical flood seasons. In the light of the objectives of this study, this is a very pleasant observation.

3. Correlations between the South region and the other regions are well preserved. This can probably be ascribed to the fact that the site model at South is a PARMA(1,1). In

fact, for a close reproduction of cross-correlations it seems advantageous that the site models be as simple as possible. (For example, if all site models were PARMA(1,0), there would be no problem in preserving the observed cross-correlations.)

4. The adjusted moment estimates yield cross-correlations of flows that are generally closer to the observed, than those corresponding to ML-estimates.

The estimation of the **G**-matrices completes the calibration of the multivariate PARMA-model.

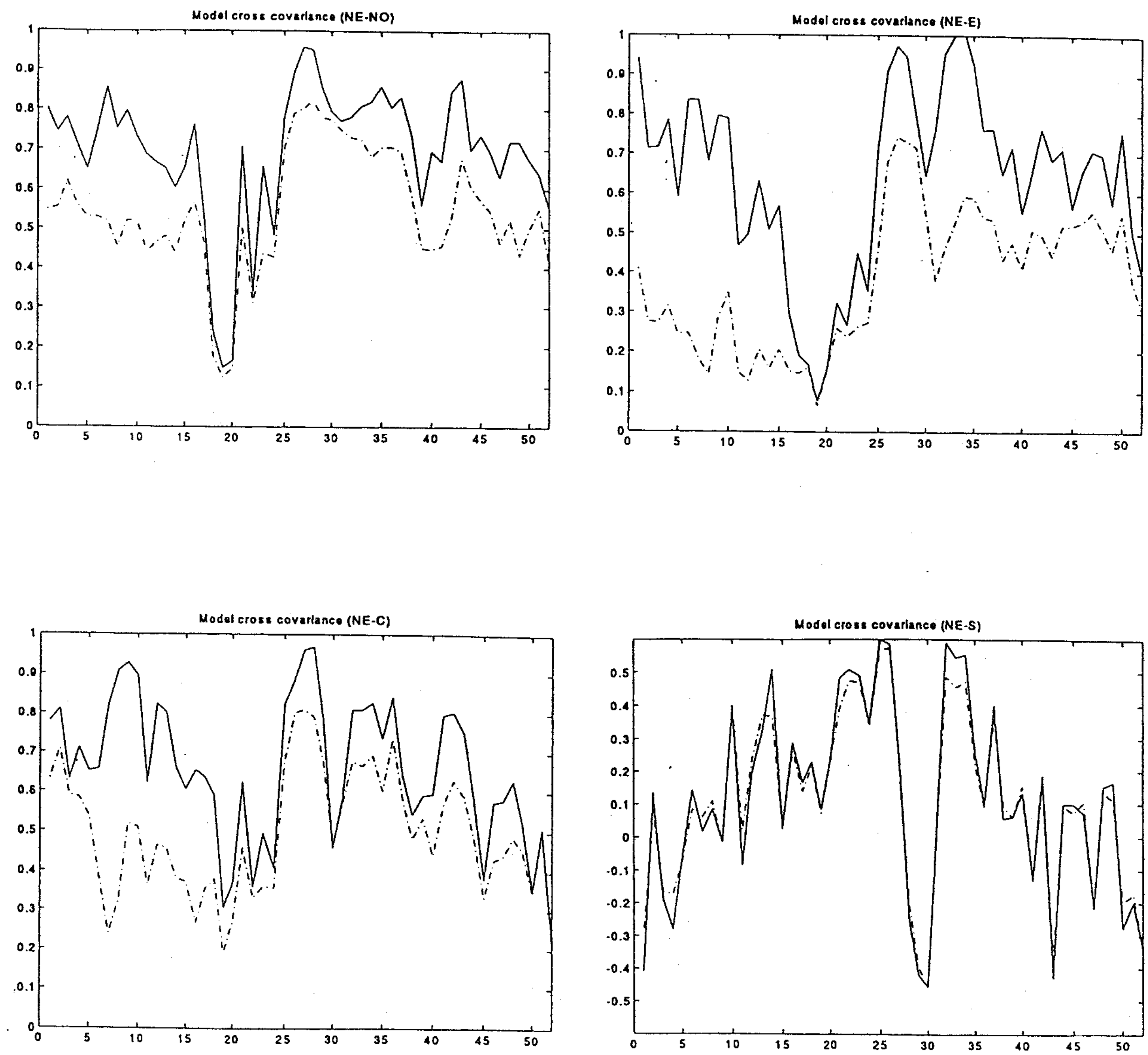


Figure 7.1 Observed and fitted cross-correlation of flows

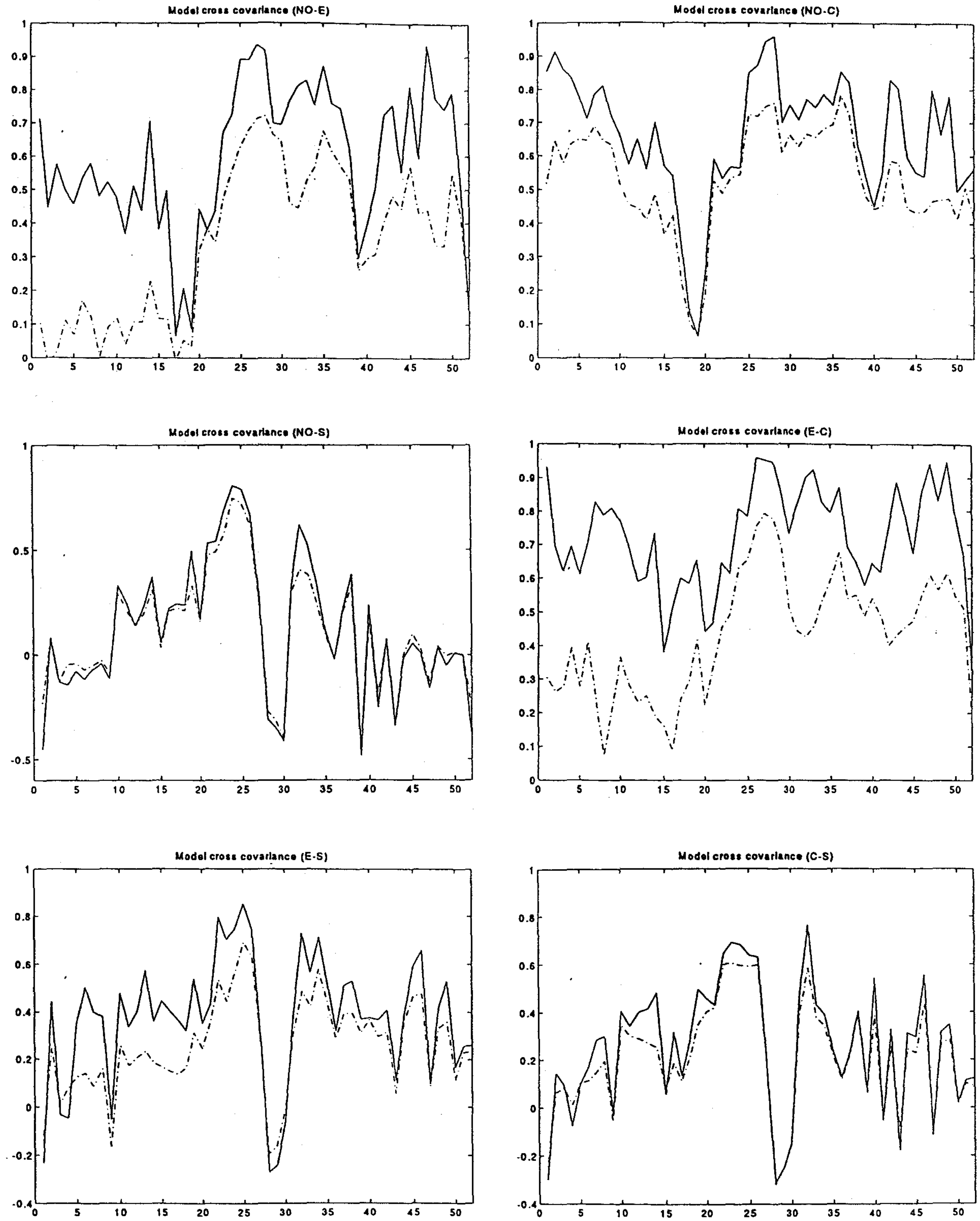


Figure 7.1 *(cont.)*

8 CONCLUDING REMARKS

The results presented in this report constitute only one element of a flow-generator developed for the Ottawa River system. It will be used in connection with a spatial disaggregation model, and a time disaggregation scheme that takes the weekly data down to a time scale of one day. Hence, when evaluating the output of the flow-generator, several other elements enter as potential sources of errors or inadequacies. It is important that each component of the generator be thoroughly tested and evaluated. We have developed several routines that permit to test and evaluate the performance of the multivariate PARMA model as applied to transformed weekly data.

Apparently, for models beyond the PARMA(1,1), the method of moments has never been used to estimate the residual cross-covariances. Although no exact feasible moment solution could be obtained in this study, the approximate solution, based on corrected G-matrices, turned out to be generally superior to the ML-method. More research should be devoted to techniques for adjusting matrices that are negative definite. We examined several options and selected the one that seemingly gave the best results. However, it cannot be excluded that even better methods can be found.

9 References

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APPENDICES

Appendix A. Periodic multivariate Yule-Walker equations

The periodic multivariate Yule-Walker equations can be deduced as follows. By definition, we have

$$\mathrm{E}\left[\mathbf{X}_\tau \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{M}_\tau(0) \tag{A1}$$

$$\mathrm{E}\left[\mathbf{X}_\tau \mathbf{X}_{\tau-1}^{\mathrm{T}}\right] = \mathbf{M}_\tau(1) \quad \text{and} \quad \mathrm{E}\left[\mathbf{X}_{\tau-1} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{M}_\tau^{\mathrm{T}}(1) \tag{A2}$$

$$\mathrm{E}\left[\boldsymbol{\varepsilon}_\tau \boldsymbol{\varepsilon}_\tau^{\mathrm{T}}\right] = \mathbf{G}_\tau \tag{A3}$$

$$\mathrm{E}\left[\mathbf{X}_{\tau-i} \boldsymbol{\varepsilon}_\tau^{\mathrm{T}}\right] = \mathbf{0} \quad \text{for } i > 0 \tag{A4}$$

The multivariate PARMA(2,2) model has the form

$$\mathbf{X}_\tau = \boldsymbol{\Phi}_{1,\tau} \mathbf{X}_{\tau-1} + \boldsymbol{\Phi}_{2,\tau} \mathbf{X}_{\tau-2} + \boldsymbol{\varepsilon}_\tau - \boldsymbol{\Theta}_{1,\tau} \boldsymbol{\varepsilon}_{\tau-1} - \boldsymbol{\Theta}_{2,\tau} \boldsymbol{\varepsilon}_{\tau-2} \tag{A5}$$

Using the above definitions, one easily shows that

$$\mathrm{E}\left[\mathbf{X}_\tau \boldsymbol{\varepsilon}_\tau^{\mathrm{T}}\right] = \mathrm{E}\left[\boldsymbol{\varepsilon}_\tau \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{G}_\tau \tag{A6}$$

$$\mathrm{E}\left[\mathbf{X}_\tau \boldsymbol{\varepsilon}_{\tau-1}^{\mathrm{T}}\right] = \left[\boldsymbol{\Phi}_{1,\tau} - \boldsymbol{\Theta}_{1,\tau}\right] \mathbf{G}_{\tau-1} \quad \text{and} \quad \mathrm{E}\left[\boldsymbol{\varepsilon}_{\tau-1} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{G}_{\tau-1}\left[\boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} - \boldsymbol{\Theta}_{1,\tau}^{\mathrm{T}}\right] \tag{A7}$$

$$\mathrm{E}\left[\mathbf{X}_\tau \boldsymbol{\varepsilon}_{\tau-2}^{\mathrm{T}}\right] = \left[\boldsymbol{\Phi}_{1,\tau-1} \boldsymbol{\Phi}_{1,\tau} - \boldsymbol{\Theta}_{1,\tau-1} \boldsymbol{\Phi}_{1,\tau} + \boldsymbol{\Phi}_{2,\tau} - \boldsymbol{\Theta}_{2,\tau}\right] \mathbf{G}_{\tau-1} \quad \text{and}$$

$$\mathrm{E}\left[\boldsymbol{\varepsilon}_{\tau-2} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{G}_{\tau-1}\left[\boldsymbol{\Phi}_{1,\tau-1}^{\mathrm{T}} \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} - \boldsymbol{\Theta}_{1,\tau-1}^{\mathrm{T}} \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} + \boldsymbol{\Phi}_{2,\tau}^{\mathrm{T}} - \boldsymbol{\Theta}_{2,\tau}^{\mathrm{T}}\right] \tag{A8}$$

The covariance matrix of $\mathbf{X}_\tau$ and $\mathbf{X}_{\tau-i}$ is obtained by multiplying (A5) by $\mathbf{X}_{\tau-i}$ and taking expectation:

$$\begin{aligned} \mathbf{M}_\tau(0) = \mathrm{E}\left[\mathbf{X}_\tau \mathbf{X}_\tau^{\mathrm{T}}\right] &= \mathbf{M}_\tau(1)\, \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} + \mathbf{M}_\tau(2)\, \boldsymbol{\Phi}_{2,\tau}^{\mathrm{T}} + \mathbf{G}_\tau - \left[\boldsymbol{\Phi}_{1,\tau} - \boldsymbol{\Theta}_{1,\tau}\right] \mathbf{G}_{\tau-1} \boldsymbol{\Theta}_{1,\tau}^{\mathrm{T}} \\ &- \left[\boldsymbol{\Phi}_{1,\tau} \boldsymbol{\Phi}_{1,\tau-1} - \boldsymbol{\Phi}_{1,\tau} \boldsymbol{\Theta}_{1,\tau-1} + \boldsymbol{\Phi}_{2,\tau} - \boldsymbol{\Theta}_{2,\tau}\right] \mathbf{G}_{\tau-2} \boldsymbol{\Theta}_{2,\tau}^{\mathrm{T}} \end{aligned} \tag{A9}$$

$$\mathbf{M}_\tau^{\mathrm{T}}(1) = \mathrm{E}\left[\mathbf{X}_{\tau-1} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{M}_{\tau-1}(0)\, \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} + \mathbf{M}_{\tau-1}(1)\, \boldsymbol{\Phi}_{2,\tau}^{\mathrm{T}} - \mathbf{G}_{\tau-1} \boldsymbol{\Theta}_{1,\tau}^{\mathrm{T}} - \left[\boldsymbol{\Phi}_{1,\tau-1} - \boldsymbol{\Theta}_{1,\tau-1}\right] \mathbf{G}_{\tau-2} \boldsymbol{\Theta}_{2,\tau}^{\mathrm{T}} \tag{A10}$$

$$\mathbf{M}_\tau^{\mathrm{T}}(2) = \mathrm{E}\left[\mathbf{X}_{\tau-2} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{M}_{\tau-1}^{\mathrm{T}}(1)\, \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} + \mathbf{M}_{\tau-2}(0)\, \boldsymbol{\Phi}_{2,\tau}^{\mathrm{T}} - \mathbf{G}_{\tau-2} \boldsymbol{\Theta}_{2,\tau}^{\mathrm{T}} \tag{A11}$$

$$\mathbf{M}_\tau^{\mathrm{T}}(k) = \mathrm{E}\left[\mathbf{X}_{\tau-2} \mathbf{X}_\tau^{\mathrm{T}}\right] = \mathbf{M}_{\tau-1}^{\mathrm{T}}(k-1)\, \boldsymbol{\Phi}_{1,\tau}^{\mathrm{T}} + \mathbf{M}_{\tau-2}(k-2)\, \boldsymbol{\Phi}_{2,\tau}^{\mathrm{T}} \qquad k > 2 \tag{A12}$$

The univariate cases (3a-d) correspond to the diagonal elements of the above matrices.

APPENDIX B MATLAB PROGRAMS

The following pages contains a listing of some of the most important routines developed in this project for calibrating and testing a multivariate PARMA(2,2) model. The following table summarizes the name and purpose of the four main functions and routines.

Table B1 MATLAB subroutines

modelcov :	The function modelcov takes as input the estimated model parameters (autoregressive, moving average and variance of residuals) of a univariate PARMA(2,2) model, or any sub-model. It returns the periodic variance and the periodic autocorrelation.
gij :	The function takes as input the estimated model parameters (autoregressive and moving average) of a bivariate PARMA(2,2) model and the observed cross-correlation between the transformed flows at the two sites. It returns a vector containing the covariance terms in the 52 covariance matrices of residuals, which produces exactly the observed cross-correlation of flows. However, the solution may not be feasible!
dij :	This routine examines each of the 52 (5 x 5) matrices for negative eigenvalues. If negative eigenvalues are present, the matrix is adjusted to make it positive semidefinite.
modelcol :	The function takes as input the estimated model parameters (autoregressive and moving average) of a bivariate PARMA(2,2) model and the cross-covariance of residuals between the two sites. The function returns the corresponding cross-correlation of transformed flows between the two sites.

```
% ****************************************************************
%    Covariance structure of PARMA(2,2) and sub-models
%                     @1994 by Peter F. Rasmussen
% ****************************************************************
%
% function [var, scr] = modelcov(phi,tht,g)
%
%
% Input:      phi (2 x 52)   : phi parameters
%             tht (2 x 52)   : tht parameters
%             g   (1 x 52)   : residual variance
%
% Output:     scr (10 x 52) : seasonal correlations up to lag 10
%             var (1 x 52)   : variance of process
% --------------------------------------------------------------

function [var, scr] = modelcov(phi,tht,g);

% ** Define constants
m=length(g);
mslag=10;

% ** Initialization of coefficient matrix
a = zeros(3*m,3*m);
clear y;

% ** Calculate some useful constants
for i=1:m
   i1 = i-1;
   i2 = i-2;
   if i==1
      i1 = m;
      i2 = m-1;
   end
   if i==2
      i2 = m;
   end
   aa(i) = tht(2,i) * g(i2);
   bb(i) = -tht(1,i) * g(i1) + aa(i) * tht(1,i1);
   cc(i) = g(i) + tht(1,i)^2 * g(i1) + aa(i) * tht(2,i);
end;

% ** fill out coefficient matrix
%-------------------------------------------------------------
%      phi(1,t) m(1,t-1) + phi(2,t) m(0,t-2) - m(2,t) = aa(t)
%-------------------------------------------------------------
for i=1:m
   if i==1
      a(i,3) = -1;
      a(i,3*m-5) = phi(2,i);
      a(i,3*m-1) = phi(1,i);
      y(i) = aa(i);
   elseif i==2
      a(i,6) = -1;
      a(i,3*m-2) = phi(2,i);
      a(i,2) = phi(1,i);
      y(i) = aa(i);
   else
```

```
      a(i,3*i) = -1;
      a(i,3*i-8) = phi(2,i);
      a(i,3*i-4) = phi(1,i);
      y(i) = aa(i);
   end;
end;

%--------------------------------------------------------
%      m(1,t) - phi(1,t) m(0,t-1) - phi(2,t) m(1,t-1)
%        = bb(t) - phi(1,t-1) aa(t)
%--------------------------------------------------------
for i=1:m
   if i==1
      a(m+i,2) = 1;
      a(m+i,3*m-2) = -phi(1,i);
      a(m+i,3*m-1) = -phi(2,i);
      y(m+i) = bb(i) - aa(i) * phi(1,m);
   else
      a(m+i,3*i-1) = 1;
      a(m+i,3*i-5) = -phi(1,i);
      a(m+i,3*i-4) = -phi(2,i);
      y(m+i) = bb(i) - aa(i) * phi(1,i-1);
   end;
end;

%-------------------------------------------------------------------------
%      m(0,t) - phi(1,t) m(1,t) - phi(2,t) m(2,t)
%        = cc(t) + phi(1,t) bb(t) - aa(t) [phi(1,t) phi(1,t-1) + phi(2,t)]
%-------------------------------------------------------------------------
for i=1:m
   if i==1
      a(2*m+i,3*i-2) = 1;
      a(2*m+i,3*i-1) = -phi(1,i);
      a(2*m+i,3*i) = -phi(2,i);
      y(2*m+i) = cc(i) - aa(i) * (phi(1,i) ...
                 * phi(1,m)+phi(2,i) ) + bb(i)*phi(1,i);
   else
      a(2*m+i,3*i-2) = 1;
      a(2*m+i,3*i-1) = -phi(1,i);
      a(2*m+i,3*i) = -phi(2,i);
      y(2*m+i) = cc(i) - aa(i) * (phi(1,i) ...
                 * phi(1,i-1) + phi(2,i) ) + bb(i)*phi(1,i);
   end;
end;

% ** Solve a * x = y for x
x = a \ y';
clear a

% ** Store solution x in scr (matrix of seasonal covariances)
clear var;
clear scr;
for i=1:m
   var(i) = x(3*i-2);
   scr(1,i) = x(3*i-1);
   scr(2,i) = x(3*i);
end;
```

```
% ** Compute seasonal covariance up to order mslag
for k=3:mslag
   for i=1:m
      if i==1
         scr(k,i) = phi(1,i)*scr(k-1,m) + phi(2,i)*scr(k-2,m-1);
      elseif i==2
         scr(k,i) = phi(1,i)*scr(k-1,i-1) + phi(2,i)*scr(k-2,m);
      else
         scr(k,i) = phi(1,i)*scr(k-1,i-1) + phi(2,i)*scr(k-2,i-2);
      end;
   end;
end;

% ** Compute seasonal correlations up to order mslag
for i=1:m
   for k=1:mslag
      if(i-k>0)
         scr(k,i) = scr(k,i) / sqrt(var(i)) / sqrt(var(i-k));
      else
         scr(k,i) = scr(k,i) / sqrt(var(i)) / sqrt(var(i+12-k));
      end;
   end;
end;
```

```
% ****************************************************************
%    Moment estimation of the periodic cross-covariance of
%    residuals at two sites
%                  @1994 by Peter F. Rasmussen
% ****************************************************************
%
% Comment:
%   For a given solution for phi and tht at two sites,
%   and observed cross correlation 'm12' of flows at the
%   same two sites, the program computes the covariance
%   of residuals 'g12' that produce the cross correlation
%   of flows.
%
%   NB  If g1 and g2 (the residual variances at the two
%       sites) do not produce a model variance of exactly
%       1, then it might be preferable to estimate the g12
%       that produces a cross covariance of
%       m12*(var1*var2)^0.5, where var1 and var2 are the
%       the model variance (these can be obtained with
%       the routine MODELCOR.M). This adjustment assures
%       the model generates flows with the observed cross
%       correlation, although not with the observed variance.
%
%   NB  It is very important to check that the covariance
%       matrix G is consistent, i.e. is positive semi-definite.
%       There is no guarantee that a moment solution exists!
%
%  function cov=gij(phi1,tht1,phi2,tht2,m12)
%
%
% Input   phi1 (2 x 52) : matrix for site 1
%         tht1 (2 x 52) : matrix for site 1
%         phi2 (2 x 52) : matrix for site 2
%         tht2 (2 x 52) : matrix for site 2
%         m12  (52)     : correlations of transformed data
%
% Output  g12 : covariance of residuals
% ------------------------------------------------------------

function cov=gij(phi1,tht1,phi2,tht2,m12)

m=52;

% ** Fill out coefficient matrix
clear A y x

for i=1:m
   i1 = i-1;
   i2 = i-2;
   if i==1
      i1 = m;
      i2 = m-1;
   elseif i==2
      i2 = m;
   end
```

```
    % ** First set of equations
    A(i,i2) = -(  (phi1(1,i)*phi1(1,i1) - phi1(1,i)*tht1(1,i1) ...
                  + phi1(2,i) - tht1(2,i)    ) * tht2(2,i)   ...
                  + tht1(2,i)*phi2(2,i)    );
    A(i,i1) = -tht2(1,i) * ( phi1(1,i) - tht1(1,i) );
    A(i,i) = 1;
    A(i,m+i1) = phi1(1,i) * phi2(2,i);
    A(i,m+i) = phi2(1,i);

    % ** Second set of equations
    A(m+i,i2) = tht2(2,i) * (  phi1(1,i1) - tht1(1,i1)   );
    A(m+i,i1) = tht2(1,i);
    A(m+i,m+i1) = -phi2(2,i);
    A(m+i,2*m+i) = 1;

    % ** Third set of equations
    A(2*m+i,i2) = tht1(2,i) * (  phi2(1,i1) - tht2(1,i1)   );
    A(2*m+i,i1) = tht1(1,i);
    A(2*m+i,m+i) = 1;
    A(2*m+i,2*m+i1) = -phi1(2,i);

    % ** Right hand side
    y(i) = m12(i) - phi1(2,i)*phi2(2,i)*m12(i2);
    y(m+i) = phi2(1,i)*m12(i1);
    y(2*m+i) = phi1(1,i)*m12(i1);
end;

% ** Solve system for vector x
% ** The elements of x are [g12(1:m) m12(1)(1:52) m21(1)(1:52)]
x = A \ y';

cov=x(1:m);
```

```
%*************************************************************
%       Adjustment of covariance matrices of residuals
%               @1994  by  Peter F. Rasmussen
%*************************************************************
%
% Comment:  The procedure for adjusting the G matrices (i.e.
%           making them non-negative definite) consists of
%           the following steps:
%
%       1   The G matrix is first decomposed into eigenvectors
%           and eigenvalues, EVEC and EVAL, using the 'eig'
%           function of MATLAB.
%
%       2   If G has negative eigenvalues, then these are set
%             to zero. An adjusted G matrix is computed as
%               CORRG = EVEC * EVAL * EVEC'
%             where EVAL is the adjusted diagonal matrix of
%             non-negative eigenvalues
%       3   An additional adjustment is made to ensure that the
%           variances in the original G matrix are preserved.
%           CORRG is multiplied from left and from right with
%           a diagonal matrix having the elements
%           sqrt( G(ii)/CORRG(ii) )
%
% Input : parameters and flow values for each of the five
%         sites must be specified in the beginning of the
%         program, as well as the model variance corrsponding
%         to each site (use program MODELCOV)
%
% Output: vectors dij, i=1,..,5   j=i,..,5   which are the
%         elements of adjusted G matrices
%
%         mmij : model cross-covariance of flows at site i and j
% -----------------------------------------------------------

% Selected LS-solution for the five sites
% RESI_VA1.MAT contains the parameters below plus the
% model variances corresponding to each solution
load resi_va1

% Define input data
phi1 = phi_ne;
tht1 = tht_ne;
g1 = g_ne;
q1 = q_ne;
m11 = var_ne;

phi2 = phi_no;
tht2 = tht_no;
g2 = g_no;
q2 = q_no;
m22 = var_no;

phi3 = phi_e;
tht3 = tht_e;
```

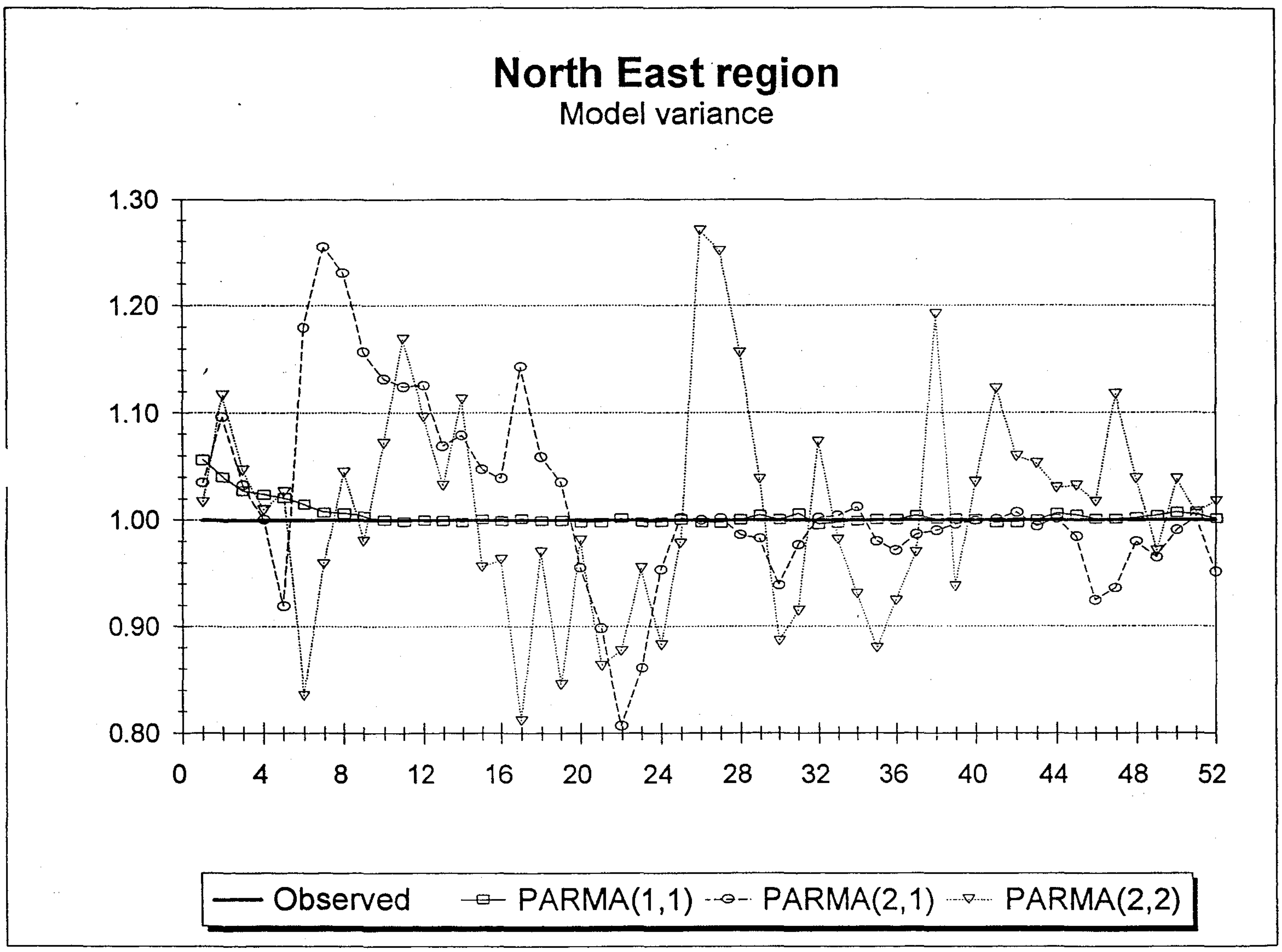
```
g3 = g_e;
q3 = q_e;
m33 = var_e;

phi4 = phi_c;
tht4 = tht_c;
g4 = g_c;
q4 = q_c;
m44 = var_c;

phi5 = phi_s;
tht5 = tht_s;
g5 = g_s;
q5 = q_s;
m55 = var_s;

% ** compute cross-correlations
m12 = cros_cor(q1,q2) .* sqrt(m11.*m22);
m13 = cros_cor(q1,q3) .* sqrt(m11.*m33);
m14 = cros_cor(q1,q4) .* sqrt(m11.*m44);
m15 = cros_cor(q1,q5) .* sqrt(m11.*m55);
m23 = cros_cor(q2,q3) .* sqrt(m22.*m33);
m24 = cros_cor(q2,q4) .* sqrt(m22.*m44);
m25 = cros_cor(q2,q5) .* sqrt(m22.*m55);
m34 = cros_cor(q3,q4) .* sqrt(m33.*m44);
m35 = cros_cor(q3,q5) .* sqrt(m33.*m55);
m45 = cros_cor(q4,q5) .* sqrt(m44.*m55);

% ** Estimate covariance matrices of residuals
g12 = gij(phi1,tht1,phi2,tht2,m12);
g13 = gij(phi1,tht1,phi3,tht3,m13);
g14 = gij(phi1,tht1,phi4,tht4,m14);
g15 = gij(phi1,tht1,phi5,tht5,m15);
g23 = gij(phi2,tht2,phi3,tht3,m23);
g24 = gij(phi2,tht2,phi4,tht4,m24);
g25 = gij(phi2,tht2,phi5,tht5,m25);
g34 = gij(phi3,tht3,phi4,tht4,m34);
g35 = gij(phi3,tht3,phi5,tht5,m35);
g45 = gij(phi4,tht4,phi5,tht5,m45);

% ** Decompose G matrices
for i=1:52
   G = [g1(i)  g12(i) g13(i) g14(i) g15(i)
        g12(i) g2(i)  g23(i) g24(i) g25(i)
        g13(i) g23(i) g3(i)  g34(i) g35(i)
        g14(i) g24(i) g34(i) g4(i)  g45(i)
        g15(i) g25(i) g35(i) g45(i) g5(i)  ];

   % ** Decompose G in eigenvectors and eigenvalues
   [evec eval] = eig(G);
   eval = real(eval);

   % ** Set negative eigenvalues equal to zero
   ix = find(eval<0);
   eval(ix)=zeros(1,length(ix));

   % ** Compute adjusted G matrix
   corrG = evec * eval * evec';
```

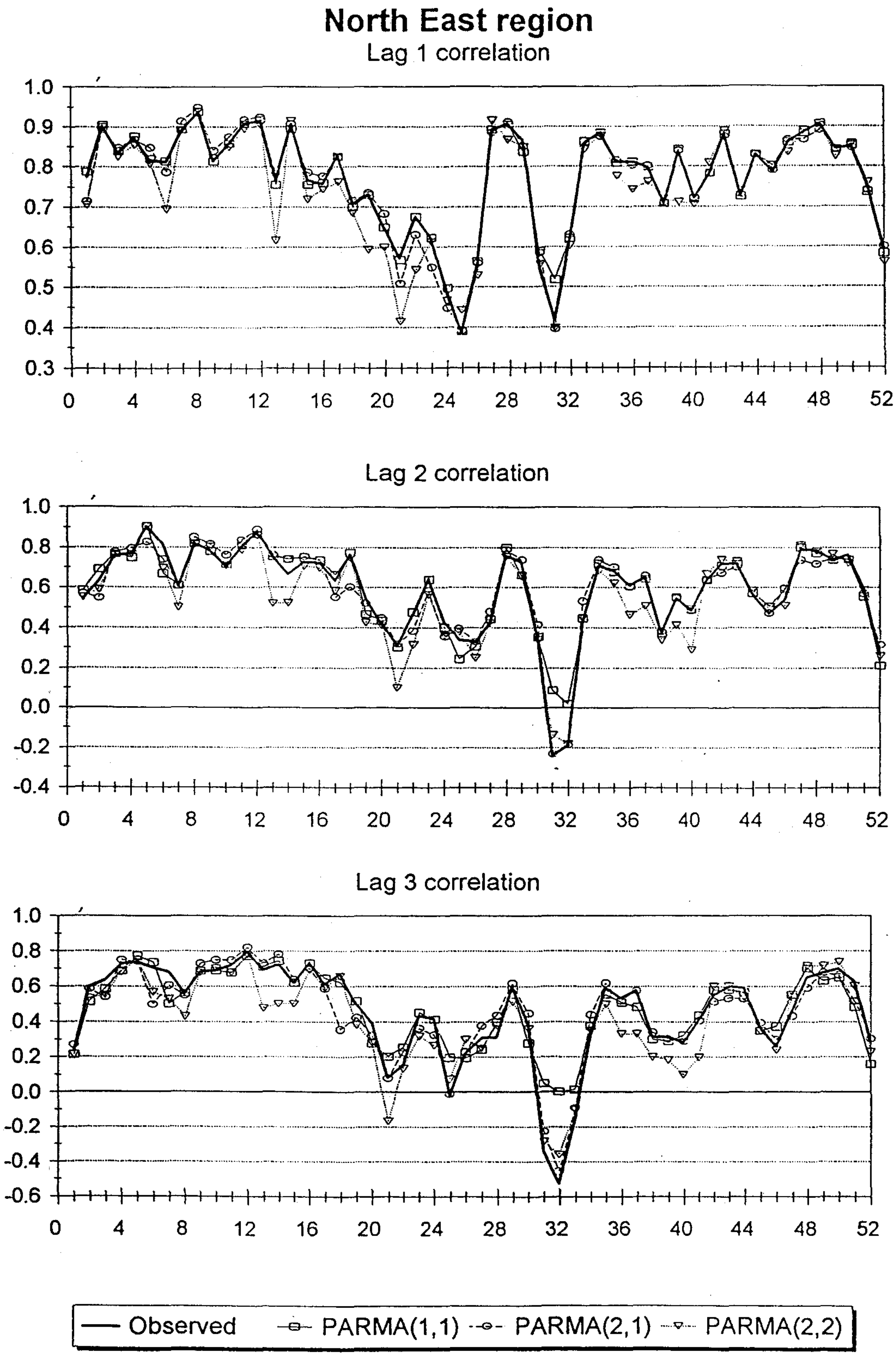
```
    % ** Adjust corrG matrix in order to preserve variances, i.e
    % the diagonal elements of matrix G
    factor = diag(sqrt(diag(G)./diag(corrG)));
    corrG = factor * corrG * factor;

    D = corrG;

    % ** Store result in 10 vectors
    if i==1
       d11 = D(1,1); d12 = D(1,2); d13 = D(1,3); d14 = D(1,4); d15 =
D(1,5);
       d22 = D(2,2); d23 = D(2,3); d24 = D(2,4); d25 = D(2,5);
       d33 = D(3,3); d34 = D(3,4); d35 = D(3,5);
       d44 = D(4,4); d45 = D(4,5);
       d55 = D(5,5);
    else
       d11 = [d11 D(1,1)];
       d12 = [d12 D(1,2)];
       d13 = [d13 D(1,3)];
       d14 = [d14 D(1,4)];
       d15 = [d15 D(1,5)];
       d22 = [d22 D(2,2)];
       d23 = [d23 D(2,3)];
       d24 = [d24 D(2,4)];
       d25 = [d25 D(2,5)];
       d33 = [d33 D(3,3)];
       d34 = [d34 D(3,4)];
       d35 = [d35 D(3,5)];
       d44 = [d44 D(4,4)];
       d45 = [d45 D(4,5)];
       d55 = [d55 D(5,5)];
    end

% ** Decompose to D matrix to B where D=BB'
    [evecD evalD]=eig(D);
    B = real(evecD*sqrt(evalD));

end

% ** Compute model crosscorrelation
mm12 = modelcol(phi1,tht1,phi2,tht2,d12);
mm13 = modelcol(phi1,tht1,phi3,tht3,d13);
mm14 = modelcol(phi1,tht1,phi4,tht4,d14);
mm15 = modelcol(phi1,tht1,phi5,tht5,d15);
mm23 = modelcol(phi2,tht2,phi3,tht3,d23);
mm24 = modelcol(phi2,tht2,phi4,tht4,d24);
mm25 = modelcol(phi2,tht2,phi5,tht5,d25);
mm34 = modelcol(phi3,tht3,phi4,tht4,d34);
mm35 = modelcol(phi3,tht3,phi5,tht5,d35);
mm45 = modelcol(phi4,tht4,phi5,tht5,d45);
```

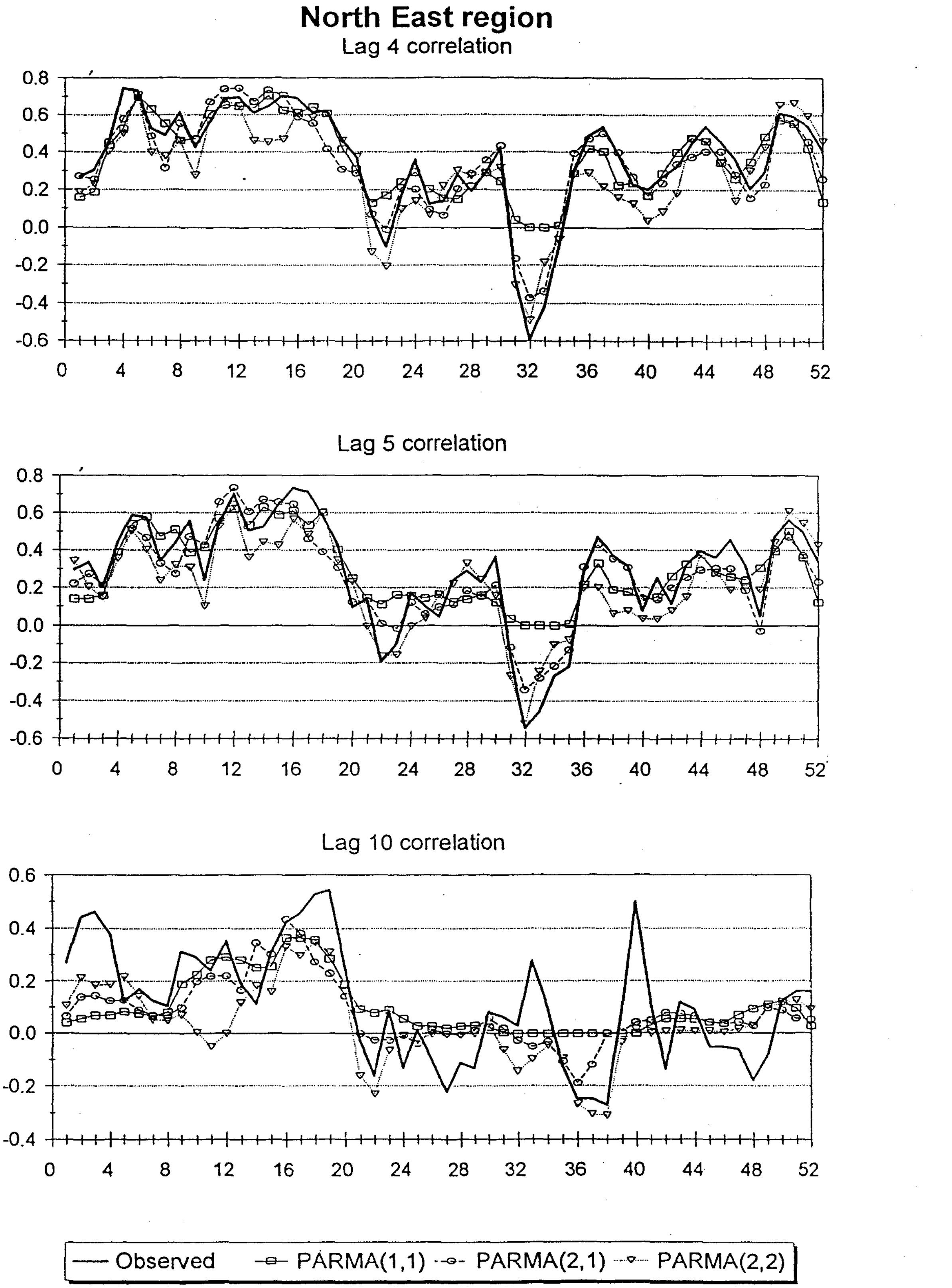
```
%*****************************************************************
%      Cross-correlations of flows at two sites corresponding
%      to a calibrated model
%                  @1994 by Peter F. Rasmussen
%*****************************************************************
%
% Comment:  For a given solution for phi and tht at two sites,
%           and an estimate of the covariance of the residuals,
%           the program permits to evaluate the correlation of
%           flows at the two sites
%           NB  Before using this program, it should be verified
%               that all G matrices are positive semi-definite.
%               If this is not the case, an ajustment must be .
%               made on G%
%
%  function m12=modelcol(phi1,tht1,phi2,tht2,g12)
%
%
% Input:  phi1 (2 x 52) : matrix for site 1
%         tht1 (2 x 52) : matrix for site 1
%         phi2 (2 x 52) : matrix for site 2
%         tht2 (2 x 52) : matrix for site 2
%         g12  (52)     : covariance of residuals
%
% Output: m12 (52) : correlations of transformed data
%-------------------------------------------------------------------

function m12=modelcol(phi1,tht1,phi2,tht2,g12)

% ** Define constant
m = 52;

% ** Initialize variables
A=zeros(4*m,4*m);
clear y

% ** Fill out coefficient matrix
for i=1:m
   i1=i-1;
   i2=i-2;
   if i==1
      i1=m;
      i2=m-1;
   end
   if i==2
      i2=m;
   end

   % ** First set of equation corresponding to m12(0)
   A(i,i2) = -phi1(2,i)*phi2(2,i);
   A(i,i) = 1;
   A(i,2*m+i1) = -phi1(1,i)*phi2(2,i);
   A(i,2*m+i) = -phi2(1,i);
   y(i) = -tht1(2,i)*g12(i2)*phi2(2,i) + g12(i) ...
          - ( phi1(1,i)-tht1(1,i) )*g12(i1)*tht2(1,i) ...
          - ( phi1(1,i)*phi1(1,i1) - phi1(1,i)*tht1(1,i1) ...
             + phi1(2,i) - tht1(2,i) ) * g12(i2) * tht2(2,i);
```

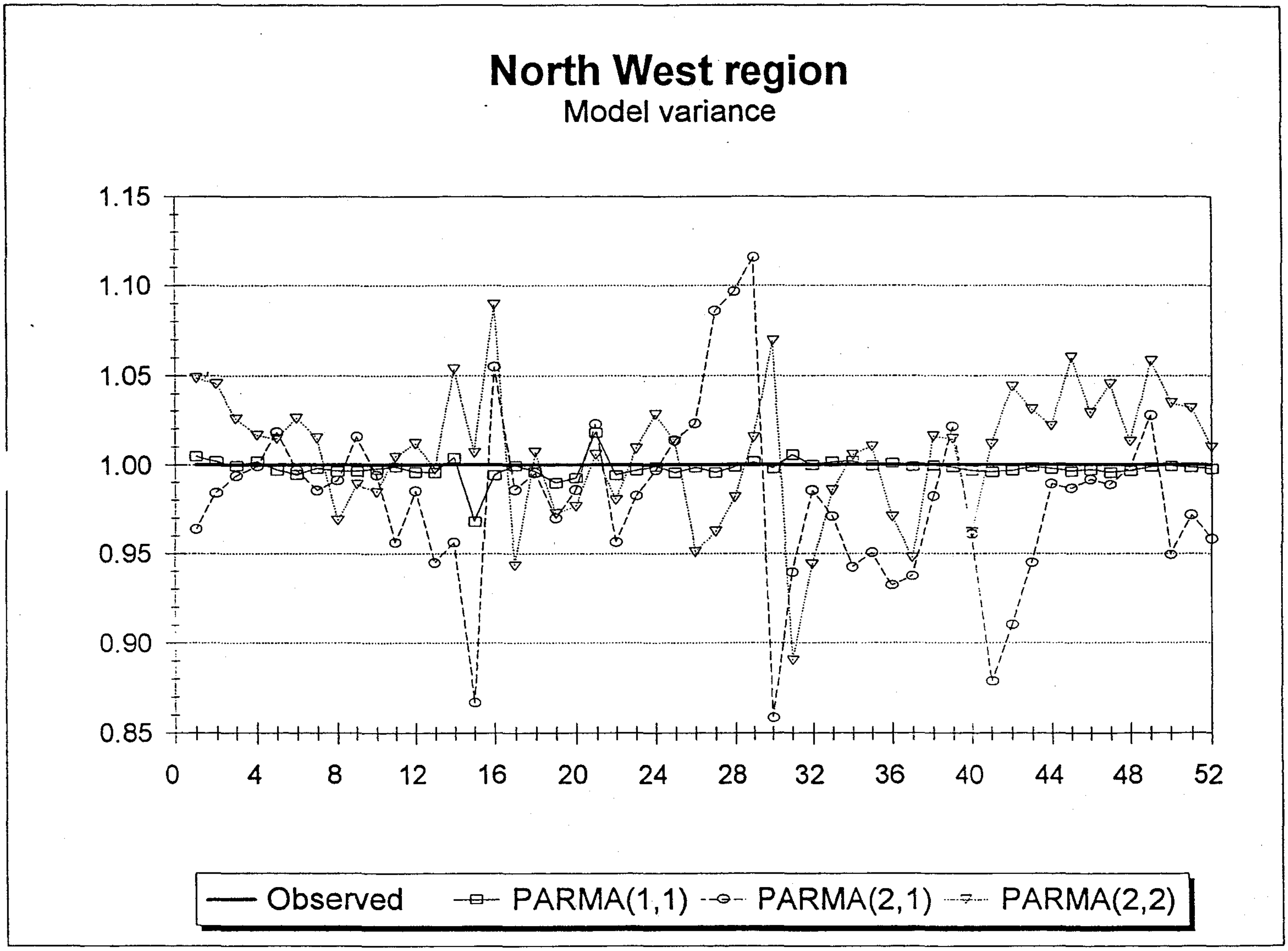
```
    % ** Second set of equation corresponding to m21(0)
    A(m+i,m+i2) = -phi2(2,i)*phi1(2,i);
    A(m+i,m+i) = 1;
    A(m+i,3*m+i1) = -phi2(1,i)*phi1(2,i);
    A(m+i,3*m+i) = -phi1(1,i);
    y(m+i) = -tht2(2,i)*g12(i2)*phi1(2,i) + g12(i) ...
           - ( phi2(1,i)-tht2(1,i) )*g12(i1)*tht1(1,i) ...
           - ( phi2(1,i)*phi2(1,i1) - phi2(1,i)*tht2(1,i1) ...
               + phi2(2,i) - tht2(2,i) ) * g12(i2) * tht1(2,i);

    % ** Third set of equations corresponding to m12(1)
    A(2*m+i,i1) = phi2(1,i);
    A(2*m+i,2*m+i1) = phi2(2,i);
    A(2*m+i,3*m+i) = -1;
    y(2*m+i) = g12(i1)*tht2(1,i) + (phi1(1,i1)-
tht1(1,i1))*g12(i2)*tht2(2,i);

    % ** Forth set of equations corresponding to m21(1)
    A(3*m+i,m+i1) = phi1(1,i);
    A(3*m+i,3*m+i1) = phi1(2,i);
    A(3*m+i,2*m+i) = -1;
    y(3*m+i) = g12(i1)*tht1(1,i) + (phi2(1,i1)-
tht2(1,i1))*g12(i2)*tht1(2,i);
end

% ** Solve system for vector x
x = A \ y';

m12 = x(1:m);
```

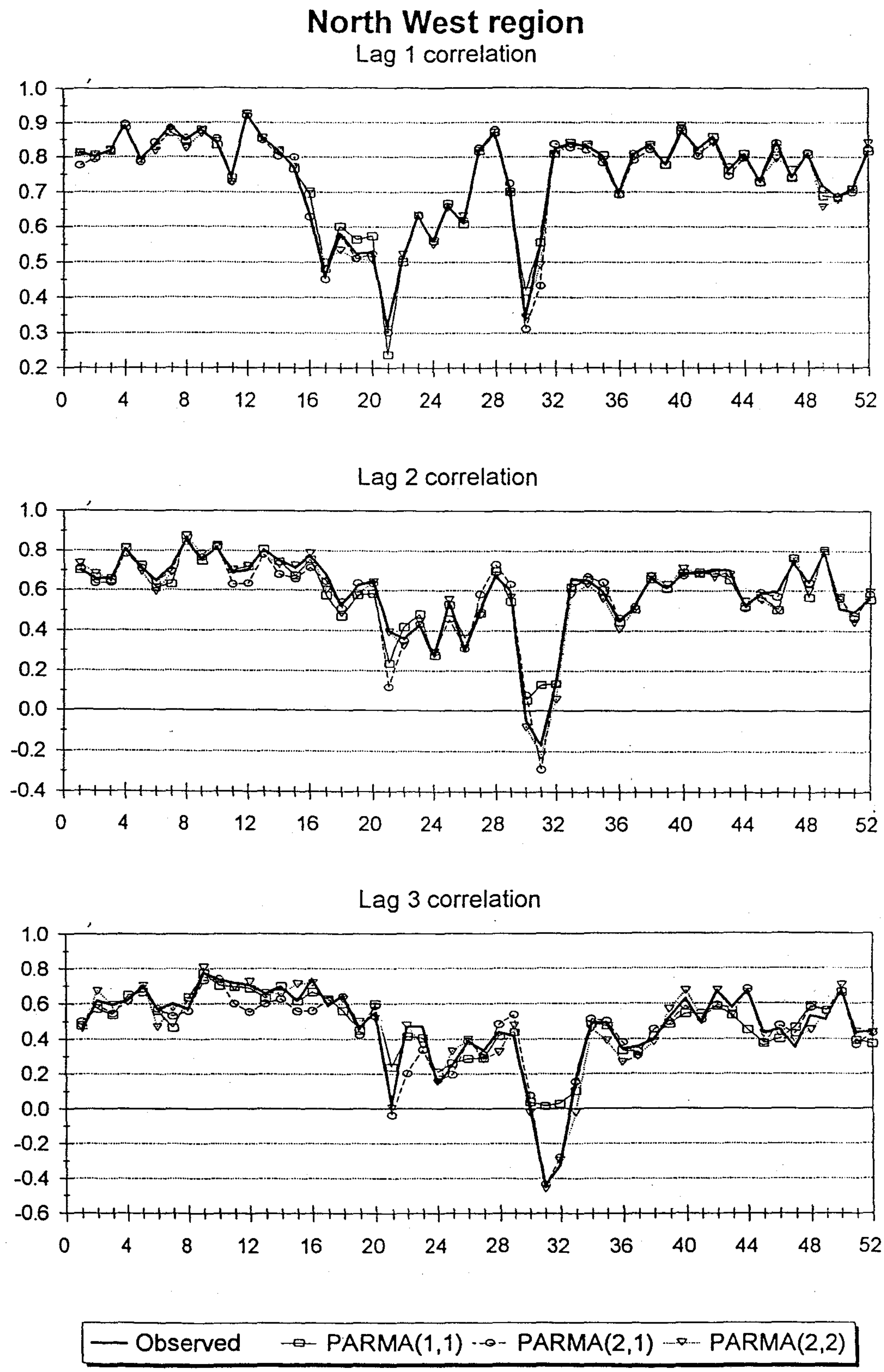
Appendix C
Periodic variance and periodic autocorrelation of fitted models

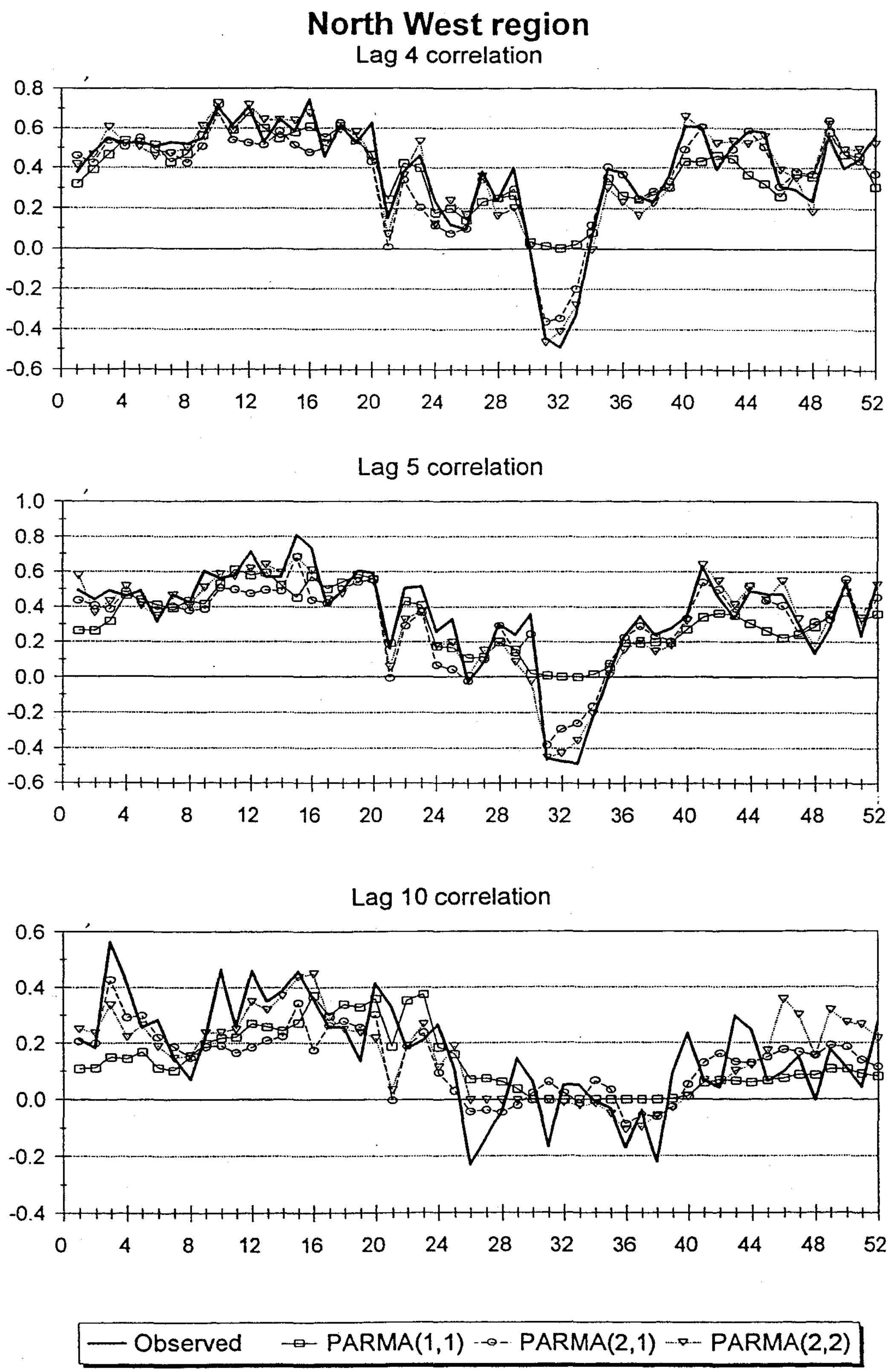

North East region
Model variance
1.30
1.20
1.10
1.00
0.90
0.80
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

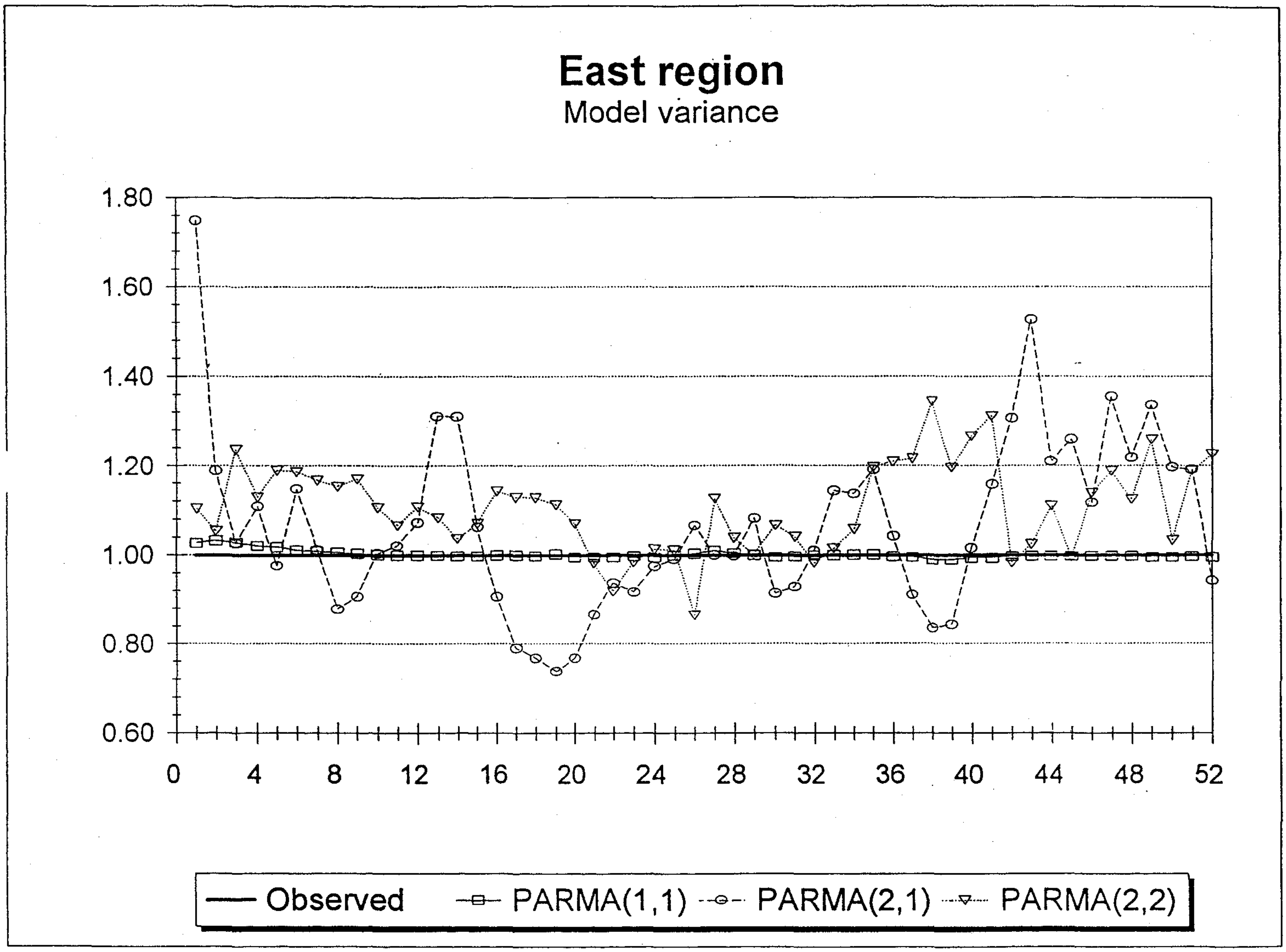

North East region
Lag 1 correlation
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 2 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 3 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

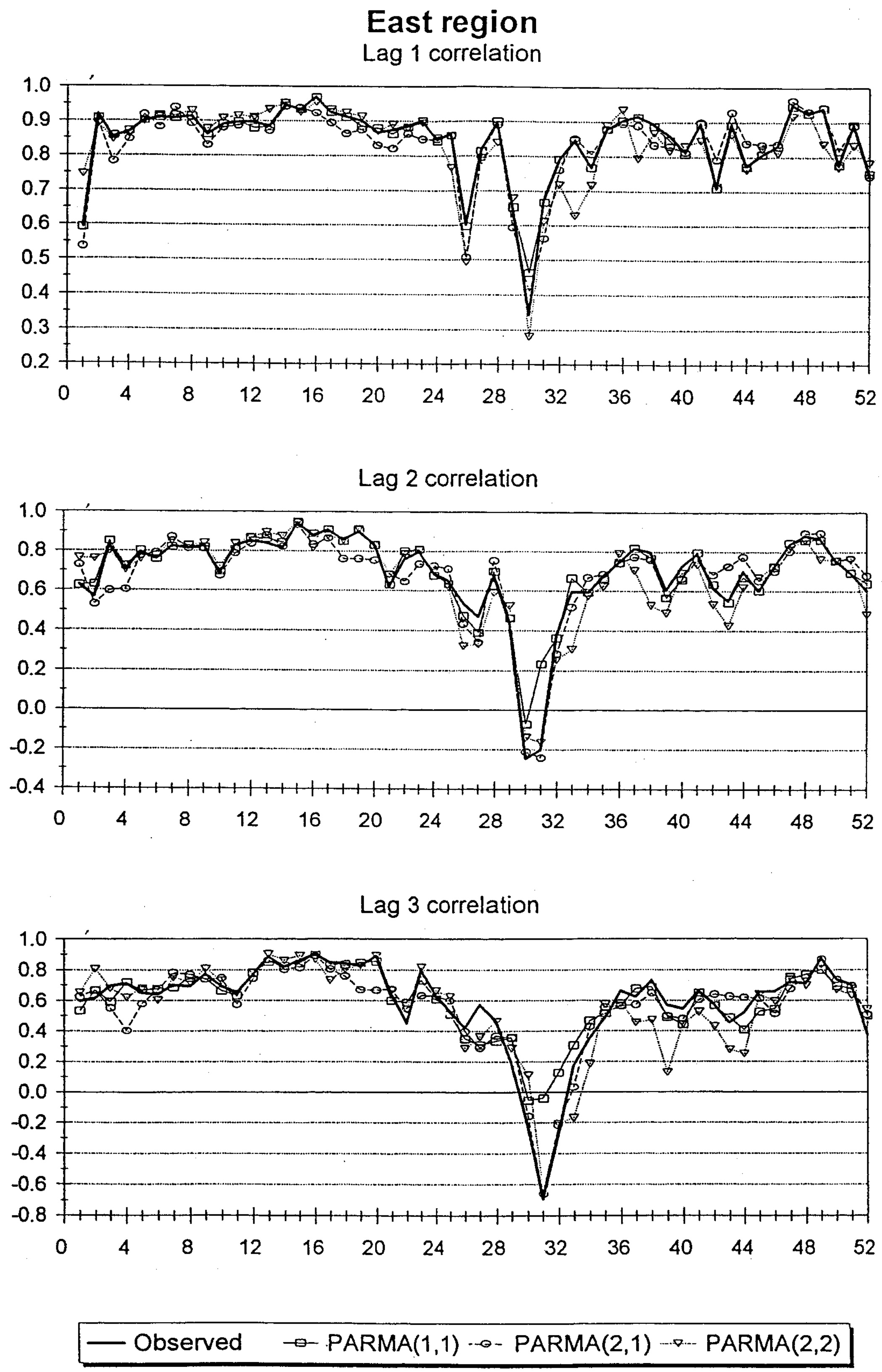

North East region
Lag 4 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 5 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 10 correlation
0.6
0.4
0.2
0.0
-0.2
-0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


North West region
Model variance
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


North West region
Lag 1 correlation
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 2 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 3 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

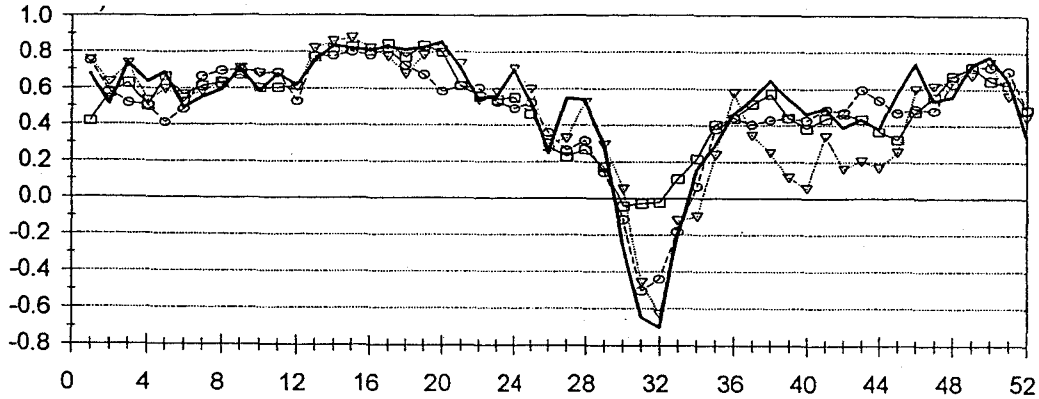

North West region
Lag 4 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 5 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 10 correlation
0.6
0.4
0.2
0.0
-0.2
-0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


East region
Model variance
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

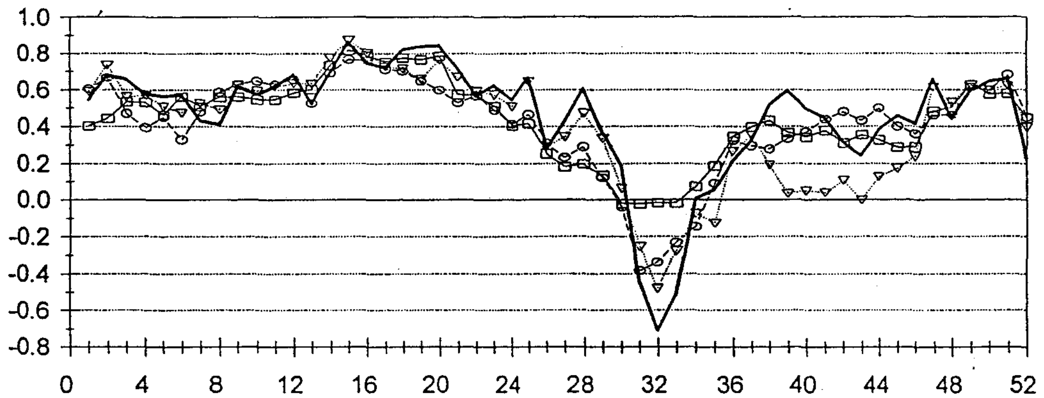

East region
Lag 1 correlation
1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 2 correlation
1.0 0.8 0.6 0.4 0.2 0.0 -0.2 -0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 3 correlation
1.0 0.8 0.6 0.4 0.2 0.0 -0.2 -0.4 -0.6 -0.8
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

East region

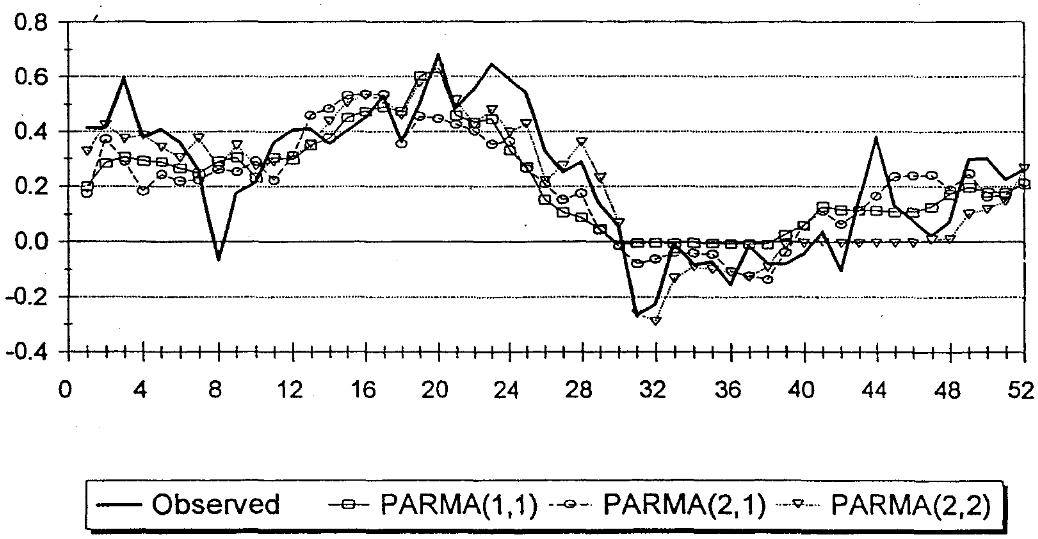
Lag 4 correlation

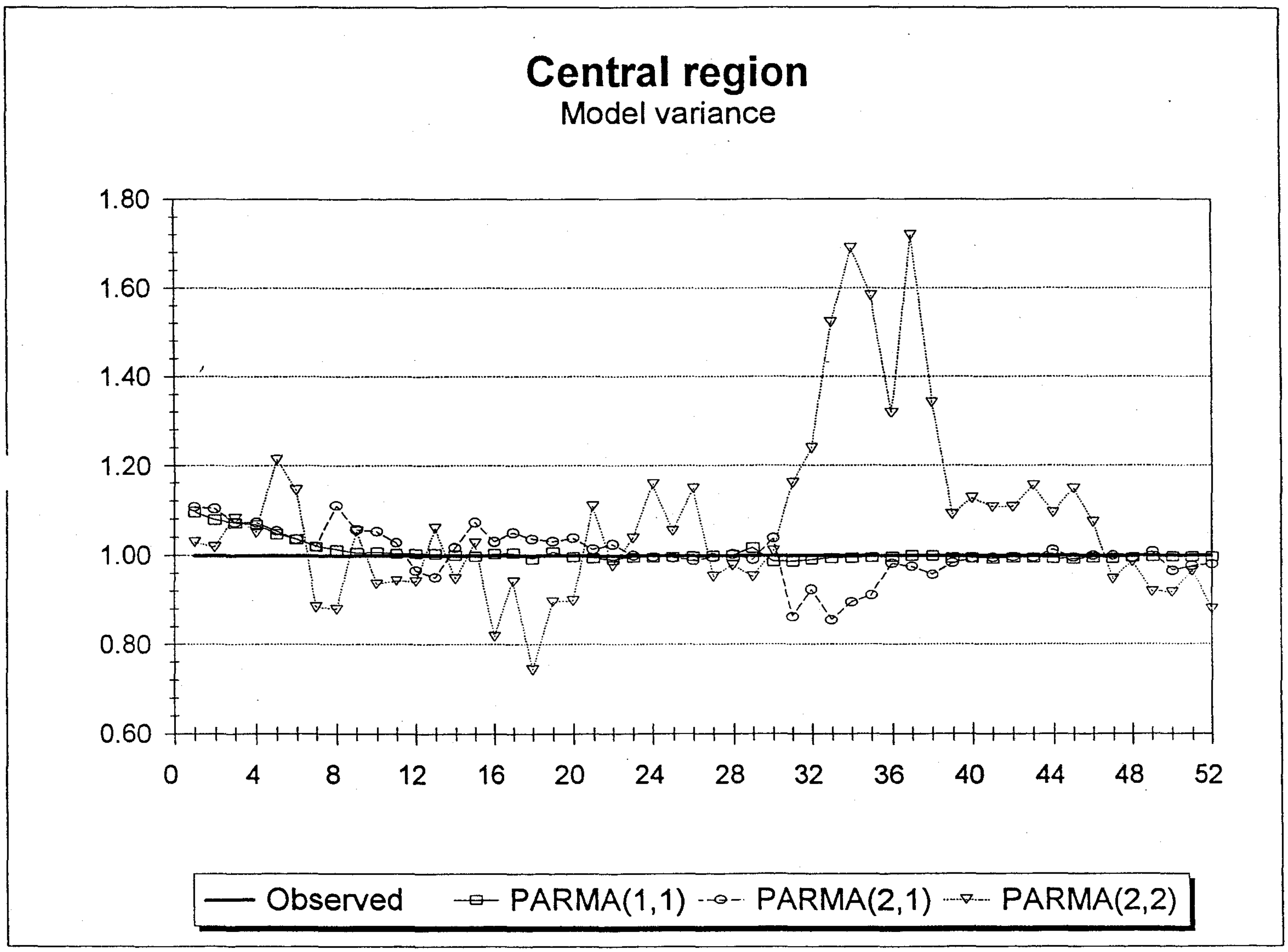


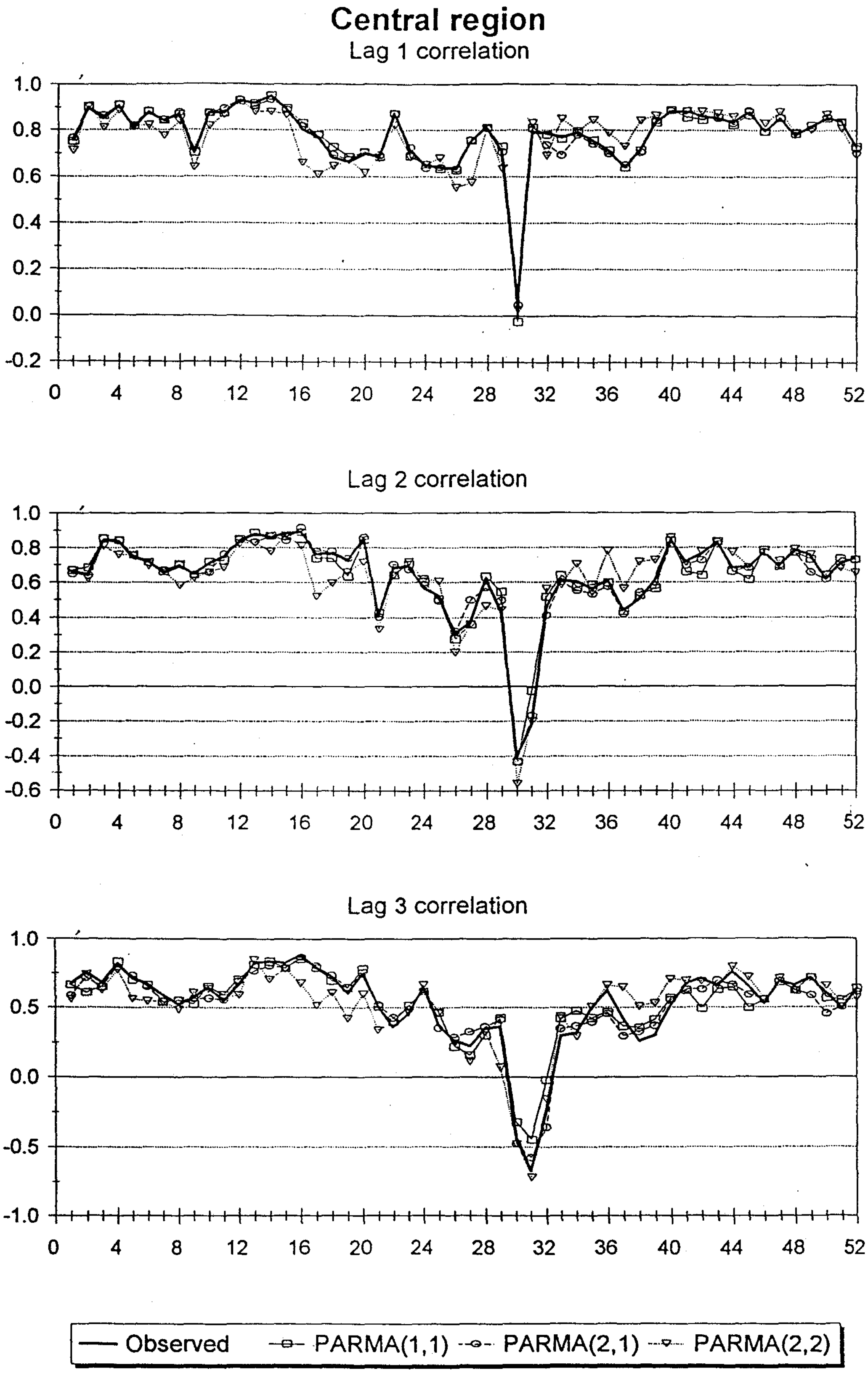
Lag 5 correlation

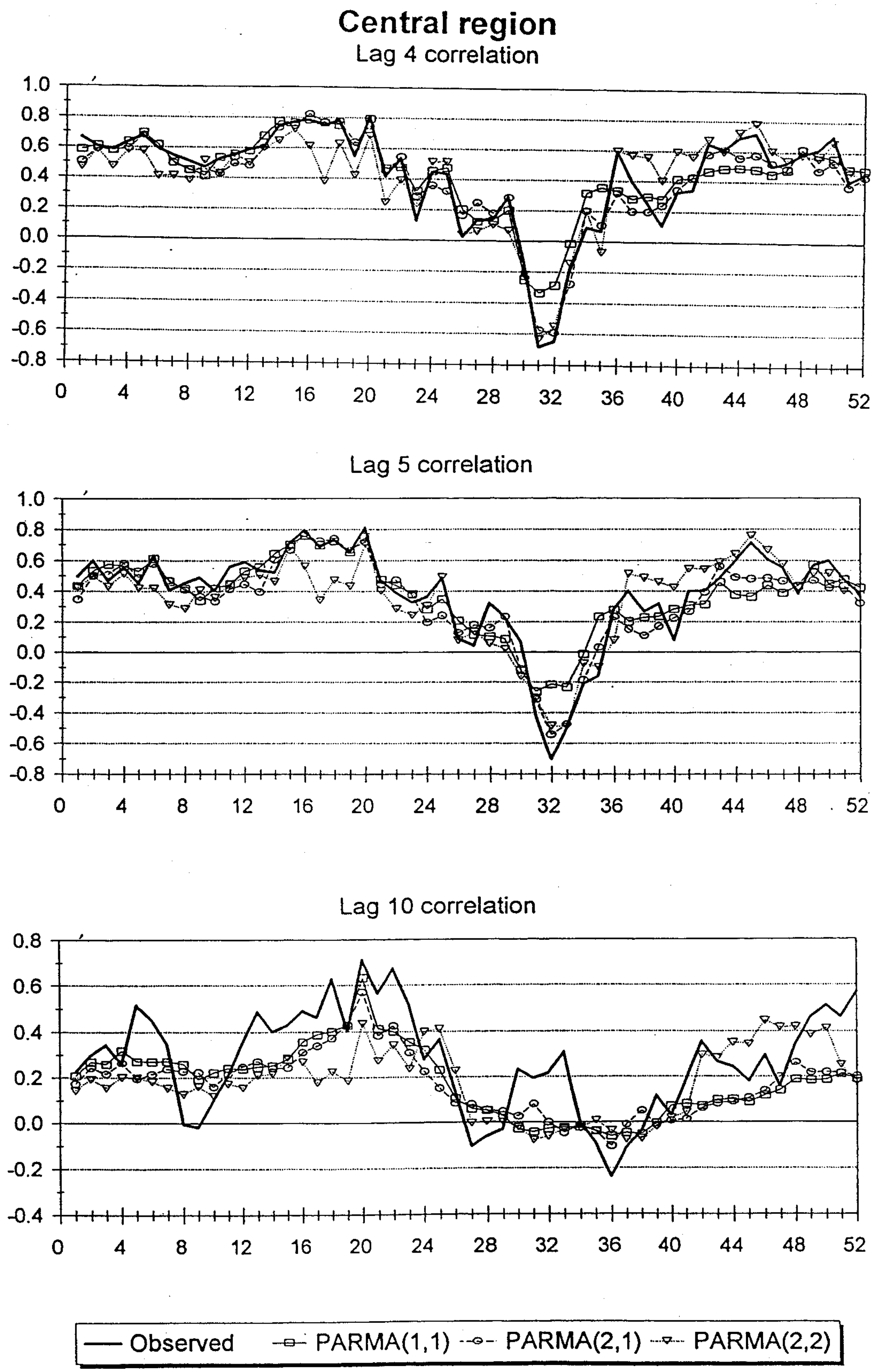


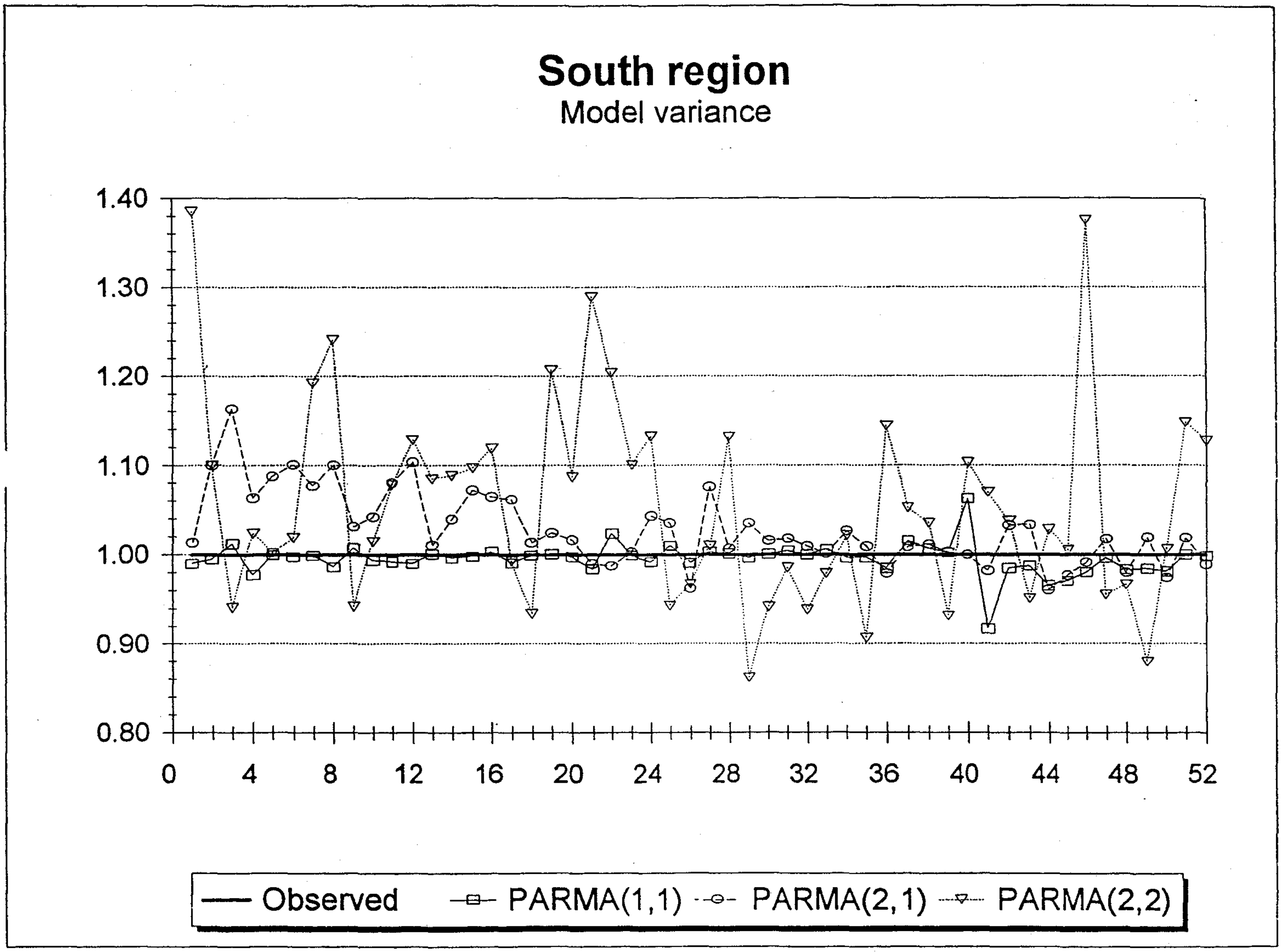
Lag 10 correlation

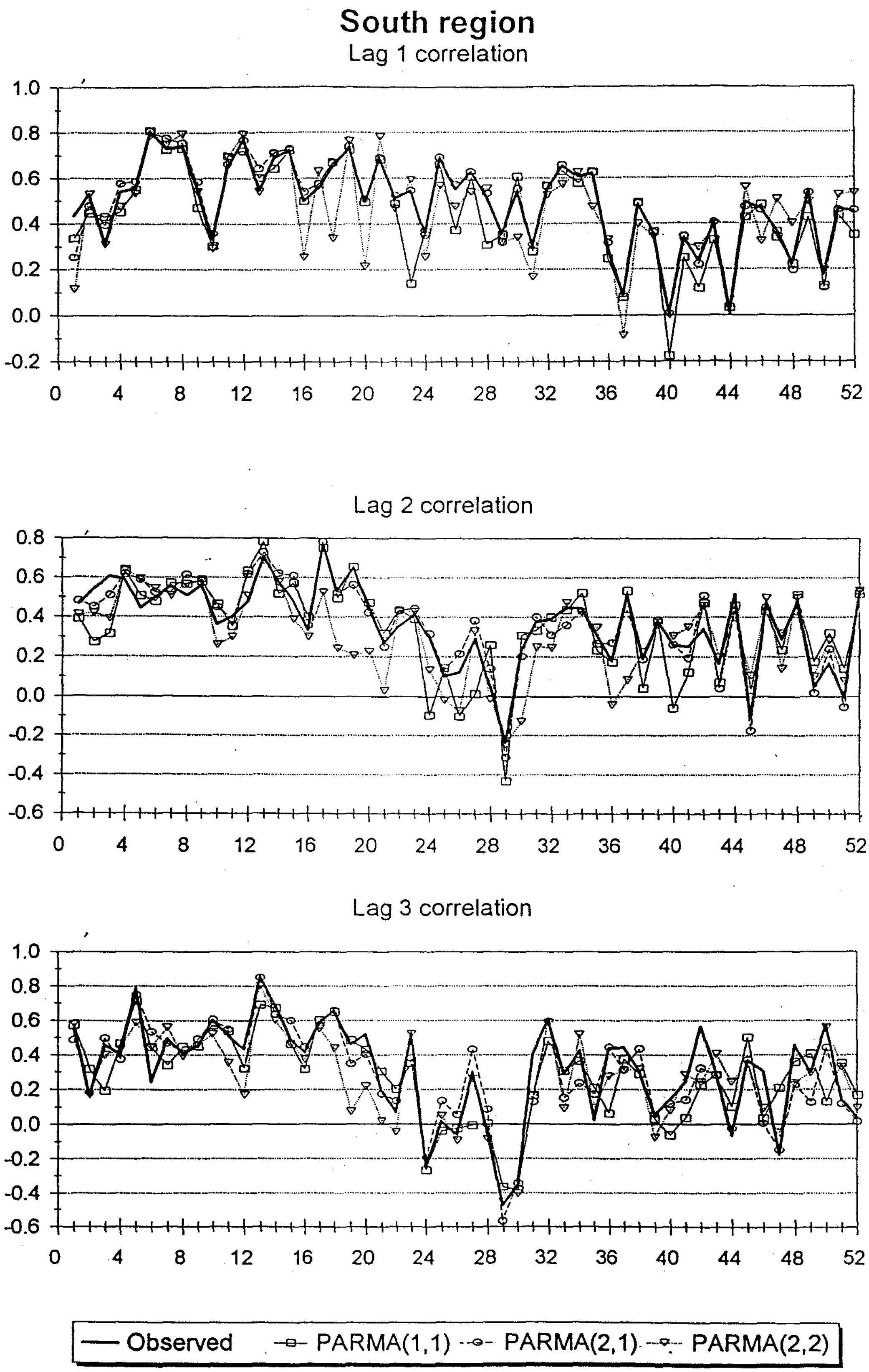


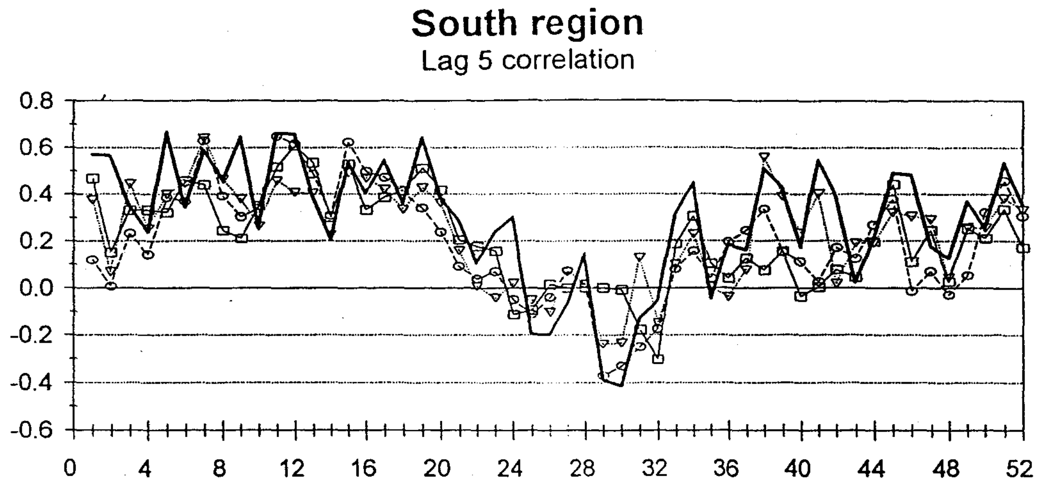

Central region
Model variance
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

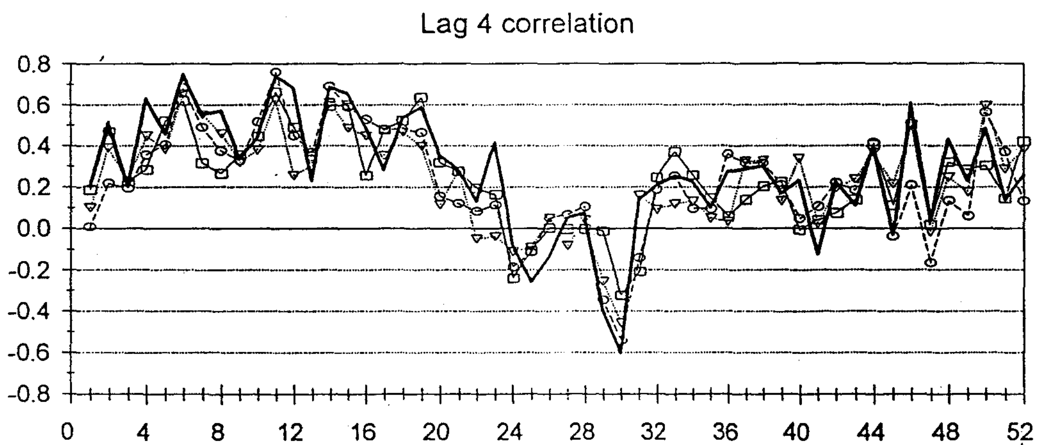

Central region
Lag 1 correlation
Lag 2 correlation
Lag 3 correlation
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)

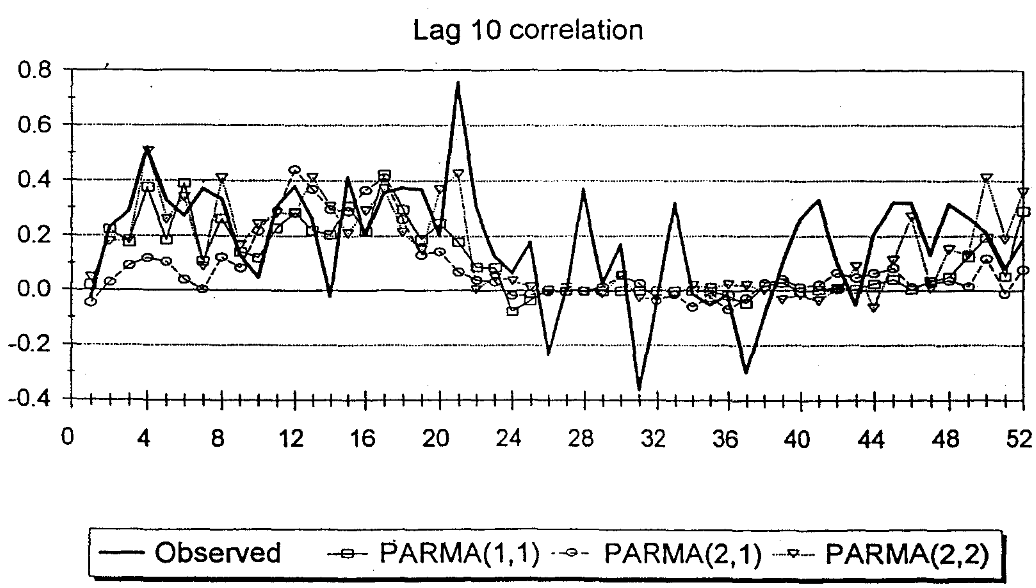

Central region
Lag 4 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 5 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 10 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


South region
Model variance
1.40
1.30
1.20
1.10
1.00
0.90
0.80
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


South region
Lag 1 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 2 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Lag 3 correlation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0 4 8 12 16 20 24 28 32 36 40 44 48 52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)


South region
Lag 5 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
4
8
12
16
20
24
28
32
36
40
44
48
52


Lag 4 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
0
4
8
12
16
20
24
28
32
36
40
44
48
52


Lag 10 correlation
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Observed
PARMA(1,1)
PARMA(2,1)
PARMA(2,2)